Some Properties on the Error-Sum Function of Alternate Sylvester Series

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ABSTRACT
The error-sum function of alternating Sylvester series is introduced. Some elementary properties of this function are studied. Also, the Hausdorff dimension of the graph of such function is determined.

Keywords: Alternating Sylvester Series; Error-Sum Function; Hausdorff Dimension

1. Introduction
For any \( x \in (0, 1] \), let \( d_i = d_i(x) \in N \) and \( T := T(x) \in (0, 1] \) be defined as
\[
d_i(x) = \left[ \frac{1}{x} \right] (x) - x, \quad T(0) := 0. \tag{1}
\]
where \([ ]\) denote the integer part. And we define the sequence \( \{d_n(x), n \geq 2\} \) as follows:
\[
d_n(x) = d_1(T^{n-1}(x)), \tag{2}
\]
where \( T^n \) denotes the \( n \)th iterate of \( T \) \( (T^0 = \text{Id}_{(0,1]} \)).

It is well known that from the algorithm (1), all \( x \in (0, 1] \) can be developed uniquely into an infinite or finite series
\[
x = \sum_{i \geq 1} (-1)^{i-1} \frac{1}{d_i(x)}. \tag{3}
\]
where \( i+1 \geq d_i(x)(d_i(x)+1) \).

In the literature [2], (3) is called the Alternating Balkema-Oppenheim expansion of \( x \) and denoted by \( x = [d_1(x), \cdots, d_n(x), \cdots] \) for short. From the algorithm, one can see that \( T \) maps irrational element into irrational element, and the series is infinite. While for rational numbers, in fact, we have \( x \in (0, 1] \) is rational if and only if its sequence of digits \( d_1(x), \cdots \), is terminate or periodic, see [1-3].

For any \( x \in (0, 1] \) and \( n \geq 1 \), define
\[
p_n(x) = \sum_{i=1}^{n} (-1)^{i-1} \frac{1}{d_i(x)}. \tag{4}
\]
and we call \( S(x) \) the error-sum function of Alternating Sylvester series. By (4), since \( d_{n+1}(x) \geq d_n(x)(d_n(x)+1) \) for all \( n \geq 1 \), then \( |S(x)| \leq 1 \) and \( S(x) \) is well defined. In this paper, we shall discuss some basic nature of \( S(x) \), also the Hausdorff dimension of the graph of \( S(x) \) is determined.

2. Some Basic Properties of \( S(x) \)

In what follows, we shall often make use of the symbolic space.

For any \( n \geq 1 \), let
\[
D_n = \{ (\sigma_1, \sigma_2, \cdots, \sigma_n) \in N^n : \sigma_{k+1} \geq \sigma_k (\sigma_k + 1) \}
\]
for all \( 1 \leq k \leq n \).

Define

\[
\]
For any \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in D_n \), write
\[
A_{\sigma} = \frac{1}{\sigma_1} - \frac{1}{\sigma_2} + \cdots + (-1)^{n-1} \frac{1}{\sigma_n},
\]
and
\[
B_{\sigma} = \frac{1}{\sigma_1} - \frac{1}{\sigma_2} + \cdots + (-1)^{n-1} \frac{1}{\sigma_n + 1}.
\]
We use \( J_{\sigma} \) to denote the following subset of \((0,1]\),
\[
J_{\sigma} = \{ x \in (0,1] : d_i(x) = \sigma_i, \quad d_i(x) = \sigma_2, \ldots, d_i(x) = \sigma_n \}.
\]
From theorem 4.14 of [8], we have \( J_{\sigma} = (A_{\sigma}, B_{\sigma}) \) when \( n \) is even, and \( J_{\sigma} = (B_{\sigma}, A_{\sigma}) \) when \( n \) is odd. Finally, define
\[
I = \{ A_{\sigma}, B_{\sigma}, \sigma \in D_n, n \geq 1 \}.
\]
**Lemma 1.** For any \( n \geq 1 \) and \( x \in (0,1] \),
1) \( \lim_{x \to 0^+} S(x) = 0; \)
2) \( -\frac{17}{30} \leq S(x) \leq 0); \)
3) \( S(x) = \sum_{i=1}^{n} \left( x - \frac{p_i(x)}{q_i(x)} \right) (+1)^n S\left(T^n(x)\right). \)

