Applications of Multivalent Functions Associated with Generalized Fractional Integral Operator

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ABSTRACT
By using a method based upon the Briot-Bouquet differential subordination, we investigate some subordination properties of the generalized fractional integral operator $J_{a,\mu}^{\lambda,z}$ which was defined by Owa, Saigo and Srivastava [1]. Some interesting further consequences are also considered.

Keywords: Multivalent Functions; Subordination; Gaussian Hypergeometric Function; Fractional Integral Operator

1. Introduction
Let $A_{\mu}(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=0}^{\infty} a_{p+\mu} z^{p+\mu}, \quad (p, n \in \mathbb{N}) \Rightarrow \{1, 2, 3, \ldots\}$$

(1.1)

which are analytic in the open unit disk $U = \{ z; \ z \in \mathbb{C} \text{ and } |z| < 1 \}$. Also let $f$ and $g$ be analytic in $U$ with $f(0) = g(0)$. Then we say that $f$ is subordinate to $g$ in $U$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists the Schwarz function $w$, analytic in $U$ such that $w(0) = 0$, $w(z) \leq 1$ and $f(z) = g(w(z))(z \in U)$. We also observe that

$$f(z) \prec g(z) \quad \text{in} \ U$$

if and only if

$$f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U)$$

whenever $g$ is univalent in $U$.

Let $a$, $b$ and $c$ be complex numbers with $c \neq 0, -1, -2, \ldots$.

Then the Gaussian/classical hypergeometric function $\,_{2}F_{1}(a, b; c; z)$ is defined by

$$\,_{2}F_{1}(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

(1.2)

where $(\eta)_k$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$\eta_k = \frac{\Gamma(\eta + k)}{\Gamma(\eta)} = \frac{1}{\eta(\eta + 1) \cdots (\eta + k - 1)}, \quad (k = 0)$$

(1.3)

The hypergeometric function $\,_{2}F_{1}(a, b; c; z)$ is analytic in $U$ and if $a$ or $b$ is a negative integer, then it reduces to a polynomial.

For each $A$ and $B$ such that $-1 \leq B < A \leq 1$, let us define the function

$$h(A,B; z) = \frac{1+A}{1+B}, \quad (z \in U).$$

(1.4)

It is well known that $h(A, B; z)$, for $-1 \leq B \leq 1$, is the conformal map of the unit disk onto the disk symmetrical respect to the real axis having the center $(1 - AB)/(1 - B^2)$ and the radius $(A - B)/(1 - B^2)$. The boundary circle cuts the real axis at the points $(1 - A)/(1 - B)$ and $(1 + A)/(1 + B)$.

Many essentially equivalent definitions of fractional calculus have been given in the literature (cf., e.g. [2,3]). We state here the following definition due to Saigo [4] (see also [1,5]).

**Definition 1.** For $\lambda > 0$, $\mu, \nu \in \mathbb{R}$, the fractional integral operator $I_{\alpha,\beta}^\mu$ is defined by

$$I_{\alpha,\beta}^\mu f(z) = \frac{\Gamma(\alpha + \beta) z^{\alpha + \beta}}{\Gamma(\alpha + \beta)} \int_0^z (z - \zeta)^{\alpha - 1} \,_{2}F_{1}(\lambda + \mu, -\nu; \lambda; 1 - \frac{\zeta}{z}) f(\zeta) d\zeta,$$

(1.5)

where $\,_{2}F_{1}$ is the Gaussian hypergeometric function defined by (1.2) and $f(z)$ is taken to be an analytic function in a simply-connected region of the $z$-plane containing the origin with the order

$$f(z) = O\left(|z|^{\epsilon}\right) \quad (z \to 0)$$

for $\epsilon > \max \{0, \mu - \nu\} - 1$, and the multiplicity of $(z - \zeta)^{\alpha - 1}$ is removed by requiring that $\log(z - \zeta)$ to be real when $z - \zeta > 0$. 
The definition (1.5) is an interesting extension of both the Riemann-Liouville and Erdélyi-Kober fractional operators in terms of Gauss’s hypergeometric functions.

