Existence of Multiple Positive Solutions for \( n \)th Order Two-Point Boundary Value Problems on Time Scales

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Received July 14, 2012; revised September 18, 2012; accepted September 26, 2012

ABSTRACT

We consider the \( n \)th order nonlinear differential equation on time scales

\[
y^{(n)}(t) + f(t, y(t)) = 0, \quad t \in [a, b],
\]

subject to the right focal type two-point boundary conditions

\[
y^{(i)}(a) = 0, \quad 0 \leq i \leq n-2
\]

\[
y^{(n)}(\sigma^{n-p}(b)) = 0, \quad (1 \leq p \leq n-1, \text{but fixed}).
\]

We establish a criterion for the existence of at least one positive solution by utilizing Krasnosel’skiĭ fixed point theorem. And then, we establish the existence of at least three positive solutions by utilizing Leggett-Williams fixed point theorem.

Keywords: Time Scale, Dynamical Equation, Positive Solution, Cone, Boundary Value Problem

1. Introduction

The study of the existence of positive solutions of boundary value problems (BVPs) for higher order differential equations on time scales has gained prominence and it is a rapidly growing field, since it arises, especially for higher order differential equations on time scales arise naturally in technical applications. Meyer [1], strictly speaking, boundary value problems for higher order differential equation on time scales are a particular class of interface problems. One example in which this is exhibited is given by Keener [2] in determining the speed of a flagellate protozoan in a viscous fluid. Another particular case of a boundary value problem for a higher order differential equation on time scales arising as an interface problem is given by Weyner, et al. [3] in dealing with a study of perfectly wetting liquids. In these applied settings, only positive solutions are meaningful. By a time scale we mean the intersection of the real interval with a given time scale. The existence of positive solutions for BVPs has been studied by many authors, first for differential equations, then finite difference equations, and recently, unifying results for dynamic equations. We list some papers, Erbe and Wang [5], and Eloe and Henderson [6,7], Ahczi and Guseinor [8], and Anderson and Avery [9], and Avery and Peterson [10], Agarwal, Regan and Wang [11], Deimling [12], Gregus [13] Guo and Lakshmikantham [14], Henderson and Ntouyas [15], Hopkins [16] and Li [17]. Recently, in 2008, Moustafa Shehed [18] obtained at least one positive solution to the boundary value problem

\[
y^{(n)}(t) + \lambda a(t)f(y(t)) = 0, 0 < t < 1,
\]

\[
y(0) = y^{(n-2)}(0) = y^{(n-1)} = y'(1) = 0.
\]

This paper considers the existence of positive solutions to \( n \)th order nonlinear differential equation on time scales by the symbol \( \sigma \). By an interval we mean the intersection of the real interval with a given time scale. The existence of positive solutions for BVPs has been studied by many authors, first for differential equations, then finite difference equations, and recently, unifying results for dynamic equations. We list some papers, Erbe and Wang [5], and Eloe and Henderson [6,7], Ahczi and Guseinor [8], and Anderson and Avery [9], and Avery and Peterson [10], Agarwal, Regan and Wang [11], Deimling [12], Gregus [13] Guo and Lakshmikantham [14], Henderson and Ntouyas [15], Hopkins [16] and Li [17]. Recently, in 2008, Moustafa Shehed [18] obtained at least one positive solution to the boundary value problem

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scales
\[ y^{(n)}(t) + f(t, y(t)) = 0, \quad t \in [a, b], \quad a < b \] subject to the right focal type boundary conditions
\[ y^{(n)}(a) = 0, \quad 0 \leq i \leq n - 2 \] holds for all \( m \leq n \in \mathbb{N} \), where \( \sum_{n_1, n_2, \ldots, n_m \in \mathbb{N}} n_1 + n_2 + \cdots + n_m \) is the set of all distinct combinations of \( \{n_1, n_2, \cdots, n_m\} \) such that the sum is equal to given \( r \).

Proof see [19].

