$G$-Design of Complete Multipartite Graph Where $G$ Is Five Points-Six Edges

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ABSTRACT
In this paper, we construct $G$-designs of complete multipartite graph, where $G$ is five points-six edges.

Keywords: Complete Multipartite Graph; Graph Design; Latin Square

1. Introduction
Let $K_n$ be a complete graph with $n$ vertices, and $G$ be a simple graph with no isolated vertex. A $G$-design (or $G$-decomposition) is a pair $(X, B)$, where $X$ is the vertex set of $K_n$ and $B$ is a collection of subgraphs of $K_n$, called blocks, such that each block is isomorphic to $G$ and any edge of $K_n$ occurs in exactly a blocks of $B$. For simplicity, such a $G$-design is denoted by $GGD(v)$. Obviously, the necessary conditions for the existence of a $G-GD(v)$ are

\[ \begin{align*}
    v & \geq \left| V(G) \right| \\
    v(v-1) & \equiv 0 \mod 2 \left| E(G) \right|,
\end{align*} \]

\[ v - 1 \equiv 0 \mod d \]

where $d$ is the greatest common divisor of the degrees of the vertices in $V(G)$.

Let $K_{n_1,n_2,\ldots,n_k}$ be a complete multipartite graph with vertex set $X = \bigcup_{i=1}^{k} X_i$, where these $X_i$ are disjoint and $|X_i| = n_i$, $1 \leq i \leq k$. For a given graph $G$, a holey $G$-design, denoted by $(X, G, \mathcal{B})$, where $X$ is the vertex set of $K_{n_1,n_2,\ldots,n_k}$, $G = \{X_1, X_2, \ldots, X_m\}$ (X called hole) and $\mathcal{B}$ is a collection of subgraphs of $K_{n_1,n_2,\ldots,n_k}$ called blocks, such that each block is isomorphic to $G$ and any edge of $K_{n_1,n_2,\ldots,n_k}$ occurs in exactly a blocks of $\mathcal{B}$. When the multipartite graph has $k_i$ partite of size $n_i$, $1 \leq i \leq r$, the holey $G$-design is denoted by $G-HD(n_1^{n_1} n_2^{n_2} \cdots n_r^{n_r})$.

When $n_1 = n_2 = \cdots = n_k = n$, the holey $G$-design is denoted by $G-HD(n^n)$ (also known as $G$-decomposition of complete multipartite graph $K_n(t)$).

On the $G$-design of existence has a lot of research. Let $k$ be the vertex number of $G$, When $k \leq 4$, J. C. Bermond proved that condition (1) is also sufficient in [1]; When $k = 5$, J. C. Bermond gives a complete solution in [2]. When $G = S_6$, $P_2$ and $C_k$, K. Ushio investigated the existence of $G$-design of complete multipartite graph in [3].

2. Fundamental Theorem and Some Direct Construction
Let $G$ be a simple graph with five points-six edges (see Graph 1). $G$ is denoted by $(a, b, c)$-$(d, e, f)$.

The lexicographic product $G_1 \otimes G_2$ of the graphs $G_1$ and $G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and an edge joining $(u_1, u_2)$ to $(v_1, v_2)$ if and only if either $u_1$ is adjacent to $v_1$ in $G_1$ or $u_1 = v_1$ and $u_2$ and $v_2$ are adjacent in $G_2$. We are only concered with a particular kind of lexicographic product, $G \times K_n$ ($K_n$ be a empty graph with $n$ vertices). Observe that

\[ K_n(t) = K_n(t) \otimes K_1. \]

Lemma 2.1. If there exists a $G-HD(l^n)$, then there exists a $G-HD((lt)^n)$ for any integer $l$.

Proof. Let $V(K_n) = \{1, 2, \ldots, l\}$. Take any $l \times l$ latin square and consider each element in the form $(\alpha, \beta, \gamma)$ where $\alpha$ denotes the row, $\beta$ the column and $\gamma$ the entry, with $1 \leq \alpha, \beta, \gamma \leq l$. We can construct $l^2$ graphs $G$.
Let $K$ be a subset of positive integers. A pairwise balanced design (PBD($v, K$)) of order $v$ with block sizes from $K$ is a pair $(\mathcal{Y}, \mathscr{B})$, where $\mathcal{Y}$ is a finite set (the point set) of cardinality $v$ and $\mathscr{B}$ is a family of subsets (blocks) of $\mathcal{Y}$ which satisfy the properties:

1. If $B \in \mathscr{B}$, then $|B| \in K$.
2. Every pair of distinct elements of $\mathcal{Y}$ occurs in exactly a blocks of $\mathscr{B}$.

Let $K$ be a set of positive integers and

$$B(K) = \left\{ v \in N \mid \exists \text{PBD}(v, K) \right\},$$

then $B(K)$ is the PBD-closure of $K$.

**Lemma 2.2** [5] If $K = \{3, 4, 5, 6, 8\}$, then

$$B(K) = \{ n \in N \mid n \geq 3 \}.$$  

**Lemma 2.3** [5] If $K = \{3, 4, 6\}$, then

$$B(K) = \{ v \in N \mid n > 3, n \equiv 0, 1 \text{(mod 3)} \}.$$  

**Lemma 2.4** [5] If $K = \{5, 9, 13, 17, 29, 33\}$, then

$$B(K) = \{ v \in N \mid n > 4, n \equiv 1 \text{(mod 4)} \}.$$  

**Lemma 2.5** If there exists a $G$-HD$(\ell)$ where $k \in \{3, 4, 5, 6, 8\}$, then there exists a $G$-HD$(\ell)$ where $n \geq 3$.