With the aid of the above definition, Owa, Saigo and Srivastava [1] defined a modification of the fractional integral operator \( I_{\alpha, \beta}^{\lambda, \mu, \nu} \) by

\[
J_{\alpha, \beta}^{\lambda, \mu, \nu} f(z) = \frac{\Gamma(p+1-\mu)\Gamma(\lambda+p+\nu)}{\Gamma(p+1)\Gamma(p+\mu-\nu)} z^\mu I_{\alpha, \beta}^{\lambda, \mu, \nu} f(z) \tag{1.6}
\]

for \( f(z) \in \mathcal{A}_\alpha(p) \) and \( \mu-\nu-p<1 \). Then it is observed that \( J_{\alpha, \beta}^{\lambda, \mu, \nu} \) also maps \( \mathcal{A}_\alpha(p) \) onto itself as follows:

\[
J_{\alpha, \beta}^{\lambda, \mu, \nu} f(z) = z^\mu + \sum_{k=0}^{\infty} \frac{(p+1)_k (p+1-\mu)_k}{(p+1-\mu)_k} \frac{(\lambda+p+\nu)_k}{(\lambda+p+1+\nu)_k} a_{p+k} z^{p+k}, \tag{1.7}
\]

\( \lambda > 1, \mu-\nu-p < 1; f \in \mathcal{A}_\alpha(p) \).

We note that \( J_{\alpha, \beta}^{\alpha, \beta, -1} f(z) = C_\alpha^{\beta} f(z), (\alpha \geq 0; \beta > -1) \), where the operator \( C_\alpha^{\beta} \) was introduced and studied by Jung, Kim and Srivastava [6] (see also [7]).

It is easily verified from (1.7) that

\[
z \left( J_{\alpha, \beta}^{\lambda, \mu, \nu} f(z) \right) = (\lambda + \nu + p) J_{\alpha, \beta}^{\lambda, \mu, \nu} f(z) - (\lambda + \nu) J_{\alpha, \beta}^{\lambda, \mu, \nu} f(z). \tag{1.8}
\]

The identity (1.8) plays an important and significant role in obtaining our results.

Recently, by using the general theory of differential subordination, several authors (see, e.g. [7-9]) considered some interesting properties of multivalent functions associated with various integral operators. In this manuscript, we shall derive some subordination properties of the fractional integral operator \( J_{\alpha, \beta}^{\lambda, \mu, \nu} \) by using the technique of differential subordination.

2. Main Results

In order to establish our results, we shall need the following lemma due to Miller and Mocanu [10].

**Lemma 1.** Let \( h(t) \) be analytic and convex univalent in \( \mathbb{U} \) with \( h(0) = 1 \), and let \( g(z) = 1 + b_2 z + b_3 z^2 + \cdots \) be analytic in \( \mathbb{U} \). If

\[
g(z) + \frac{1}{c} z g'(z) < h(z), \tag{2.1}
\]

then for \( c \neq 0 \) and \( \text{Re} c \geq 0 \),

\[
g(z) < \frac{c}{n} \sum_{n=0}^{\infty} f^{(n)}(0) h(t) dt. \tag{2.2}
\]

We begin by proving the following theorem.

**Theorem 1.** Let \( -1 \leq B < A \leq 1, \lambda > 1, \lambda + \nu > -p, \mu - \nu - p < 1, \mu - 1 < p \) and \( 0 < \alpha < 1 \), and let

\[
f(z) = z^\mu + \sum_{k=0}^{\infty} a_{p+k} z^{p+k} \in \mathcal{A}_\alpha(p). \tag{2.3}
\]

Suppose that

\[
\sum_{k=0}^{\infty} c_k |a_{p+k}| \leq 1,
\]

where

\[
c_k = 1 - B \frac{\lambda + p + v + k(1 - \alpha)}{A - B} \frac{1}{(p+1)_k} \frac{1}{(p+1-\mu)_k} \frac{1}{(\lambda + p + v + k)_k} \tag{2.4}
\]

and \( (\eta)_k \) is given by (1.3).

1) If \( -1 \leq B < 0 \), then

\[
(1 - \alpha) J_{\alpha, \beta}^{1-\lambda, \mu, \nu} f(z) \left( \frac{z^\mu}{A} \right) + \alpha J_{\alpha, \beta}^{\lambda, \mu, \nu} f(z) \left( \frac{z^\mu}{A} \right) \leq h(A, B, z). \tag{2.5}
\]

2) If \( -1 \leq B < 0 \) and \( \gamma \geq 1 \), then

\[
\text{Re} \left\{ \left( \frac{J_{\alpha, \beta}^{\lambda, \mu, \nu} f(z)}{z^\mu} \right)^{1/\gamma} \right\} > \left( \frac{\lambda + v + p}{n(1-\alpha)} \right)^{1/\gamma} \left( \frac{1 - Au}{1 - Bu} \right)^{1/\gamma}. \tag{2.6}
\]