We denote
\[ \omega(t, s) = \sum_{j=0}^{n-1} [(t-s)^{n-1-j}] \]
and

**Theorem 2.2.** Green's function for the homogeneous BVP
\[ -y^{(n)} = 0, \]
with the boundary conditions (2), (3) is given by
\[ G(t, s) = \left\{ \begin{array}{ll} G_1(t, s), & t \leq s \\ G_1(t, s) - \frac{1}{(n-1)!} \prod_{j=1}^{n-1} (t-\sigma^j(s)), & \sigma(s) \leq t \end{array} \right. \]
where
\[ G_1(t, s) = \frac{\omega(\sigma^{n-r}(b), \sigma^j(s)) \prod_{i=1}^{n} (t-\sigma^i(a))}{(n-1)! \omega(\sigma^{n-r}(b), \sigma^1(a))} \]
for all \((t, s) \in [a, \sigma^n(b)] \times [a, b]\).

**Proof:** It is easy to check that the BVP \( -y^{(n)} = 0 \) with the boundary conditions (2) and (3) has only trivial solution. Let \( y(t, s) \) be the Cauchy function for \( -y^{(n)} = 0 \), and is given by
\[ y(t, s) = \frac{1}{(n-1)!} \prod_{j=1}^{n} (t-\sigma^j(s)) \]
For each fixed \( s \in [a, b] \), let \( u(\cdot, s) \) be the unique solution of the BVP
\[ -u^{(n)}(\cdot, s) = 0, \quad u^{(n)}(a, s) = 0, \quad 0 \leq i \leq n - 2 \]
and
\[ u^{(n)}(\sigma^{n-r}(b), s) = -\frac{1}{(n-1)!} \omega(\sigma^{n-r}(b), \sigma^1(s)). \]
Since
\[ y_1(t) = 1, \quad y_2(t) = \int_a^t \Delta r, \cdots, \Delta r, \cdots, \Delta r \]
are the solutions of \( -u^{(n)} = 0 \),
\[ u(t, s) = \alpha_1(s) \cdot 1 + \alpha_2(s) \cdot \int_a^t \Delta r + \cdots + \alpha_n(s) \cdot \int_a^{\Delta r \cdots \Delta r} \Delta r \cdots \Delta r \]

## 2. Green's Function and Bounds

In this section, first we state a Lemma to compute delta derivatives for \( r^k \), next, construct a Green's function for homogeneous two point BVP \( -y^{(n)} = 0 \) with (2), (3) and estimate the bounds to the Green's function.

**Lemma 2.1.** Let \( n \in \mathbb{N} \), define a function \( f : T \rightarrow \mathbb{R} \) by \( f(t) = r^t \), if we assume that the conditions (A2) and (A3) are satisfied, then
\[ f^{(n)}(t) = \frac{n!}{(n-1)!} \sum_{i=1}^{n} \omega_i \]
\[ \omega_i = \prod_{j=1}^{n} \frac{r^{(i-1)}}{(i-1)!} \prod_{k=1}^{n} \frac{r^{(n-1)k}}{(n-1)!} \]
\[ \forall t \in T^{(n)} \]

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By using boundary conditions, \( u^{(i)}(a) = 0 \), \( 0 \leq i \leq n-2 \), we have \( \alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 0 \). Therefore
\[
u(t) = \alpha_n \sum_{i=1}^{n-1} \frac{\omega(\sigma_{n-i}(b), \sigma_i(s))}{(n-1)!} \sigma_i(a)\prod_{i=1}^{n-1} (t-\sigma_i(a)).
\]
Since,
\[
u^{(i)}(\sigma_{n-i}(b), s) = -\nu^{(i)}(\sigma_{n-i}(b), s),
\]
It follows that
\[
\alpha_n = \frac{\omega(\sigma_{n-i}(b), \sigma_i(s))}{(n-1)!} \omega(\sigma_{n-i}(b), \sigma_{n-i}(a))
\]
Hence \( G(t,s) \) has the form for \( t \leq s \),
\[
G(t,s) = \frac{\omega(\sigma_{n-i}(b), \sigma_i(s))}{(n-1)!} \sigma_i(a)\prod_{i=1}^{n-1} (t-\sigma_i(a)).
\]
And for \( t \geq \sigma(s) \), \( G(t,s) = \nu(t) + u(t,s) \). It follows that
\[
G(t,s) = \bar{G}_1(t,s) - \frac{1}{(n-1)!} \sigma_i(a)\prod_{i=1}^{n-1} (t-\sigma_i(s)),
\]
where
\[
G_1(t,s) = \frac{\omega(\sigma_{n-i}(b), \sigma_i(s))}{(n-1)!} \sigma_i(a)\prod_{i=1}^{n-1} (t-\sigma_i(a)).
\]

