Existence and Nonexistence of Global Solutions of a Fully Nonlinear Parabolic Equation

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ABSTRACT
In the paper, we study the global existence of weak solution of the fully nonlinear parabolic problem (1.1)-(1.3) with nonlinear boundary conditions for the situation without strong absorption terms. Also, we consider the blow up of global solution of the problem (1.1)-(1.3) by using the convexity method.

Keywords: Nonlinear Parabolic Equation; Blow Up; Convexity Method

1. Introduction
In this paper, we consider the following fully nonlinear parabolic problem:

\[ u_t = \Delta \varphi(u(x,t)) - \lambda f(u(x,t)) + (x,t) \in \Omega \times (0,T), \quad (1.1) \]

\[ \frac{\partial \varphi(u(x,t))}{\partial \nu} = g(u(x,t)), (x,t) \in \partial \Omega \times (0,T), \quad (1.2) \]

\[ u(x,0) = u_0(x), x \in \Omega, \quad (1.3) \]

where \( \Omega \) is a bounded open domain with smooth boundary \( \partial \Omega \), \( \partial / \partial \nu \) is differentiation in the direction of the outward unit normal to \( \partial \Omega \), \( \lambda > 0 \) and \( u_0(x) \in L^2(\Omega) \).

Denote \( \varphi(u(x,t)) \), \( f(u(x,t)) \) and \( g(u(x,t)) \) by \( \varphi(u), f(u) \), respectively. Also, we need the following conditions:

(D1) \( f(s) \) and \( g(s) \) are local Lipschitz continuous with respect to \( s \);

(D2) \( f(s) \) and \( g(s) \) are positive for all \( s \);

(D3) \( \varphi(s) \in C^1(\mathbb{R}) \) and \( \varphi'(s) > 0 \) with \( \varphi(0) = 0 \).

The problem (1.1)-(1.3) appears in mathematical models of a number of areas of science such as gas dynamics, fluid flow, porous media and biological populations, one can see [1-9]. As for the case of semi-linear or degenerate equations with a nonlinear boundary condition which can be taken as the special case of the problem (1.1)-(1.3), the behavior properties of the above mentioned such as existence and uniqueness, blow up of some special problems, have been established by [2,10-17] and so on.

In this paper, we study the conditions for global existence and blow up of the problem (1.1)-(1.3). The remaining parts of the paper are organized as follows. In Section 2, we give the global solvability condition for the situations with and without strong absorption terms. Finally, we obtain the condition of blowing up of global solution by the convexity method in [18,19].

2. Global Existence
Firstly, we give the definition of weak solution as follows:

**Definition 2.1.** Given \( u_0(x) \in L^2(\Omega) \), if \( u(x,t) \in C([0,T]; L^2(\Omega)) \cap L^\infty(0,T) \) satisfies

\[ \int_{\Omega} (\nabla \varphi(u) \nabla \phi - u \phi + \lambda f(u) \phi) \, dx \, dt \]

\[ \int_{\Omega} g(u) \phi \, dx \, dt = \int_{\Omega} u_0(x) \phi(x,0) \, dx \]

for any test function \( \phi \in L^2(0,T; H^1(\Omega)) \cap W^{1,1}(0,T; L^\infty(\Omega)) \) with \( \phi(T) = 0 \), then \( u(x,t) \) is called by a weak solution of the problem (1.1), (1.2).

The local existence and uniqueness of weak solution of the problem (1.1)-(1.3), one can see [20]. For the global existence of weak solution, we have the following result:

**Theorem 2.1.** Assume that there exist strictly non-decreasing positive functions \( H(s) \) and \( H'(s) \) such that...
\[ \varphi'(s) H(s) \geq g(s) \text{ for } s \geq s_0 > 0, \quad (2.2) \]

\[ H'(s) \geq \varphi'(s) l - (\varphi'(s) H'(s) + \varphi^*(s) H(s)) M^2_2 + \lambda f(s)/H(s), \quad (2.3) \]

where

\[ M_1 = \max_{\bar{h}} h(x), \]
\[ M_2 = \max_\bar{h} \|\nabla h(x)\| \]

and \( h(x) \) satisfies

\[ \Delta h(x) = 1 \text{ in } \Omega, \quad \frac{\partial h(x)}{\partial \nu} = 1 \text{ on } \partial \Omega, \quad (2.5) \]

Then the solution of the problem (1.1)-(1.3) is global.