**Proof.** 1) Since \( d_{j+1}(x) \geq d_j(x) \left(d_j(x) + 1\right) \) and \( d_j(x) \geq 1 \), so when \( n \geq 3 \), we can get
\[
d_{n+1} > d_n^2 \geq \cdots > d_2^{n-1},
\]
accordingly
\[
d_n > d_2^{n-1} \geq \left(d_2^{n-1}(d_2^{n-1} + 1)^2\right)^{n-2},
\]
we write \( a(x) = d_i(x)^2 \left(d_i(x) + 1\right)^2 \), so \( d_n > a(x)^{-2}. \)

Now \( d_{n+1}(x) = \left[ \frac{1}{T^n(x)} \right] \) implies
\[
\frac{1}{d_{n+1}(x) + 1} < T^n(x) \leq \frac{1}{d_{n+1}(x)}, \quad \text{for} \ 0 < T^n(x) \leq 1.
\]
Thus
\[
S(x) = \sum_{i=1}^{n} \left( x - \frac{p_i(x)}{q_i(x)} \right) (+1)^n S\left(T^n(x)\right).
\]

2) From 1) we know that
\[
d_{i+1} > d_i^2 \geq \cdots > d_2^{n-1},
\]
from the definition of \( d_i(x) \) we also know that \( d_i \geq 1 \), so \( d_2 > d_2(d_2 + 1) \geq 2 \),
\[
d_{n+1} > d_2^{n-1} \geq 4^{n-1},
\]
thus
\[
S(x) = \sum_{i=1}^{n} \left( x - \frac{p_i(x)}{q_i(x)} \right) \geq -\frac{1}{2} \sum_{i=2}^{n} \frac{1}{4^{i-2}} = -\frac{17}{30}.
\]

3) Since as \( n > m \),
\[
\frac{p_n(x)}{q_n(x)} - \frac{p_m(x)}{q_m(x)} = \left(-1\right)^m \frac{p_k(x)}{q_k(x)} \left(T^n(x)\right).
\]

Thus
\[
S(x) = \sum_{i=1}^{n} \left( x - \frac{p_i(x)}{q_i(x)} \right)
\]
\[
= \sum_{i=1}^{n} \left( x - \frac{p_i(x)}{q_i(x)} \right) + \sum_{i=1}^{n} \left( x - \frac{p_i(x)}{q_i(x)} + \frac{p_i(x)}{q_i(x)} - \frac{p_i(x)}{q_i(x)} \right)
\]
\[
= \sum_{i=1}^{n} \left( x - \frac{p_i(x)}{q_i(x)} \right) - (-1)^n \frac{p_n(x)}{q_n(x)} \left(T^n(x)\right)
\]
\[
= \sum_{i=1}^{n} \left( x - \frac{p_i(x)}{q_i(x)} \right) + (-1)^n \sum_{i=1}^{n} \frac{T^n(x)}{q_i(x)}
\]
\[
= \sum_{i=1}^{n} \left( x - \frac{p_i(x)}{q_i(x)} \right) + (-1)^n S\left(T^n(x)\right).
\]

Let
\[
I' = I \backslash \{1\}.
\]

**Proposition 2.** For any \( x \in I' \), if \( x = [d_1(x), \ldots, d_{2k+1}(x)] \), then \( S(x) \) is left continuous but not right continuous. If \( x = [d_1(x), \ldots, d_{2k}(x)] \), then \( S(x) \) is right continuous but not left continuous.

**Proof.** For any \( n \geq 1 \) and \( \sigma \in D_n \), write \( x_1 = A_{\sigma} \), \( x_2 = B_{\sigma} \), where \( A_{\sigma}, B_{\sigma} \) are given by (6) and (7).

Case 1, \( n = 2k + 1 \), then
\[
x_1 = \frac{1}{\sigma_1} - \frac{1}{\sigma_2} + \cdots + \frac{1}{\sigma_{2k+1}} \quad (13)
\]
\[
x_2 = \frac{1}{\sigma_1} - \frac{1}{\sigma_2} + \cdots + \frac{1}{\sigma_{2k+1} + 1} \quad (14)
\]
and \( J_{\sigma} = (B_{\sigma}, A_{\sigma}) \). For any \( x_1 \in J_{\sigma} \), since when \( \sigma_{2k+1} = \sigma_{2k} (\sigma_{2k} + 1) \),
\[
\frac{1}{\sigma_1} - \frac{1}{\sigma_2} + \cdots - \frac{1}{\sigma_{2k}} + \frac{1}{\sigma_{2k+1}} = \frac{1}{\sigma_1} - \frac{1}{\sigma_2} + \cdots - \frac{1}{\sigma_{2k}} + \frac{1}{\sigma_{2k+1}} \geq \frac{1}{\sigma_2} + 1.
\]

This situation is included in Case II, so we can take \(\sigma_{2k+1} > \sigma_2 (\sigma_{2k} + 1)\) and \(x' = x - \frac{1}{\alpha}\) for some \(\alpha \geq \sigma_{2k+1} (\sigma_{2k} + 1)\).