**Lemma 2.3.** For \( (t,s) \in [a, \sigma^*(b)] \times [a, \sigma(b)] \), we have
\[
G(t,s) \leq G(\sigma^*(b), s).
\]
**Proof.** For \( a \leq t \leq s \leq \sigma^*(b) \), we have
\[
G(t,s) = \bar{G}_1(t,s) \leq \bar{G}_1(\sigma^*(b), s) = G(\sigma^*(b), s).
\]
Similarly, for \( a \leq \sigma(s) \leq t \leq \sigma^*(b) \), we have
\[
G(t,s) \leq \frac{\sigma_i(a)}{\sigma_i(s)} G(\sigma^*(b), s).
\]
Thus, we have
\[
G(t,s) \leq G(\sigma^*(b), s),
\]
for all \( (t,s) \in [a, \sigma^*(b)] \times [a, \sigma(b)] \). \( \square \)

**Lemma 2.4.** Let \( I = \left[ \frac{3a+\sigma^*(b)}{4}, \frac{a+3\sigma^*(b)}{4} \right] \). For \( (t,s) \in I \times [a, \sigma(b)] \), we have
\[
G(t,s) \geq \frac{1}{p \cdot 4^{n-2}} G(\sigma^*(b), s).
\]
**Proof.** The Green's function \( G(t,s) \) for the homogeneous BVP corresponding to (1)-(3) is positive on \( [a, \sigma^*(b)] \times [a, \sigma(b)] \).

For \( a \leq t \leq s < \sigma^*(b) \) and \( t \in I \), we have
\[
G(t,s) \geq \frac{1}{p \cdot 4^{n-2}} G(\sigma^*(b), s).
\]
Similarly, for \( a \leq \sigma(s) \leq t < \sigma^*(b) \) and \( t \in I \) we have
\[
G(t,s) \geq \frac{1}{p \cdot 4^{n-2}} G(\sigma^*(b), s).
\]

3. Existence of at Least One Positive Solution

In this section, we establish a criteria for the existence of at least one positive solution of the BVP (1)-(3). Let \( y(t) \) be the solution of the BVP (1)-(3), and is given by
\[
y(t) = \int_a^y G(t,s) f(s,y(s)) \Delta s,
\]
for all \( t \in [a, \sigma^*(b)] \).

Define \( \beta = \{ y : y \in C[a, \sigma^*(b)] \} \) with the norm
\[
\| y \| = \max_{t \in [a, \sigma^*(b)]} | y(t) |
\]
Then \( (\beta, \| \|) \) is a Banach space. Define a set \( \kappa \) by
\[
\kappa = \{ y \in \beta : y(t) \geq 0 \text{ on } [a, \sigma^*(b)] \}
\]
and
\[
\kappa = \frac{1}{p \cdot 4^{n-2}} \| y \|.
\]
We define the operator \( T : \kappa \rightarrow \beta \) by
\[
(Ty)(t) = \int_a^y G(t,s) f(s,y(s)) \Delta s.
\]
for all \( t \in [a, \sigma^*(b)] \).