**Proof.** Let \( \bar{u}(x,t) = \psi(\delta(t) + h(x)) \), where \( \psi \) is the solution of

\[ \bar{u}_t - \Delta \phi(\bar{u}) + \lambda f(\bar{u}) = H(\bar{u}) \delta'(t) - \Delta \phi(\bar{u}) + \lambda f(\bar{u}) \]
\[ = H(\bar{u}) \delta'(t) - \left[ \varphi'(\bar{u}) H(\bar{u}) l + \left( \varphi'(\bar{u}) \psi' \delta(t) + h(x) \right) + \varphi^*(\bar{u}) H^2(\bar{u}) \right] \nabla h^2 + \lambda f(\bar{u}) \]
\[ \geq H(\bar{u}) \delta'(t) - \varphi'(\bar{u}) l - \left( \varphi'(\bar{u}) H'(\bar{u}) + \varphi^*(\bar{u}) H(\bar{u}) M^2_2 + \lambda f(\bar{u})/H(\bar{u}) \right) \geq 0. \]

Using (2.2), (2.5) and (2.6), we obtain

\[ \frac{\partial \phi(\bar{u})}{\partial \nu} - g(\bar{u}) = \nabla \phi(\bar{u}) \cdot n_0 - g(\bar{u}) = \varphi'(\bar{u}) \nabla \psi(\delta(t) + h(x)) \cdot n_0 - g(\bar{u}) \]
\[ = \varphi'(\bar{u}) H(\bar{u}) \nabla h \cdot n_0 - g(\bar{u}) = \varphi'(\bar{u}) H(\bar{u}) \frac{\partial h}{\partial \nu} - g(\bar{u}) = \varphi'(\bar{u}) H(\bar{u}) - g(\bar{u}) \geq 0. \]

From (2.9) and (2.10), we see that \( \pi(x,t) \) is a sup-solution to the problem (1.1)-(1.3) defined for all \( t \geq 0 \) with \( \pi(x,0) \geq u_0(x) \). By using the sup- and sub-solution argument (c.f. [7]), we know that the solution of the problem (1.1)-(1.3) is global.

**Remark 2.1.** If the conditions (2.2) and (2.3) hold, the problem (1.1)-(1.3) is called by the problem without strong absorption terms.

### 3. Blow Up

In the section, we use the convexity method (see [18,19]) to show that the global solution blows up in finite time under some suitable condition. To this end, we define

\[ E(t) = -\frac{1}{2} \int_{\Omega} \| \nabla u \|^2 \, dx + \frac{1}{2} \int_{\Omega} \int_{0}^{t} \varphi'(s) g(s) \, ds \, dx - \lambda \int_{\Omega} \int_{0}^{t} \varphi'(s) f(s) \, ds \, dx \]

and

\[ F(t) = \int_{\Omega} \int_{0}^{t} \phi(z) \, dz \, dx. \]

Suppose that following conditions hold:

\[ (D4) \text{ if } g, \varphi \text{ and } f \text{ satisfy the following inequalities} \]
\[ g(s) \int_{0}^{t} \varphi'(z) \, dz \geq 2 \int_{0}^{t} \varphi'(z) g(z) \, dz \]
\[ \text{and} \]
\[ 2 f(s) \int_{0}^{t} \varphi'(z) \, dz \geq 2 \int_{0}^{t} \varphi'(z) f(z) \, dz. \]

