Uniqueness of Radial Solutions for Elliptic Equation Involving the Pucci Operator*

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ABSTRACT
The solution of a nonlinear elliptic equation involving Pucci maximal operator and super linear nonlinearity is studied. Uniqueness results of positive radial solutions in the annulus with Dirichlet boundary condition are obtained. The main tool is Lane-Emden transformation and Koffman type analysis. This is a generalization of the corresponding classical results involving Laplace operator.

Keywords: Pucci Operator; Radial Solution; Uniqueness; Super Linear

1. Introduction
We study the nonlinear elliptic equation

\[ M_{\lambda,\Lambda}^+ \left(D^2 u \right) + f(u) = 0, \quad (1) \]

where \( 0 < \lambda < \Lambda, \) \( M_{\lambda,\Lambda}^+ \) is Pucci maximal operator, the potential \( f \) is super linear with some further constraints.

Using \( \mu_i, i = 1, \cdots, n \) to denote the eigenvalues of \( D^2 u, \) then explicitly, the Pucci operator \( M_{\lambda,\Lambda}^+ \) is given by

\[ M_{\lambda,\Lambda}^+ \left(D^2 u \right) = \sum_{\mu_i > 0} \lambda \mu_i + \sum_{\mu_i < 0} \mu_i. \]

For more detailed discussion, see for example [1,2]. This equation has been extensively studied, see [3-5], etc. and the references therein.

Normalize \( \lambda \) to be 1 for simplicity. We will in this paper investigate the uniqueness of positive radial solution of (1) in the annulus

\[ \Omega := \{ x \in \mathbb{R}^n : a < |x| < b \} \]

with Dirichlet boundary condition. In this case, Equation (1) reduces to

\[ \Lambda(u') u''(r) + \Lambda(u') \frac{n-1}{r} u'(r) + f(u) = 0, r \in (a, b). \quad (2) \]

where

\[ \Lambda(s) = \begin{cases} \Lambda, & \text{for } s > 0, \\ 1, & \text{for } s \leq 0. \end{cases} \]

Throughout the paper, we assume \( n \geq 2. \) Note that \( \Lambda > 1. \) Now we could state our main results.

\[ \text{Theorem 1. Suppose } \frac{b}{a} \text{ is small enough and } \]

\[ tf'(t) > f(t) > 0 \quad \text{for } t > 0. \]

Then (2) has at most one positive solution with Dirichlet boundary condition.

If instead of the smallness of \( \frac{b}{a} \) we assume further growing condition on \( f, \) then we have the following

\[ \text{Theorem 2. Suppose that } \quad \frac{(2\Lambda - 1)n - 2\Lambda + 2}{(2\Lambda - 1)n - 2\Lambda} f(t) > tf'(t) > \mu f(t) > 0, \]

where

\[ \mu = \max \left\{ 1, \frac{-\frac{2\Lambda - 1}{n+2\Lambda + 2} \right\}. \]

Then (2) has at most one positive solution with Dirichlet boundary condition.

In the case \( \Lambda = 1, \) the Pucci operator reduces to the usual Laplace operator, and the corresponding unique results are proved by Ni and Nussbaum in [6].

We also remark that the above theorems could be generalized to nonlinearities \( f, \) which also depends on \( r. \) We will not pursue this further in this paper.

2. Lane-Emden Transformation and Uniqueness of the Radial Solutions

2.1. Proof of Theorem 1
We shall perform a Lane-Emden type transformation to Equation (2). Let us introduce a new function
\[ w(s) := u(r), \]

where \( s = r^a \), with

\[ \alpha := 1 - \Lambda(n - 1) < 0. \]

Then \( w \) satisfies

\[ w''(s) + ms^{-1}w'(s) + \frac{f(w)s^{\frac{2}{n}}}{\Lambda(u''(r))\alpha^2} = 0, \quad s \in (s_1, s_2), \tag{3} \]

where we have denoted

\[ m(s) := \frac{\Lambda(u''(r))(\alpha - 1) + \Lambda(u'(r))(n - 1)}{\Lambda(u'(r))\alpha} \geq 0, \]

and \( (s_1, s_2) = (b^a, a^a) \). Note that \( m \) may not be continuous at the points where \( u'(r) = 0 \) or \( u''(r) = 0 \). Additionally, if \( u''(r) < 0 \) and \( u''(r) > 0 \), then \( m(s) = 0 \).

**Lemma 3.** Let \( w \) be a positive solution of (3) with \( w(s_1) = w(s_2) = 0 \). Then there exists \( \xi \in (s_1, s_2) \) such that \( w'(\xi) = 0 \), and

\[ w'(s) > 0, \quad \text{in } (s_1, \xi), \]

\[ w'(s) < 0, \quad \text{in } (\xi, s_2). \]

**Proof.** If \( w'(\xi) = 0 \) for some \( \xi \in (s_1, s_2) \), then

\[ w''(\xi) = -ms^{-1}w'(\xi) - \frac{f(w)s^{\frac{2}{n}}}{\Lambda(u''(r))\alpha^2} < 0. \]

The conclusion of the lemma follows immediately from this inequality. \( \blacksquare \)

Given \( d > 0 \), the solution of (3) with \( w(s_1) = 0 \), and \( w'(s_1) = d \) will be denoted by \( w(s, d) \). Let

\[ \varphi(s, d) = \partial_d w(s, d). \]

By standard argument, we know that positive solution of (3) with Dirichlet boundary condition is unique if we could show that

\[ \varphi(s_2, d_0) < 0, \]

whenever \( w(s, d_0) \) is a positive solution to (3) with \( w(s_1, d_0) = w(s_2, d_0) = 0 \).