For any \(n \geq 1\) and \(\sigma \in D_n\), write \(\alpha = \max\{A_\sigma, B_\sigma\}\) and \(\alpha = \min\{A_\sigma, B_\sigma\}\). Then for any \(x \in J_\sigma\), if \(n = 2k + 1\), then

\[
S(x') - S(x) = \sum_{i=1}^{2k} \left( x' - p_i(x') \right) + \left( x'' - p_{2k+1}(x'') \right) q_i(x') = \sum_{i=1}^{2k} \left( x' - p_i(x') \right) q_i(x') + S(T^{2k+2}(x'))
\]

and this implies \(S(x)\) is not right continuous at \(x_i\). For \(x_2 = \frac{1}{\sigma_1} - \frac{1}{\sigma_2} + \cdots + \frac{1}{\sigma_{2k+1} + 1}\),

\[
S(x'_2) = S(x_2) - \frac{1}{\sigma_{2k+1} (\sigma_{2k+1} + 1)}.
\]

Following the same line as above, we have

\[
S(x'_2) = S(x_2) - \frac{1}{\sigma_{2k+1} (\sigma_{2k+1} + 1)}.
\]

Case II \(n = 2k\)

Let

\[
y_1 = \frac{1}{\sigma_1} - \frac{1}{\sigma_2} - \cdots - \frac{1}{\sigma_{2k}}, \quad y_2 = \frac{1}{\sigma_1} - \frac{1}{\sigma_2} + \cdots - \frac{1}{\sigma_{2k} + 1}
\]

Following the same line as above, we have

\[
S(y'_1) = S(y_1) + \frac{1}{\sigma_{2k} (\sigma_{2k} + 1)}
\]

and \(S(y'_1), S(y'_2)\) is right continuous.

**Corollary 3.** For any \(n \geq 1\) and \(\sigma \in D_n\), write \(\alpha = \max\{A_\sigma, B_\sigma\}\) and \(\alpha = \min\{A_\sigma, B_\sigma\}\). Then for any \(x \in J_\sigma\), if \(n = 2k + 1\), then

\[
S'(\alpha_2) < S(x) \leq S(\alpha_1),
\]

where \(S'(\alpha_2) = S(\alpha_2) - \frac{1}{\sigma_{2k+1} (\sigma_{2k+1} + 1)}\).

From the corollary, for any \(\sigma \in D_n\)

\[
\sup_{x,y \in J_\sigma} |S(x) - S(y)| = \frac{n}{\sigma_n (\sigma_n + 1)} = n\lambda(J_\sigma)
\]

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where \( \lambda (J_\sigma) \) is the Lebesgue measure of \( J_\sigma \).

**Theorem 4.** \( S(x) \) is continuous on \( (0,1]\) \( J^\prime \).

Proof. For any \( x \in (0,1]\) \( J' \) and \( x \neq 0 \), let \( x = (d_1(x), \ldots, d_k(x), \ldots) \) be its Alternating Sylvester expansion. For any \( n \geq 1 \), write \( \sigma^n = (d_1(x), \ldots, d_k(x)) \). By (Corollary 3), for any \( y \) \( J_{\sigma^n} \), we have

\[
|S(x) - S(y)| \leq n \lambda (J_{\sigma^n}) \to 0, \quad \text{as} \to \infty.
\]

Write \( I_0 = \{C_\sigma\} \), where

\[
C_\sigma = \frac{1}{\sigma_1} - \frac{1}{\sigma_2} + \cdots + \frac{1}{\sigma_{2n+1}}
\]

**Theorem 5.** If \( 0 < a < b < 1, S(a) < y < S(b) \), then there exists \( c \in (a,b) \setminus \{I_0\} \), such that \( S(c) = y \).

Proof. Set \( g(x) = S(x) - y \), then \( g(x) \) has the same continuity as \( S(x) \). Write

\[
E = \{x \mid g(x) < 0, x \in [a,b]\}, \quad x_0 = sup E.
\]

trivially, \( a \in E \), then the set is well defined.

If \( b = [\sigma_1, \sigma_2, \ldots, \sigma_{2n+1}] \), then by the left continuity of \( S(b) \), we have

\[
\lim_{x \to b^-} g(x) = g(b) > 0,
\]

As a result, there exists a \( \delta > 0 \) such that for any \( x \in (b-\delta, b), g(x) > 0 \).