**The result is sharp. Proof.**

1) If we set

\[
L = (1 - \alpha) J_{\alpha, \beta}^{1-\lambda, \mu, \nu} f(z) \left( \frac{z^\mu}{A} \right) + \alpha J_{\alpha, \beta}^{\lambda, \mu, \nu} f(z) \left( \frac{z^\mu}{A} \right),
\]

then, from (1.7) we see that

\[
L = 1 + \sum_{k=0}^{\infty} \frac{[\lambda + p + v + k(1 - \alpha)](p+1)_k (p+1-\mu)_k}{(\lambda + p + v)_k (p+1-\mu)_k} a_{p+k} z^k. \tag{2.7}
\]

For \( -1 \leq B < 0 \) and \( z \in \mathbb{U} \), it follows from (2.3) that

\[
\left| \frac{L - 1}{A - BL} \right| = \sum_{k=0}^{\infty} \frac{[\lambda + p + v + k(1 - \alpha)](p+1)_k (p+1-\mu)_k}{(\lambda + p + v)_k (p+1-\mu)_k} a_{p+k} z^k \leq \frac{\sum_{k=0}^{\infty} c_k |a_{p+k}|}{1 - B + \sum_{k=0}^{\infty} c_k |a_{p+k}|} \leq 1, \tag{2.8}
\]
which implies that
\[(1-\alpha)\frac{J_{0,\alpha}^{\lambda,\mu,v}f(z)}{zp} + \alpha \frac{J_{0,\alpha}^{\lambda,\mu,v}f(z)}{zp} < h(A,B;z).\]

2) Let
\[g(z) = \frac{J_{0,\alpha}^{\lambda,\mu,v}f(z)}{zp}, \quad (f \in A_\alpha(p)). \quad (2.9)\]

Then the function \(g(z) = 1 + b_1z^n + b_{21}z^{n+1} + \cdots\) is analytic in \(U\). Using (1.8) and (2.9), we have
\[\frac{J_{0,\alpha}^{\lambda,\mu,v}f(z)}{zp} = g(z) + \frac{1}{\lambda + v + p}zg'(z). \quad (2.10)\]

From (2.5), (2.9) and (2.10) we obtain
\[(1-\alpha)\frac{J_{0,\alpha}^{\lambda-1,\mu,v}f(z)}{zp} + \alpha \frac{J_{0,\alpha}^{\lambda,\mu,v}f(z)}{zp} = g(z) + \frac{1-\alpha}{\lambda + v + p}zg'(z) < h(A,B;z).\]

Thus, by applying Lemma 1, we observe that
\[g(z) < \frac{\lambda + v + p}{n(1-\alpha)} \left[ \int_0^{\lambda + v + p} \left( \frac{1}{n(a-\alpha)} \left( 1 + At \right) \right) \right] \frac{z^{\lambda + v + p}}{zp} \quad (z \in U).\]

Since \(Re(w(z)) \geq (Re(w))^{1/2}\) for \(Re(w) > 0\) and \(\gamma \geq 1\), from (2.12) we see that the inequality (2.6) holds.

To prove sharpness, we take \(f(z) \in A_\alpha(p)\) defined by
\[J_{0,\alpha}^{\lambda,\mu,v}f(z) = \frac{\lambda + v + p}{n(1-\alpha)} \left[ \int_0^{\lambda + v + p} \left( \frac{1}{n(a-\alpha)} \left( 1 + At \right) \right) \right] \frac{z^{\lambda + v + p}}{zp} \quad (z \in U).\]

Thus, the proof of Theorem 1 is evidently completed.

**Theorem 2.** Let \(-1 \leq B < A \leq 1\), \(\lambda > 1\), \(\lambda + v > p\), \(\mu - v - p < 1\), \(\mu - 1 < p\) and \(0 < \alpha < 1\). Suppose that
\[f(z) = z^n + \sum_{k=0}^{\infty} a_k z^{k+1} \in A_\alpha(p), \quad s_k(z) = z^n + \sum_{k=0}^{m-2} a_k z^{k+1} (m \geq 2).\]

If the sequence \(\{c_k\}\) is nondecreasing with
\[c_k > \frac{1-B}{(A-B)(\lambda + p + v)}(k \geq n), \quad (2.13)\]
where \(c_k\) is given by (2.4) and satisfies the condition (2.3), then
\[Re \left\{ \frac{J_{0,\alpha}^{\lambda,\mu,v}f(z)}{s_m(z)} \right\} > 0, \quad (2.14)\]
and
\[Re \left\{ \frac{s_m(z)}{J_{0,\alpha}^{\lambda,\mu,v}f(z)} \right\} > 0. \quad (2.15)\]

Each of the bounds in (2.14) and (2.15) is best possible for \(m \in \mathbb{N}\).