The functions \( \varphi \) and \( w \) satisfy the following equations:

\[ \left( s^m w' \right)' + \frac{f(w)s^{\frac{2}{n}}}{\Lambda(u''(r))\alpha^2} = 0, \]

\[ \left( s^m \varphi' \right)' + \frac{f(w)s^{\frac{2}{n}}}{\Lambda(u''(r))\alpha^2} \varphi = 0. \]

The initial condition satisfied by \( \varphi \) is: \( \varphi(s_1) = 0 \), \( \varphi'(s_1) = 1 \).

Now let \( d_0 \) be a positive constant such that \( w(s, d_0) \) is a positive solution to (3) with \( w(s_1, d_0) = w(s_2, d_0) = 0 \). To show that \( \varphi(s_1, d_0) < 0 \), let us first prove that \( \varphi(s, d) \) must vanish at some point in the interval \( (s_1, s_2) \). In the following, we write \( \varphi(s, d) \) simply as \( \varphi(s) \).

**Lemma 4.** There exists \( \xi \in (s_1, s_2) \) such that \( \varphi(\xi) = 0 \).

**Proof.** Let us consider the function

\[ \eta_1(s) := s^m w'(s) \varphi(s) - s^m \varphi'(s) w(s). \]

We have

\[ \eta_1'(s) = \left( s^m w' \right)' \varphi - \left( s^m \varphi' \right) w + \frac{s^{\frac{2}{n}}}{\Lambda(u''(r))\alpha^2} \left[ f'(w)w - f(w) \right]. \]

We remark that \( \eta_1 \) is indeed not everywhere differentiable, since \( m \) is not continuous. It however could be shown that the jump points of \( m \) are isolated. Here by \( \eta_1 \), we mean the derivative of \( \eta_1 \) at the point where it is differentiable. The same remark applies to the functions \( \eta_1 \) and \( \eta_2 \) below.

Now if \( \varphi(s) > 0 \) for \( s \in (s_1, s_2) \), then

\[ \eta_1(s) > 0, \quad s \in (s_1, s_2). \]

Since \( \eta_1(s_1) = 0 \), we infer that \( \eta_1(s_2) > 0 \).

It follows that

\[ w'(s_2) \varphi(s_2) > 0. \]

This is a contradiction, since \( w'(s_2) \leq 0 \) and \( \varphi(s_2) \geq 0 \). \( \blacksquare \)

With the above lemma at hand, we wish to show that in the interval \( (s_1, s_2) \), \( \varphi \) vanishes at only one point \( \xi \). For this purpose, let us define functions \( g(s) := w'(s) \) and \( h(s) := (s - s_1)w'(s) \). Put

\[ \eta_2(s) := s^m g'(s) \varphi(s) - s^m \varphi'(s) g(s), \]

\[ \eta_3(s) := s^m h'(s) \varphi(s) - s^m \varphi'(s) h(s), \]

and

\[ \Phi(w, s) := \frac{f(w)s^{\frac{2}{n}}}{\Lambda(u''(r))\alpha^2}, \]

**Lemma 5.** We have

\[ \eta_2'(s) = s^m \left( m s^2 g - \Phi \right) \varphi, \tag{4} \]

\[ \eta_3'(s) = s^m \left( -m s^2 g - (s - s_1) \Phi - 2 f(w)s^{\frac{2}{n}} - \frac{f(w)s^{\frac{2}{n}}}{\Lambda(u''(r))\alpha^2} \right) \varphi. \tag{5} \]
By (5) using the fact that \(0, \alpha, \beta \in \mathbb{R}\), we have:

\[
\eta'_2(s) - \Phi_s \phi'(\xi) \leq 0
\]

This implies that

\[
g'(\xi)\phi'(\xi) - g(\xi)\phi'(\xi) < 0
\]

but this contradicts with \(g(\xi_1) = 0\), \(\phi'(\xi_1) > 0\), and \(g(\xi_2) < 0\). This finishes the proof.

