A $\varphi_x$- and Open $C^*_D$-Filters Process of Compactifications and Any Hausdorff Compactification

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ABSTRACT

By means of a characterization of compact spaces in terms of open $C^*_D$-filters induced by a $D \subseteq C^*(Y)$, a $\varphi_x$- and open $C^*_D$-filters process of compactifications of an arbitrary topological space $Y$ is obtained in Sec. 3 by embedding $Y$ as a dense subspace of $(Y^w, \mathcal{J}_B)$ or $(Y^w, \mathcal{J}_B)$, where $Y^w = Y_E \cup Y_S$, $Y^w = Y_E \cup Y_T$, $Y_E = \{N|N$ is a $\varphi_x$-filter, $x \in Y\}$, $Y_S = \{F|F$ is an open $C^*_D$-filter that does not converge in $Y\}$, $Y_T = \{\tilde{A}|\tilde{A}$ is a basic open $C^*_D$-filter that does not converge in $Y\}$, $\mathcal{J}_B$ is the topology induced by the base $\mathcal{B} = \{U^*|U$ is open in $Y, U \neq \phi\}$ and $U^* = \{F \in Y^w+(Y^w) \cup F\}$. Furthermore, an arbitrary Hausdorff compactification $(Z, h)$ of a Tychonoff space $X$ can be obtained from a $D \subseteq C^*(X)$ by the similar process in Sec. 3.

Keywords: Net; Open Filter; Open $C^*_D$-Filter Base; Basic Open $C^*_D$-Filter; Open $C^*_D$-Filter; $\varphi$-Filter; $\varphi_x$-Filter; Tychonoff Space; Normal $T_1$-Space; Compact Space; Compactifications; Stone-Cech Compactification; Wallman Compactification

1. Introduction

Throughout this paper, $[T]^{-\alpha}$ denotes the collection of all finite subsets of the set $T$. For the other notations and terminologies in General Topology which are not explicitly defined in this paper, the readers will be referred to the Ref. [1].

For an arbitrary topological space $Y$, let $C^*(Y)$ be the set of bounded real-valued continuous functions on $Y$, $D \subseteq C^*(Y)$. It is shown in Sec. 2 that there exists a unique $r_t \in \text{Cl}(r(Y))$ for each $r \in D$ such that for any $H \in [D]^{-\alpha}$, $\varepsilon > 0$, $\bigcap_{t \in H} f^{-1}((r_t - \varepsilon, r_t + \varepsilon)) \neq \phi$. Let $V_{t} = \bigcap_{t \in H} f^{-1}((r_t - \varepsilon, r_t + \varepsilon)) \neq \phi$ for any $H \in [D]^{-\alpha}$, $\varepsilon > 0$. $V_{t}$ is called an open $C^*_D$-filter base. An open filter $E_{x}$ on $Y$ containing an open $C^*_D$-filter base $V_{t}$ is called an open $C^*_D$-filter. An open filter $A_{y}$ on $Y$ generated by an open $C^*_D$-filter base $V_{t}$ is called a basic open $C^*_D$-filter. By a characterization of compact spaces in Sec. 2 and the $\varphi_x$- and open $C^*_D$-filters process of compactifications in Sec. 3, $Y$ can be embedded as a dense subspace of $(Y^w, \mathcal{J}_B)$ or $(Y^w, \mathcal{J}_B)$, where $Y^w = Y_E \cup Y_S$, $Y^w = Y_E \cup Y_T$, $Y_E = \{N|N$ is a $\varphi_x$-filter, $x \in Y\}$, $Y_S = \{F|F$ is an open $C^*_D$-filter that does not converge in $Y\}$, $Y_T = \{\tilde{A}|\tilde{A}$ is a basic open $C^*_D$-filter that does not converge in $Y\}$, $\mathcal{J}_B$ is the topology induced by the base $\mathcal{B} = \{U^*|U \neq \phi, U$ is open in $Y\}$ and $U^* = \{F \in Y^w+(Y^w) \cup F\}$. Furthermore an arbitrary Hausdorff compactification $(Z, h)$ of a Tychonoff space $X$ can be obtained from a $D \subseteq C^*(X)$ by the similar process in Sec. 3.