Then the existence of a constant \( I_0 \) and a convexity function \( \psi(s) > 0 \) such that

\[ \int_{0}^{\infty} \frac{ds}{\psi(s)} < +\infty \]

and

\[ \frac{2E(0) + \frac{\lambda}{2} \int_{0}^{t} \varphi'(z) f(z) \, dz}{\psi(0)} > 0 \]

with
Proof. Multiplying (1.1) by \( \varphi(u) \) and integrating by parts over \( \Omega \), we have

\[
0 \leq \int_{\Omega} \varphi'(u) \varphi(u) \, dx + \frac{1}{2} \int_{\Omega} \varphi'(u) \varphi'(u) \, dx - \lambda \int_{\Omega} \varphi''(u) \varphi(u) \, dx
\]

(3.7)

Lemma 3.1. If the condition (D4) holds, then \( E(t) \geq E(0) \), i.e., \( \int_{\Omega} \varphi''(u) \varphi(u) \, dx \geq 0 \).

Proof. Multiplying (1.1) by \( \varphi(u) \) and integrating by parts over \( \Omega \), we have

\[
0 \leq \int_{\Omega} \varphi'(u) \varphi(u) \, dx + \frac{1}{2} \int_{\Omega} \varphi'(u) \varphi'(u) \, dx - \lambda \int_{\Omega} \varphi''(u) \varphi(u) \, dx
\]

(3.8)

Using (3.8), we have

\[
\frac{d}{dt} \left[ -\int_{\Omega} \varphi'(u) \varphi(u) \, dx + \frac{1}{2} \int_{\Omega} \varphi'(s) \varphi(s) \, ds \right] \geq 0.
\]

(3.9)

Using (3.9) and (3.1), we have \( dE(t)/dt \geq 0 \). So, we obtain \( E(t) \geq E(0) \).

Theorem 3.1. Suppose that the conditions (D4) and (D5) hold, then the solution of the problem (1.1)-(1.3) blows up in finite time.

Proof. Using (3.2), we have

\[
\frac{dF(t)}{dt} = \int_{\Omega} \left( \int_{\partial \Omega} \varphi'(z) \, dz \right) u \, dx = \int_{\Omega} \left( \int_{\partial \Omega} \varphi'(z) \, dz \right) \Delta \varphi(u) \, dx - \lambda \int_{\Omega} \varphi''(z) \varphi(z) \, dz \left( \frac{F(t)}{\|\Omega\|} \right)
\]

(3.10)

Since \( \varphi(u) = \int_{0}^{u} \varphi'(s) \, ds \), so we have

\[
\varphi(u) = \int_{0}^{u} \varphi'(s) \, ds
\]

(3.11)

Using (3.12) and Lemma 3.1, we obtain

\[
F'(t) \geq 2E(0) + \lambda \int_{\Omega} \varphi'(z) \varphi(z) \, dz - \lambda \int_{\Omega} \varphi''(z) \varphi(z) \, dz + \lambda \int_{\Omega} \varphi'(z) \varphi(z) \, dz \left( \frac{F(t)}{\|\Omega\|} \right)
\]

(3.13)

From the condition (D5), we see

\[
F'(t) \geq \int_{\Omega} \varphi' \left( \int_{\partial \Omega} \varphi(z) \, dz \right) \, dx > 0.
\]

(3.14)

Using the Jensen’s inequality, we get

\[
F'(t) \geq \left\| \varphi' \right\|_{L^1(\Omega)} \left( \int_{\Omega} \varphi(z) \, dz \right) \left( \frac{F(t)}{\|\Omega\|} \right)
\]

(3.15)

Hence, we have

\[
F'(t) \geq \left| \varphi' \right|_{L^1(\Omega)} \left( \int_{\partial \Omega} \varphi(z) \, dz \right) \left( \frac{F(t)}{\|\Omega\|} \right)
\]

(3.16)
\[
\int_{E(0)}^{+\infty} \frac{dy}{\varphi(y)} < +\infty. \tag{3.19}
\]

Therefore, there exists \( T_0 \) such that
\[
\lim_{t \to T_0} F(t) = \lim_{t \to T_0} \int_{D} \left( \int_{\Omega} \varphi(z) \, dz \right) \, dx = +\infty. \tag{3.20}
\]

From (3.20), we know that the solution of the problem (1.1)-(1.3) must blow up in finite time.

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REFERENCES