### 2.2. Proof of Theorem 2

Similar arguments as that of Theorem 1 could be used to prove Theorem 2. In this case, we shall make the following transform:

\[
u(r) = r^\alpha w(s), \quad s = r^\beta,
\]

where

\[
\alpha = 1 - \Lambda(n-1) < 0,
\]

and \(\beta = 2n - 2\alpha > 0\). Then

\[
u^*(r) = \beta^\alpha r^\beta w^*(s) + \beta(\alpha + \beta - 1) r^\alpha s w'(s)
\]

and

\[
u^*(s) + m_1 s^{-\alpha} w^*(s) + m_2 s^{-\alpha} w + \frac{2}{\Lambda(u^*(r))} \frac{f(s^{-\alpha} w)}{\beta^2} = 0,
\]

where

\[
m_1(s) := \frac{\Lambda(u^*(r))(2\alpha + \beta - 1) + \Lambda(u^*(r))(n-1)}{\Lambda(u^*(r)) \beta},
\]

\[
m_2(s) := \frac{\alpha \left[ \Lambda(u^*(r))(\alpha - 1) + \Lambda(u^*(r))(n-1) \right]}{\Lambda(u^*(r)) \beta^2}.
\]

By the definition of \(\alpha, \beta\), one could verify that \(m_2 \geq 0\). Note that \(m_1 \) and \(m_2 \) are step functions and not continuous.

Let \(w(s, d)\) be the solution of (8) with \(w(s, d) = 0\) and \(w(s, d) = d\). Now similar as in the proof of Theorem 1, we suppose \(w = w(s, d_0)\) is a positive solution with Dirichlet boundary condition and \(\phi = \partial_s w(s, d_0)\). We have the following lemma, whose proof will be omitted.

**Lemma 6.** There exists \(\bar{s} \in (s_1, s_2)\) such that

\[
w'(\bar{s}) = 0, \text{ and}
\]

\[
w'(s) > 0 \text{ in } (s_1, \bar{s}),
\]
Consider the function \( \eta_s = s^n w \varphi - s^n \varphi' w \), then

\[
\eta_s(s) = \frac{s \beta^{-1}}{\Lambda(u^*(r)) \beta^2} \varphi \left[ f' \left( s^{-1}w \right) s^{-1}w - f \left( s^{-1}w \right) \right].
\]

From this we infer that the function \( \varphi \) must change sign in the interval \((s_1, s_2)\), similar as that of Theorem 1.

Now let us define

\[
\eta_s(s) = s^n [m_s w - 2m_s s^{-3} w - \Phi_s],
\]

and

\[
\eta_s(s) = s^n h(s) \varphi(s) - s^n h(s) \varphi'(s),
\]

where \( g = w' \) and \( h = (s-s_1)g \). Moreover, denote

\[
\Phi(w,s) := \frac{s \beta^{-1}}{\Lambda(u^*(r)) \beta^2} f \left( s^{-1}w \right).
\]

**Lemma 7.** There holds

\[
\eta_s(s) = s^n \left[ m_s w - 2m_s s^{-3} w - \Phi_s \right] \varphi,
\]

**Proof.** Direct calculation shows

\[
g^r(s) + m_s s^{-1} g'(s) + m_s s^{-2} g + \frac{s \beta^{-1}}{\Lambda(u^*(r)) \beta^2} f' \left( s^{-1}w \right) s^{-1} g = m_s w^2 g + 2m_s s^{-3} w - \Phi_s,
\]

and

\[
(s^m h)' + m_s s^{-2} h = -s_s m_s s^{-2} g - 2s_s m_s s^{-3} w - (s-s_1) \Phi_s - \frac{m_s \beta^{-1}}{\Lambda(u^*(r)) \beta^2} f' \left( s^{-1}w \right) s^{-1} h - \frac{m_s \beta^{-1}}{\Lambda(u^*(r)) \beta^2} f \left( s^{-1}w \right).
\]

This then leads to the desired identity. ■

Now with the help of this lemma, we could prove Theorem 2.

**Proof of Theorem 2.** First we show the first zero \( \xi \) of \( \varphi \) is in the interval \([s_1, s_2]\) Otherwise, since

\[
(s-s_1) \Phi_s + \frac{m_s \beta^{-1}}{\Lambda(u^*(r)) \beta^2} f' \left( s^{-1}w \right) s^{-1} h - \frac{m_s \beta^{-1}}{\Lambda(u^*(r)) \beta^2} f \left( s^{-1}w \right) \geq 0,
\]

one could then use the fact that \( m_s \geq 0 \) in \((s_1, s_2)\) to deduce that in \((s_1, \xi)\),

\[
\eta_s(s) < 0.
\]

But this contradicts with \( \eta_s(s_1) = 0 \) and \( \eta_s(\xi) \geq 0 \). Now if the second zero \( \xi' \) of \( \varphi \) is in \((\xi, s_2)\). Then since

\[
\Lambda(u^*(r)) \beta^2 \Phi_s = \frac{2 \beta^{-1}}{\Lambda(u^*(r)) \beta^2} f \left( s^{-1}w \right) - s^{-1} w f' \left( s^{-1}w \right) < 0,
\]

one could use \( m_s \leq 0 \) in \((\xi, s_2)\) to deduce that \( \eta_s(\xi') < 0 \) in \((\xi, \xi')\), which contradicts with \( \eta_2(\xi) = 0 \).
\[ \eta_2(x) \geq 0. \]

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REFERENCES