2. Open $C^*_D$-Filters and a Characterization of Compact Spaces

Let $Y$, $C^*(Y)$ and $D$ be the sets that are defined in Sec. 1.

**Theorem 2.1** Let $F$ be a filter on a topological space $Y$. For each $f \in D$, there exists a $r_t \in \text{Cl}(f(Y))$ such that $f^{-1}((r_t - \varepsilon, r_t + \varepsilon)) \cup F \neq \phi$ for any $F \in F$ and any $\varepsilon > 0$. (See Thm 2.1 in the Ref. [2, p. 1164].)

**Proof.** If the conclusion is not true, then there is an $f \in D$ such that for each $r_t \in \text{Cl}(f(Y))$, there exist an $F_t \in F$ and an $\varepsilon_t > 0$ such that $f^{-1}((r_t - \varepsilon_t, r_t + \varepsilon_t)) \cap F = \phi$. Since $\text{Cl}(f(Y))$ is compact and $\text{Cl}(f(Y)) \subseteq \bigcup\{(r_t - \varepsilon_t, r_t + \varepsilon_t)|r_t \in \text{Cl}(f(Y))\}$, there exist $r_1, \ldots, r_n$ in $\text{Cl}(f(Y))$ such that $Y = f^{-1}(\text{Cl}(f(Y))) = \bigcup\{f^{-1}((r_t - \varepsilon_t, r_t + \varepsilon_t))|r_t \in \text{Cl}(f(Y))\}$, but $\bigcap_{t=1}^{n} f^{-1}((r_t - \varepsilon_t, r_t + \varepsilon_t)) = \phi$, contradicting that $\phi \neq F$. □

**Corollary 2.2** Let $Q$ be an open ultrafilter on $Y$. For
each \( f \in D \), there exists a unique \( r_f \in \text{Cl}(f(Y)) \) such that (1) for any \( H \in [D]^\omega \), any \( \varepsilon > 0 \), \( \cap_{\varepsilon \in H} f^{-1}((t_r - \varepsilon, r_t + \varepsilon)) \neq \emptyset \) if \( S \subseteq T \) and (2) for any \( H \in [D]^\omega \), any \( \varepsilon > 0 \), \( \cap_{\varepsilon \in H} f^{-1}((t_r - \varepsilon, r_t + \varepsilon)) \neq \phi \) (See Cor. 2.2 in the Ref. [2, p. 1164]).

Therefore, for a given open ultrafilter \( Q \), \( Q \) contains a unique open filter base \( V = \{ \cap_{\varepsilon \in H} f^{-1}((t_r - \varepsilon, r_t + \varepsilon)) \} \) \( \cap_{\varepsilon \in H} f^{-1}((t_r - \varepsilon, r_t + \varepsilon)) \neq \emptyset \) for any \( H \in [D]^\omega \), \( \varepsilon > 0 \), \( \cap_{\varepsilon \in H} f^{-1}((t_r - \varepsilon, r_t + \varepsilon)) \neq \phi \) (See Cor. 2.2 in the Ref. [2, p. 1164]).

**Proof.** 1) \( \Rightarrow \) 2) is obvious by Lemma 2.8 1) above and Thm. 12.17 (a) in the Ref. [1, p. 81]. For \( \Rightarrow \) 1), Let \( \{x_i\} \) be a D-net in \( Y \), let \( F = \{O\} \) is open, and \( \{x_i\} \) is eventually in \( O \). Clearly, \( F \) is an open filter. For each \( f \in D \), let \( t_r = \lim f(x_i) \), then \( \{x_i\} \) is eventually in \( f^{-1}((t_r - \varepsilon, r_t + \varepsilon)) \) for any \( \varepsilon > 0 \), i.e., for each \( f \in D \), any \( \varepsilon > 0 \), \( f^{-1}((t_r - \varepsilon, r_t + \varepsilon)) \) \( \subseteq \mathcal{F} \), so \( F \) is an open \( C^*_p \)-filter. 2) implies that \( F \) converges to a point \( x \). Thus, for any open nhood \( U \) of \( x \), \( U \in \mathcal{F} \); i.e., \( \{x_i\} \) is eventually in \( U \). So \( \{x_i\} \) converges to \( x \). \( \square \)