**Proof.** Let $X$ be an element set and $Z_i$ be a modulo $t$ residual additive group. For $K = \{3, 4, 5, 6, 8\}$, take $Y = X \times Z_t$ by applying Lemma 2.2, we assume that $(X, \mathcal{A})$ be a $KBD(n, k)$. In the $A$, we take a block $A$, for $|A| = k \in K$, as there exists a $G$-HD$(\ell)$, let $A \times Z_t$ be the vertex set of $G$-HD$(\ell)$ and block set be $\mathcal{B}_A$, a $\mathcal{B} = \cup \mathcal{B}_A (A \in \mathcal{A})$, so $(Y, \mathcal{B})$ be a $G$-HD$(n)$.

Similar to the proof of Lemma 2.5, We have the following conclusions.

**Lemma 2.6** If there exists a $G$-HD$(\ell)$ for $k \in \{3, 4, 6\}$, then there exists a $G$-HD$(\ell)$ for $n = 0, 1 \text{ (mod 3)}$ and $n > 3$.

**Lemma 2.7** If there exists a $G$-HD$(\ell)$ for $k \in \{5, 9, 13, 17, 29, 33\}$, then there exists a $G$-HD$(\ell)$ for $n \equiv 1 \text{ (mod 4)}$ and $n > 4$.

**Lemma 2.8** [2] For $n \equiv 1, 9 \text{ (mod 12)}$, there exists a $(n, G, 1)$-GD.

**Lemma 2.9** There exists a $G$-HD$(2^9)$.

**Proof.** Take $X = \{a, b, c, d, e, f\}$ and $G = \{\{a, b\}, \{c, d\}, \{e, f\}\}$, we list vertex set and blocks below

$$\mathcal{B} : \{(a, b, f) - (f, c, b)(c, a, e) - (e, c, d)\}$$

**Lemma 2.10** There exists a $G$-HD$(2^6)$.

**Proof.** Take $X = \{0, 1, 2, 3\}$, and $G = \{\{0, 1\}, \{2, 3\}\}$, we list vertex set and blocks below

$$\mathcal{B} : \{(3, 0, 1) - (1, 2, 4) + 2 \mod 4\}$$

**Lemma 2.11** There exists a $G$-HD$(2^6)$.  

**Proof.** Take $X = \{0, 5, 1, 6, 2, 7, 3, 8, 4, 9\}$, and $G = \{\{0, 1\}, \{2, 3\}\}$, we list vertex set and blocks below

$$\mathcal{B} : \{(1, i, 3, 3, 1) - (i, 4, i, 3, 1)\}$$
For \( t \equiv 0 \pmod{6} \) and \( n \geq 3 \):
1) \( t \equiv 0 \pmod{6} \) and \( n \equiv 0, 1 \pmod{3} \),
2) \( t \equiv 0 \pmod{6} \), \( t \not\equiv 0 \pmod{3} \) and \( n \equiv 1 \pmod{4} \),
3) \( t \equiv 0 \pmod{6} \), \( t \not\equiv 0 \pmod{3} \) and \( n \equiv 1 \pmod{9} \).

**Proof.** Necessary conditions are obviously, we prove the sufficient conditions.

1) For \( t \equiv 0 \pmod{6} \) and \( n \geq 3 \), by applying Lemma 2.17 and 2.5.,

\[ \text{Lemma 2.16 There exists a } G\text{-HD}(3^{2k}) \]

2) For \( t \equiv 0 \pmod{2} \), \( t \not\equiv 0 \pmod{3} \) and \( n \equiv 0, 1 \pmod{3} \) by applying Lemma 2.9, 2.10, 2.11 and 2.12.

3) For \( t \equiv 0 \pmod{3} \), \( t \not\equiv 0 \pmod{2} \) and \( n \equiv 1 \pmod{4} \) by applying Lemma 2.12, 2.14, 2.15, 2.16, 2.18 and 2.7.

4) For \( t \equiv 0 \pmod{2} \), \( t \not\equiv 0 \pmod{3} \) and \( n \equiv 1, 9 \pmod{12} \), by applying Lemma 2.1 and 2.8.

3. **G-Designs of Complete Multipartite Graph**

**Theorem 3.1** The necessary conditions for the existence of a \( G\text{-HD}(t^n) \) are sufficient for the following \( n \) and \( t \):

- 1) \( t \equiv 0 \pmod{6} \) and \( n \geq 3 \);
- 2) \( t \equiv 0 \pmod{6} \), \( t \not\equiv 0 \pmod{3} \) and \( n \equiv 0, 1 \pmod{3} \),
- 3) \( t \equiv 0 \pmod{6} \), \( t \not\equiv 0 \pmod{3} \) and \( n \equiv 1 \pmod{4} \),
- 4) \( t \equiv 0 \pmod{6} \), \( t \not\equiv 0 \pmod{3} \) and \( n \equiv 1, 9 \pmod{12} \),

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