**Proof.** We prove the bound in (2.14). The bound in (2.15) is immediately obtained from (2.14) and will be omitted. Let
\[h(z) = \frac{J_{0,\alpha}^{\lambda,\mu,v}f(z)}{s_m(z)} (f \in A_\alpha(p); z \in U).\]

Then, from (1.7) we observe that
\[h(z) = 1 + \sum_{k=0}^{m-2} \delta_k a_{k+1} z^k + \sum_{k=m}^{\infty} \delta_k a_{k+1} z^k, \quad (z \in U).\]

where, for convenience,
\[\delta_k = \frac{(p+1)(p-1+\mu+v)}{(p+1-\mu)(\lambda + p + v)}. \quad (2.16)\]

It is easily seen from (2.4) and (2.13) that \(c_k \geq 1\) and
\[\delta_k > \frac{(A-B)(\lambda + p + v)}{1-B}(k \geq n). \quad (2.16)\]

Hence, by applying (2.3) and (2.16), we have
\[h(z) = 1 + \sum_{k=0}^{m-2} \delta_k a_{k+1} z^k + \sum_{k=m}^{\infty} \delta_k a_{k+1} z^k \leq \sum_{k=m}^{\infty} (1 - \delta_k - 1) |a_{k+1}| + \sum_{k=m}^{\infty} \delta_k |a_{k+1}| \leq 1 (z \in U). \quad (2.15)\]
which readily yields the inequality (2.14). If we take \( f(z) = z^\beta - z^{\beta + \alpha} \), then
\[
\frac{f(z)}{a_\alpha(z)} = 1 - z^{a + \alpha} \to 0 \text{ as } z \to 1^+.
\]

This shows that the bound in (2.14) is best possible for each \( m \), which proves Theorem 2.

Finally, we consider the generalized Bernardi-Livera-Livingston integral operator \( L_{\alpha} f(z) \) given by (cf. [11-13])

\[
L_{\alpha}(f)(z) := \frac{\sigma + \mu}{\sigma} \int_0^z f(t) \, dt, \quad f \in \mathcal{A}_\alpha(p); \sigma > -p.
\]

(2.17)

**Theorem 3.** Let \(-1 \leq B < A \leq 1\), \( \sigma > -p \), \( \lambda > 1 \), \( \lambda + \nu > -p \), \( \mu - \nu < p \), \( \mu - 1 < p \) and \( 0 < \alpha < 1 \), and let \( f(z) = z^\beta + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \in \mathcal{A}_\alpha(p) \). Suppose that

\[
\sum_{k=1}^{\infty} |a_{p+k}| \leq 1,
\]

(2.18)

where

\[
d_k = \frac{1 - B}{A - B} \left[ \frac{\sigma + p + k(1 - \alpha)}{(p + 1)_{\lambda + p + \nu}} \right] \frac{\mu + 1}{k (p + 1 - \mu)_{\lambda + p + \nu}}
\]

and \( \eta_k \) is given by (1.3).

1) If \(-1 \leq B < 0 \), then

\[
(1 - \alpha) \frac{\mathcal{J}_{0, \alpha}^{\lambda, \mu, \nu} f(z)}{z^\beta} + \frac{\alpha}{\sigma} \frac{\mathcal{J}_{0, \alpha}^{\lambda, \mu, \nu} L_{\alpha}(f)(z)}{z^\beta} \preceq h(A, B; z).
\]

(2.19)

2) If \(-1 \leq B < 0 \) and \( \gamma \geq 1 \), then

\[
\text{Re} \left[ \frac{\left( \frac{\mathcal{J}_{0, \alpha}^{\lambda, \mu, \nu} f(z)}{z^\beta} \right)^{1/\gamma}}{z^\beta} \left\{ \frac{\sigma + p}{\eta_{\alpha(1 - \alpha)}} \left( 1 - Au \right) \right\}^{1/\gamma} \right] \geq 1 - \frac{1}{1 - Bu} \quad (z \in U).
\]

(2.20)

The result is sharp.

Proof. 1) If we put

\[
M = (1 - \alpha) \frac{\mathcal{J}_{0, \alpha}^{\lambda, \mu, \nu} f(z)}{z^\beta} + \frac{\alpha}{\sigma} \frac{\mathcal{J}_{0, \alpha}^{\lambda, \mu, \nu} L_{\alpha}(f)(z)}{z^\beta},
\]

then, from (1.7) and (2.17) we have

\[
M = 1 + \sum_{k=\infty}^{\infty} \frac{\sigma + p + k(1 - \alpha)}{(p + 1)_{\lambda + p + \nu}} \frac{\mu + 1}{k (p + 1 - \mu)_{\lambda + p + \nu}} \cdot a_{p+k} z^{p+k}.
\]

Therefore, by using same techniques as in the proof of Theorem 1 1), we obtain the desired result.

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REFERENCES


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