**Corollary 2.10.** If every open \( C^*_p \)-filter \( E \) on \( Y \) converges in \( Y \), then \( Y \) is compact.

3. An Open \( C^*_p \)-Filter Process of Compactification

For each \( x \in Y \), let \( N_x = \{\{x_i\} \cup O \} \) is open, \( x \in O \). \( N_x \) is a \( \sigma \)-filter (See 12E in the Ref. [1, p. 83] for its definition and convergence) with \( O = N_x \). For each \( x \in Y \), \( N_x \) is called a \( \sigma \)-filter. Let \( Y = N_x \), \( \mathcal{F} = \{N_x\} \) is a \( \sigma \)-filter, \( x \in Y \), \( Y_x = \mathcal{E} \) is an open \( \mathcal{C}^*_p \)-filter that does not converge in \( Y \). \( Y_x = \mathcal{E} \cup S_y \) and \( Y_x \) is a basic open \( \mathcal{C}^*_p \)-filter that does not converge in \( Y \), \( Y_x = \mathcal{E} \cup S_y \) and \( Y_x \) is a basic open \( \mathcal{C}^*_p \)-filter that does not converge in \( Y \).

**Lemma 3.11.** For each \( F \in Y_x^w \) (or \( Y_x^w \)), there is a unique \( \{x_i\} \subseteq \text{Cl}(f(Y)) \) for each \( f \in D \) such that \( f^{-1}((t_r - \varepsilon, r_t + \varepsilon)) \subseteq V \), \( x \in F \) for all \( \varepsilon > 0 \).

**Proof.** If \( F = N_x \) for an \( x \in Y \), then for each \( f \in D \), \( f^{-1}((t_r - \varepsilon, r_t + \varepsilon)) \subseteq V \subseteq N_x \) for all \( \varepsilon > 0 \), where \( t_r = \lim f(x_i) \). If \( F = \emptyset \), then there is an open \( F \)-filter base \( V \), as the \( x \) in \( F \) defined in Section 1, such that for each \( f \in D \), \( f^{-1}((t_r - \varepsilon, r_t + \varepsilon)) \subseteq V \subseteq \emptyset \) for all \( \varepsilon > 0 \). The uniqueness of \( t_r \) for each \( f \in D \) follows from Cor. 2.2. \( \square \)

**Definition 3.12.** For any open set \( U \) \( \neq \emptyset \) in \( Y \), define \( U^* = \{f \in Y_x^w \cup Y_x^w | f \subseteq U \} \).

**Lemma 3.13.** 1) For any open set \( U \subseteq Y \), \( U \neq \emptyset \Rightarrow U^* \neq \emptyset \).

"Proof. 1) If \( U \neq \emptyset \), pick an \( x \in U \), then \( U \subseteq N_x \Rightarrow U = U^* \); i.e., \( U \neq \emptyset \); if \( U \neq \emptyset \), pick a \( f \in U \), then \( U \subseteq U^* \)."

2) \( \Rightarrow \) 1) is obvious from Def. 3.12. \( \square \)

**Lemma 3.14.** For any two nonempty open sets \( S \) and \( T \) in \( Y \), \( S \subseteq T \) iff \( S^* \subseteq T^* \) and (2) \( S \cap T^* = S^* \cap T^* \) if \( S \subseteq T \).

"Proof. 1) \( \Rightarrow \) 2) By (1), \( S \cap T^* = S^* \cap T^* \Rightarrow S^* \subseteq T^* \). For \( S \subseteq T \), there is a \( y \in (S - T) \Rightarrow y \in N_y \in \mathcal{C}^*_p \Rightarrow S^* \subseteq T^* \). By 1) above, \( S \cap T^* \subseteq S^* \cap T^* \). If \( S^* \subseteq T^* \), then \( S \subseteq T^* \subseteq T^* \); i.e., \( S \subseteq T^* \).

3) By the definition of the \( C^*_p \)-filter process, \( S \subseteq T \). \( \square \)

**Proposition 3.15.** \( B = \{U^*U \neq \emptyset \text{ is an open set in } Y \} \) is a base for \( Y_x^w \) (or \( Y_x^w \)).

"Proof. (a) In Thm. 5.3 in the Ref. [1, p. 38]: For each \( f \in Y_x^w \) (or \( Y_x^w \)), pick a \( O \in F \). Then \( O \neq \emptyset \), \( f \in O \) and \( O^* \in B \). Thus \( Y_x^w \) (or \( Y_x^w \)) = \{U^*U \in B \}.