If \( b = [\sigma_1, \sigma_2, \ldots, \sigma_{2n}] \), since \( g(b) \) is not left continuous, then \( \exists \delta > 0 \) such that for any \( x \in (b-\delta, b), g(x) > 0 \), that is \( x_0 \neq b \).

Following the same line as above, we can prove \( x_0 < a \).

Now we shall prove that \( g(x_0) \leq 0 \). We can choose \( x_\varepsilon \in E \) such that \( x_\varepsilon \to x_0 \), if \( x_\varepsilon = [\sigma_1, \sigma_2, \ldots, \sigma_{2n+1}] \), then

\[
g(x_\varepsilon) = lim_{x_\varepsilon \to x_0} g(x_\varepsilon) \leq 0,
\]

if \( x_0 = [\sigma_1, \sigma_2, \ldots, \sigma_{2n}] \), then

\[
g(x_0) + \frac{1}{(\sigma_{2n} - 1)\sigma_{2n}} = \lim_{x_\varepsilon \to x_0} g(x_\varepsilon) \leq 0
\]

In both case \( g(x_0) \leq 0 \). Following the same line as above, we can prove \( g(x_0) = 0 \), and \( x_0 \neq [\sigma_1, \sigma_2, \ldots, \sigma_{2n+1}] \).

Therefore, there exists \( c \in (a,b) \setminus \{I_0\} \), such that \( S(c) = y \).

**Theorem 6.** \( \int_0^1 S(x) dx + \int_0^1 \frac{1}{H(x)} S(x) dx = \frac{9 - \pi^2}{6} \),

and \( \int_0^1 S(x) dx = -0.1250 \).

Proof.

\[
\int_0^1 S(x) dx = \sum_{d_i=1}^{\infty} \frac{1}{d_i} \int_0^{\frac{1}{d_i}} S(x) dx
\]

\[
= \sum_{d_i=1}^{\infty} \int_0^{\frac{1}{d_i}} \left( x - \frac{1}{d_i} \right) S(T(x)) dx
\]

\[
= \sum_{d_i=1}^{\infty} \int_0^{\frac{1}{d_i}} x dx - \sum_{d_i=1}^{\infty} \frac{1}{d_i} \int_0^1 dx - \sum_{d_i=1}^{\infty} \frac{1}{d_i} S(T(x)) dx
\]

Thus,

\[
\int_0^1 S(x) dx + \sum_{d_i=1}^{\infty} \frac{1}{H(x)} S(x) dx = \frac{3}{2} - \sum_{d_i=1}^{\infty} \frac{1}{d_i^2} = \frac{9 - \pi^2}{6}.
\]

Through the MATLAB program we can get the definite integration

\[
\int_0^1 S(x) dx = -0.1250.
\]

3. Hausdorff Dimension of Graph for \( S(x) \)

Write

\[
Gr(S) = \{(x, S(x)) \mid x \in (0,1]\}.
\]

**Theorem 7.** \( \dim_H Gr(S) = 1 \).

Proof. For any \( n \geq 1 \), \( \{J_\sigma \times S(J_\sigma) \mid \sigma \in D_x \} \) is a covering of \( Gr(S) \). From (Cor 3), \( J_\sigma \times S(J_\sigma) \) can be covered by \( n \) squares with side of length \( \lambda (J_\sigma) \). For any \( \varepsilon > 0 \),

\[
H^{1+\varepsilon} (Gr(S)) \leq \liminf_{\sigma \to 0} \sum_{\sigma \in D_x} n \left( \sqrt{2} \right)^{1+\varepsilon} \lambda (J_\sigma)^{1+\varepsilon}
\]

\[
\leq \liminf_{\sigma \to 0} n \left( \sqrt{2} \right)^{1+\varepsilon} 2^{-\varepsilon} \sum_{\sigma \in D_x} n \left( \sqrt{2} \right)^{1+\varepsilon}
\]

\[
= \liminf_{\sigma \to 0} n \left( \sqrt{2} \right)^{1+\varepsilon} 2^{-\varepsilon} = 0.
\]

Thus, \( \dim_H Gr(S) \leq 1 \)

Since

\[
|\text{Proj}(x, S(x)) - \text{Proj}(y, S(y))| \leq d(x, S(x), (y, S(y))
\]

then

\[
1 = \lambda ((0,1)) = H^1((0,1)] = H^1(\text{Proj}(G, (S))) \leq 1 (G, (S))
\]

so \( \dim_H Gr(S) = 1 \).
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