**Theorem 3.1.** (Krasnosel'ski) Let \( \beta \) be a Banach space, \( K \subseteq \beta \) be a cone, and suppose that \( \Omega_1 \), \( \Omega_2 \) are open subsets of \( \beta \) with \( 0 \in \Omega_2 \) and \( \Omega_2 \subset \Omega_1 \).

Suppose further that \( T : K \cap (\Omega_2 \setminus \Omega_1) \rightarrow K \) is completely continuous operator such that either
1) \( \|Tu\| \leq \|u\| \), \( u \in K \cap \partial \Omega_2 \) and \( \|Tu\| \geq \|u\| \), \( u \in K \cap \partial \Omega_3 \), or
2) \( \|Tu\| \geq \|u\| \), \( u \in K \cap \partial \Omega_2 \) and \( \|Tu\| \leq \|u\| \), \( u \in K \cap \partial \Omega_3 \) holds. Then \( T \) has a fixed point in \( K \cap (\Omega_2 \setminus \Omega_1) \).

**Theorem 3.2.** If \( f^0 = 0 \) and \( f_\infty = \infty \), then the BVP (1)-(3) has at least one positive solution that lies in \( \kappa \).

*Proof:* We seek a fixed point of \( T \) in \( \kappa \). We prove this by showing the conditions in Theorem 3.1 hold.

First, if \( y \in \kappa \), then

\[
(Ty)(t) = \int_a^b G(t, s) f(s, y(s)) ds
\]

so that

\[
\|Ty\| \leq \int_a^b G\left(\sigma^\mu(b), s\right) f(s, y(s)) ds.
\]

Next, if \( y \in \kappa \), then

\[
\begin{align*}
(Ty)(t) &= \int_a^b G(t, s) f(s, y(s)) ds \\
&\geq \frac{1}{p \cdot 4^{n-1}} \int_a^b G\left(\sigma^\mu(b), s\right) f(s, y(s)) ds \\
&\geq \frac{1}{p \cdot 4^{n-1}} \|\|Tu\| \|, t \in I.
\end{align*}
\]

Hence, \( T : \kappa \to \kappa \). Standard argument involving the Arzelà-Ascoli theorem shows that \( T \) is completely continuous operator. Since \( f^0 = 0 \), there exist \( \eta_1 > 0 \) and \( H_1 > 0 \) such that \( \max_{\|\|a, a, \sigma(a)\|} \|\|f(t, y)\|\| \leq \eta_1 \), for \( 0 < y \leq H_1 \), and \( \eta_1 \int_a^b G\left(\sigma^\mu(b), s\right) ds \leq 1 \). Let us choose \( y \in \kappa \) with \( \|\| = H_1 \). Then, we have from Lemma 2.3,

\[
(Ty)(t) = \int_a^b G(t, s) f(s, y(s)) ds
\]

\[
\leq \int_a^b G\left(\sigma^\mu(b), s\right) f(s, y(s)) ds
\]

\[
\leq \int_a^b G\left(\sigma^\mu(b), s\right) \eta_1 y(s) ds
\]

\[
\leq \eta_1 \int_a^b G\left(\sigma^\mu(b), s\right) \|\| ds \leq \|\|, t \in [a, \sigma^\mu(b)]
\]

Therefore, \( \|Ty\| \leq \|\| \). Hence, if we set

\[
\Omega_1 = \{y \in \beta : \|\| < H_1\}.
\]

Then

\[
\|Ty\| \leq \|\|, \text{ for } y \in \kappa \cap \partial \Omega_2.
\]

Since \( f_\infty = \infty \), there exist \( \eta_2 \) and \( H_2 > 0 \) such that

\[
\min_{\|\|a, \sigma(a)\|} \|\|f(t, y)\|\| \geq \eta_2, \text{ for } y \geq H_2 \]

and define

\[
\Omega_2 = \{y \in \beta : \|\| < H_2\}.
\]