(b) If \( f \in S^* \cap T^* \), then \( S \subseteq T \subseteq T^* \), then \( S \cap T \subseteq F \); i.e., \( S \subseteq T \). \( \square \)
≠ S ∩ T ∈ ℱ, (S ∩ T)* ∈ ℋ and ℱ ∈ (S ∩ T)* ⊆ S ∩ T* ∈ ℋ. □

Equip Y^w (or Y^w) with the topology induced by ℋ. For each f in D, define f"* (or Y^w) → ℜ by f"*(ℱ) = r_0 if f"*(t(e, t, e) + ε) ⊆ V, otherwise f"*(ℱ) = r_0 for all ε > 0. By Lemma 3.11, for each f in D, f"* is well-defined and f"*(Y^w) (or f"*(Y^w) ⊆ Cl(f (Y))) is finite. f"* is a bounded real-valued function on Y^w (or Y^w) such that f"*(x) = f(x) for all x ∈ Y.

Proposition 3.16 For each f in D, let t ∈ Cl(f (Y)). For any δ, ε with 0 < δ < ε < 1, [f"*(1(δ, 1 + δ)])^* ⊆ f"*(1(ε, 1 + ε)).

Proof. 1) If f"*(1(δ, 1 + δ))]^* then f"*(1(δ, 1 + δ)) ⊆ ℱ. If f"*(ℱ) = e, then f"*(1(ε, 1 + ε)) ⊆ ℱ for all ε > 0. Since (1(δ, 1 + δ)) ⊆ (1(ε, 1 + ε)), f"*(ℱ) = e, i.e., F ∈ f"*(1(ε, 1 + ε)), hence f"*(ℱ) ⊆ (1(ε, 1 + ε)). Since f"*(1(ε, 1 + ε)) ⊆ ℱ for all ε > 0, this implies f"*(ℱ) = e for all ε > 0. Hence f"*(ℱ) = e for all ℱ in T, i.e., F ∈ f"*(1(ε, 1 + ε)).

Proposition 3.17 For each f in D, f"* is a bounded real-valued continuous function on Y^w (or Y^w).

Proof. For any f"* ∈ Y^w (or Y^w), let f"*(ℱ) = t. We show that for any ε > 0, there is a U * ∈ ℋ such that f"* ∈ U* ⊆ f"*(1(ε, 1 + ε)). Let U = f"*(1(ε, 2 + ε)) = f"*(1(ε, 2 + ε)). Since f"*(1(ε, 2 + ε)) ⊆ ℱ for all ε > 0, this implies f"*(ℱ) = e for all ℱ in T, i.e., F ∈ f"*(1(ε, 2 + ε)). Pick by Prop. 3.16 1), F ∈ U* ⊆ f"*(1(ε, 2 + ε)). Thus f"* is continuous on Y^w (or Y^w).

Lemma 18 Let k: Y → Y^w (or Y^w) be defined by k(x) = N_k. Then k is well-defined, one-one and k(U) = U for all nonempty open set U in Y and all U ∈ ℋ, i.e., k is continuous; 2) f"* ◦ k = f for all f in D; 3) k(Y) is dense in Y^w (or Y^w).

Proof. 1) For any x, y in Y, x = y implies k(x) = k(y). So k(x) = k(y). 2) k(U) = U for all nonempty open set U in Y and all U ∈ ℋ, i.e., k(U) = U for all nonempty open set U in Y, U ∈ ℋ, i.e., k is continuous.

Theorem 3.20 (Y^w, k) is a compactification of Y.

Proof. Case 1: If L_k converges to a point p in Y, let U be any open set in Y such that k(p) ∈ U in ℋ. By Lemma 3.18 1), p ∈ U = k(U), thus U ∈ ℋ, i.e., U ∈ ℋ_1. This implies that L_k converges to k(p) in Y^w. Case 2: If L_k does not converge in Y then L_k ∈ Y^w. For any U ∈ ℋ such that L_k ∈ U, U ∈ ℋ and therefore U ∈ ℋ_1. This shows that L_k converges to L_k in Y^w. By Cor. 2.10, Y^w is compact and by Lemma 3.18 3), (Y^w, k) is a compactification of Y.

Lemma 20 For any open set U in L_k = A_k, U ∈ ℋ_1.

Proof. If U ∈ ℋ_1, then there exist a H ∈ D^w, an ε > 0 such that E = (∩ H^w)ε(t_e, t_e + ε) ∈ V, and E ⊆ U. By Lemma 3.14 and Prop. 3.16 2) that F = (∩ H^w)ε(t_e, t_e + ε) ⊆ E^w ⊆ U* and F ∈ ℋ_1. Thus U = E^w.

Theorem 3.22 (Y^w, k) is a compactification of Y.

Proof. Case 1: If L_k = A_k converges to a point p in Y, let U be any open set in Y such that k(p) ∈ U, Lemma 3.18 1) implies that p ∈ U, thus U = L_k = A_k, and L_k does not converge in Y then L_k = A_k ∈ Y^w. For any U* + B such that A_k ∈ U* + B and L_k ∈ L_k and therefore U* ∈ ℋ_1. This shows that E^w converges to E^w in Y^w. By Cor. 2.10, Y^w is compact and by Lemma 3.18 3), (Y^w, k) is a compactification of Y.

4. An Arbitrary Hausdorff Compactification of a Tychonoff Space

For an arbitrary Hausdorff compactification (Z, h) of a Tychonoff space X, let D = {f|f is h o f, f ∈ D = C(Z)}. Then D ⊆ C^*(X), D separates points of X and the topology on X is the weak topology induced by D. For each x ∈ X, let V_x as the V_x defined in Section 2, be the open C^*_X-filter base at x induced by D. Obviously, we can easily get Lemma 4.21 as follows:

Lemma 4.21 G_0 = V_{x} [x ∈ X] is a base for the topology on X and for each x ∈ X, V_x is an open nhood base at x.
Let $X^W = \{ \hat{A} \mid \hat{A} \text{ is a basic open } C^*_p \text{-filter on } X \}$. For each $\hat{A} \in X^W$, let $V_\lambda$, as the $V_\lambda$ defined in Sec. 1, be the open $C^*_p \text{-filter base that generates } \hat{A}$. If $\hat{A}$ converges to an $x \in X$, then for each $f \in D$, $x \in Cl(f^{-1}(t_\lambda - d, t_\lambda + e)) \subseteq f^\ast\circ I^{-1}(t_\lambda - d, t_\lambda + e) \subseteq f^\ast((t_\lambda - d, t_\lambda + e))$ for all $\lambda > 0$.

Proof. For any open set $U \subset X$, define $U^* \mathrel{\overset{\text{def}}{=} } \{ \hat{A} \in X^W \mid U \circ I^{-1}(t_\lambda - d, t_\lambda + e) \}

The following. For any two empty sets $S$ and $T$ in $X$, 1) $S \subset \mathcal{I}$ if $S \circ I^{-1}(t_\lambda - d, t_\lambda + e)$, and 2) $(S \cap T)^* = S^* \cap T^*$ if $S \cap T \neq \phi$.

For any $f \in D$, let $t \in \mathcal{I}(f(X))$. For any $\delta, e$ with $0 < \delta < e$, 1) $[f^\ast((t_\lambda - \delta, t_\lambda + \epsilon))]^* \subseteq f^\ast((t_\lambda - \delta, t_\lambda + \epsilon))$, 2) $f^\ast((t_\lambda - \delta, t_\lambda + \epsilon)) \subseteq f^\ast((t_\lambda - \delta, t_\lambda + \epsilon))$.

For any $f \in D$, $f^\ast$ is a bounded real-valued continuous function on $X^W$.

For each basic open $C^*_p \text{-filter } \hat{A} \in X^W$, let $V_\lambda$, as the $V_\lambda$ defined in Sec. 1, be the open $C^*_p \text{-filter base that generates } \hat{A}$. Since $h^{-1}: h(X) \rightarrow X$ is one-one, $f^\ast \circ h$ and $h(X)$ is dense in $Z$, so $h^{-1}(\mathcal{I}(f(X))) = f^{-1}(t_\lambda - d, t_\lambda + e) \subseteq f^{-1}(t_\lambda - d, t_\lambda + e)$.