If \( y \in \kappa \cap \partial \Omega_2 \), so that \( \|\| = H_2 \), then

\[
\min_{\|\|a, \sigma(a)\|} \|\|f(t, y)\|\| \geq \frac{1}{p \cdot 4^{n-1}} \|\| .
\]

And we have

\[
\begin{align*}
(Ty)(t) &= \int_a^b G(t, s) f(s, y(s)) ds \\
&\geq \frac{1}{p \cdot 4^{n-1}} \int_a^b G\left(\sigma^\mu(b), s\right) f(s, y(s)) ds \\
&\geq \frac{1}{p \cdot 4^{n-1}} \int_a^b G\left(\sigma^\mu(b), s\right) \eta_1 y(s) ds \\
&\geq \frac{\eta_2}{p \cdot 4^{n-1}} \int_a^b G\left(\sigma^\mu(b), s\right) \|\| ds \geq \|\|.
\end{align*}
\]

Thus, \( \|Ty\| \geq \|\| \), and so

\[
\|Ty\| \geq \|\|, \text{ for } y \in \kappa \cap \partial \Omega_2.
\]

An application of Theorem 3.1 to (10) and (11) yields a fixed point of \( T \) that lies in \( \kappa \cap \partial \Omega_2 \). This fixed point is a solution of the BVP (1)-(3).

**Theorem 3.3.** If \( f_\infty = \infty \) and \( f^0 = 0 \), then the BVP (1)-(3) has at least one positive solution that lies in \( \kappa \).

*Proof:* Let \( T \) be the cone preserving, completely continuous operator defined as in (9). Since \( f_\infty = \infty \), there exist \( \eta_1 > 0 \) and \( J_1 > 0 \) such that

\[
\min_{\|\|a, \sigma(a)\|} \|\|f(t, y)\|\| \geq \eta_1, \text{ for } 0 < y \leq J_1 \]

and

\[
\eta_1 \int_a^b G\left(\sigma^\mu(b), s\right) ds \geq 1. \text{ In this case, define}
\]

\[
\Omega_1 = \{y \in \beta : \|\| < J_1\}.
\]

Then, for \( y \in \kappa \cap \partial \Omega_2 \), we have

\[
f(s, y(s)) \geq \eta_1 y(s), \text{ } s \in I \}
\]
Choosing \( y \in \kappa \cap \partial \Omega_2 \),

\[
(T_y)(t) \geq \int_a^b G(t, s) f \left( s, y(s) \right) ds
\]

and so

\[
\|T_y\| \leq \|v\|, \text{ for } y \in \kappa \cap \partial \Omega_2. \tag{14}
\]

An application of Theorem 3.1, to (12), (13) and (14) yields a fixed point of \( T \) that lies in \( \kappa \cap (\Omega_1 \setminus \Omega_2) \). This fixed point is a solution of the BVP (1)-(3). \( \square \)

4. Existence of Multiple Positive Solutions

In this section, we establish the existence of at least three positive solutions to the BVP (1)-(3).

Let \( E \) be a real Banach space with cone \( P \). A map \( S : P \to [0, \infty) \) is said to be a nonnegative continuous concave functional on \( P \), if \( S \) is continuous and

\[
S(\lambda x + (1-\lambda)y) \geq \lambda S(x) + (1-\lambda)S(y),
\]

for all \( x, y \in P \) and \( \lambda \in [0,1] \). Let \( a' \) and \( b' \) be two real numbers such that \( 0 < a' < b' \) and \( S \) be a nonnegative continuous concave functional on \( P \). We define the following convex sets

\[
P_\alpha = \{ y \in P : \|y\| < a' \},
\]

\[
P(S, a', b') = \{ y \in P : a' \leq S(y) \leq b' \}.
\]

We now state the famous Leggett-Williams fixed point theorem

**Theorem 4.1.** See ref. [20] Let \( T : \overline{P} \to \overline{P} \) be completely continuous and \( S \) be a nonnegative continuous concave functional on \( P \) such that \( S(y) \leq \|y\| \) for all \( y \in \overline{P} \). Suppose that there exist \( a', b', c' \), and \( d' \) with \( 0 < d' < a' < b' \leq c' \) such that

1. \( \{ y \in P(S, a', b') : S(y) > a' \} \neq \emptyset \) and \( S(Ty) > a \) for \( y \in P(S, a', b') \),
2. \( \|Ty\| < d' \) for \( \|y\| < d' \),
3. \( S(Ty) > a' \) for \( y \in P(S, a', c') \) with \( \|Ty\| > b' \).

Then \( T \) has at least three fixed points \( y_1, y_2, y_3 \) in \( \overline{P} \) satisfying

\[
\|y_1\| < d', a' < S(y_2), \|y_3\| > d', S(y_3) < a'.
\]

For convenience, we let
Thus, in view of (16) we see that $P$ is completely continuous and fixed points of $P$ are solutions of the BVP (1)-(3).

Proof. Let the Banach space $E = C[a, \sigma^n(b)]$ be equipped with the norm
\[
\|y\| = \max_{t \in [a, \sigma^n(b)]} |y(t)|.
\]

We denote
\[
P = \left\{ y \in E : y(t) \geq 0, t \in [a, \sigma^n(b)] \right\}.
\]

Then, it is obvious that $P$ is a cone in $E$. For $y \in P$, we define
\[
S(y) = \min_{t \in [a, \sigma^n(b)]} |y(t)|,
\]
and
\[
Ty(t) = \int_a^b G(t, s)f(s, y(s))\Delta s, t \in [a, \sigma^n(b)].
\]

It is easy to check that $S$ is a nonnegative continuous concave functional on $P$ with $S(y) \leq \|y\|$ for $y \in P$ and that $T : P \to P$ is completely continuous and fixed points of $T$ are solutions of the BVP (1)-(3). First, we prove that if there exists a positive number $r$ such that
\[
f(t, y(t)) < \frac{r}{D} \quad \text{for} \quad y \in [0, r],
\]
then $T : P \to P$. Indeed, if $y \in P$, then for $t \in [a, \sigma^n(b)],$
\[
(Ty)(t) = \int_a^b G(t, s)f(s, y(s))\Delta s
\]
\[
< \frac{r}{D} \int_a^b G(t, s)\Delta s
\]
\[
\leq \frac{r}{D} \max_{t \in [a, \sigma^n(b)]} \int_a^b G(t, s)\Delta s = r.
\]

Thus, $\|Ty\| < r$, that is, $Ty \in P$. Hence, we have shown that if (15) and (17) hold, then $T$ maps $P_d$ into $P_d$ and $P_\gamma$ into $P$. Next, we show that
\[
\left\{ y \in P \left( S, d_1, \frac{d_1}{\gamma} \right) : S(y) > d_1 \right\} \neq \emptyset \quad \text{and} \quad S(Ty) > d_1
\]
for all $y \in P \left( S, d_1, \frac{d_1}{\gamma} \right)$. In fact, the constant function
\[
\frac{d_1}{\gamma} + \frac{d_1}{2} \in \left\{ y \in P \left( S, d_1, \frac{d_1}{\gamma} \right) : S(y) > d_1 \right\},
\]
for all $t \in I$. Thus, in view of (16) we see that
\[
S(Ty) = \min_{t \in [a, \sigma^n(b)]} \int_a^b G(t, s)f(s, y(s))\Delta s
\]
\[
\geq \min_{t \in [a, \sigma^n(b)]} \int_a^b G(t, s)f(s, y(s))\Delta s
\]
\[
> \frac{d_1}{\gamma} \min_{t \in [a, \sigma^n(b)]} \int_a^b G(t, s)\Delta s = d_1,
\]
as required. Finally, we show that if $y \in P(S, d_1, c)$ and $\|Ty\| > d_1$, then $S(Ty) > d_1$. To see this, we suppose that $y \in P(S, d_1, c)$ and $\|Ty\| > d_1$, then, by Lemma 2.4, we have
\[
S(Ty) = \min_{t \in [a, \sigma^n(b)]} \int_a^b G(t, s)f(s, y(s))\Delta s
\]
\[
\geq \gamma \int_a^b G(\sigma^n(b), s)f(s, y(s))\Delta s
\]
\[
\geq \gamma \max_{t \in [a, \sigma^n(b)]} \int_a^b G(t, s)f(s, y(s))\Delta s.
\]

Thus, $S(Ty) > d_1$. Hence, $T$ has at least three fixed points, that is, the BVP (1)-(3) has at least three positive solutions $y_1$, $y_2$ and $y_3$ such that

\[
\frac{d_1}{\gamma} + \frac{d_1}{2} \in \left\{ y \in P \left( S, d_1, \frac{d_1}{\gamma} \right) : S(y) > d_1 \right\},
\]

To sum up the above, all the hypotheses of Theorem 4.1 are satisfied. Hence $T$ has at least three fixed points, that is, the BVP (1)-(3) has at least three positive solutions $y_1$, $y_2$ and $y_3$ such that
5. Examples

Now, we give some examples to illustrate the main result.

Example 1
Consider the following boundary value problem
\[
\begin{align*}
y^{(5)} + y^{3} \left( t^{2} y + |y|^{2} \right) &= 0, \quad t \in [0, \pi] \\
y(0) = y^{(2)}(0) - y^{(3)}(\pi) &= 0.
\end{align*}
\] (18)

The Green’s function for the homogeneous boundary value problem is given by
\[
G(t, s) = \begin{cases}
\frac{t^{2}}{2}, & t < s \\
\frac{t^{2} - (t-s)^{2}}{2}, & s < t.
\end{cases}
\]

It is easy to see that all the conditions of Theorem 3.2 hold. It follows from Theorem 3.2, the BVP (18) has at least one positive solution.

Example 2
Consider the following boundary value problem
\[
\begin{align*}
y^{(5)} + y^{3} + y^{2} &= 0, \quad t \in [0, 1] \\
y(0) = y^{(2)}(0) = y^{(3)}(1) &= 0.
\end{align*}
\] (19)

The Green’s function for the homogeneous boundary value problem is given by
\[
G(t, s) = \begin{cases}
\frac{t^{2}}{24}, & t < s \\
\frac{t^{2} - (t-s)^{4}}{24}, & s < t.
\end{cases}
\]

It is easy to see that all the conditions of Theorem 3.3 hold. It follows from Theorem 3.3, the BVP (19) has at least one positive solution.

Example 3
Consider the following boundary value problem on time scale
\[
T = \{0\} \cup \left\{ \frac{1}{2^{n}}, n \in \mathbb{N} \right\} \cup \left[ \frac{1}{2} \right] \setminus \{1/2\}
\]
\[
\begin{align*}
y^{(6)} + f(t, y) &= 0, \quad t \in [0, 1] \cap T \\
y(0) = y^{(2)}(0) &= y^{(3)}(0) = 0 \\
y^{(4)}(0) &= y^{(5)}(0) = y^{(6)}(1) &= 0.
\end{align*}
\] (20)

where

\[
f(t, y) = \begin{cases}
\frac{100(y+1)}{16(4y^{2}+999)}, & y \in \left[ 0, \frac{1}{2} \right] \\
181499.98 + 90749.981, & y \in \left[ \frac{1}{2}, 1 \right] \\
90749.98 + y, & y \in \left[ 1, 3.072 \right] \\
211659 y - 28800, & y \in \left[ 3.072, 4000 \right] \\
2880000 y + 100, & y \in \left[ 4000, \infty \right].
\end{cases}
\]

6. Conclusion

In this paper, we have established the existence of positive solutions for higher order boundary value problems on time scales which unifies the results on continuous intervals and discrete intervals, by using Leggett-Williams fixed point theorem. These results are rapidly arising in the field of modelling and determination of flagellate protozoan in a viscous fluid in further research.

REFERENCES