Proof. For any open set $U \subset X$, let $V_\lambda$, as the $V_\lambda$ defined in Sec. 1, be the open $C^*_p \text{-filter base that generates } \hat{A}$. Since $h^{-1}: h(X) \rightarrow X$ is one-one, $f^\ast \circ h$ and $h(X)$ is dense in $Z$, so $h^{-1}(\mathcal{I}(f(X))) = f^{-1}(t_\lambda - d, t_\lambda + e) \subseteq f^{-1}(t_\lambda - d, t_\lambda + e)$.

For any open $C^*_p \text{-filter } \hat{E}_* \text{ on } X^W$, let $V_\lambda^* = \{ t_\lambda \mid \mathcal{I}(t_\lambda - d, t_\lambda + e) \}$.

The topology on $X^W$ is the weak topology induced by $D^\ast$.

For each $\hat{A} \in X^W$, let $V_\lambda$, as the $V_\lambda$ defined in Sec. 1, be the open $C^*_p \text{-filter base that generates } \hat{A}$, and let $U \subset B$ such that $\hat{A} \in U^*$, then $U \in \hat{A}$. So there exist a $H \in \{ \mathcal{I} \}^{>0}$, an $e > 0$ such that $\mathcal{I}(f(X)) \subseteq f^{-1}(t_\lambda - d, t_\lambda + e) \subseteq f^{-1}(t_\lambda - d, t_\lambda + e)$.

For any open $C^*_p \text{-filter } \hat{E}_* \text{ on } X^W$, let $V_\lambda^* = \{ t_\lambda \mid \mathcal{I}(t_\lambda - d, t_\lambda + e) \}$.

The topology on $X^W$ is the weak topology induced by $D^\ast$.

For each $\hat{A} \in X^W$, let $V_\lambda$, as the $V_\lambda$ defined in Sec. 1, be the open $C^*_p \text{-filter base that generates } \hat{A}$, and let $U \subset B$ such that $\hat{A} \in U^*$, then $U \in \hat{A}$. So there exist a $H \in \{ \mathcal{I} \}^{>0}$, an $e > 0$ such that $\mathcal{I}(f(X)) \subseteq f^{-1}(t_\lambda - d, t_\lambda + e) \subseteq f^{-1}(t_\lambda - d, t_\lambda + e)$.

For any open $C^*_p \text{-filter } \hat{E}_* \text{ on } X^W$, let $V_\lambda^* = \{ t_\lambda \mid \mathcal{I}(t_\lambda - d, t_\lambda + e) \}$.

The topology on $X^W$ is the weak topology induced by $D^\ast$.
Since \( f^* = z \). Hence, \( \mathcal{H} \) is a well-defined open topology induced by \( f^* \) iff (d): \( \{ \cap_{c \in H} \mathcal{F}^{-1}(\cap_{s \in H} f^{-1}(f(z) - e, f(z) + e)) \neq \emptyset \} \) is a well-defined open \( C^*_0 \)-filter base on \( X \). Let \( A_x \) be the basic open \( C^*_0 \)-filter on \( X \) generated by \( V_x \). If \( z_0 \) is the \( w \)-point in \( Z \) induced by \( A_x \). Then \( \mathcal{F}(z_0) = \mathcal{F}(z) = f^*(A_x) \) for all \( f \in \mathcal{D} \) and \( f^* \in \mathcal{D}^* \). This implies that \( z = z_0 \) in \( Z \). So, for any \( z \in Z \), there is a unique \( A_x \) in \( X^W \) such that \( \mathcal{H}(A_x) = z \). Hence, \( \mathcal{H} \) is well-defined, one-one and onto.

**Theorem 5.27** \((X^W, k)\) is homeomorphic to \((Z, h)\) under the mapping \( \mathcal{H} \) such that \( \mathcal{H}(k(x)) = h(x) \).

**Proof.** Since the topologies on \( Z \) and \( X^W \) are the weak topologies induced by \( \mathcal{D} \) and \( \mathcal{D}^* \), respectively, to show the continuity of \( \mathcal{H} \), it is enough to show that for any \( f \in \mathcal{D} \) (or \( f^* \in \mathcal{D}^* \)), any \( \varepsilon > 0 \), \( \mathcal{H}^{-1}(f^{-1}((t_r - \varepsilon, t_r + \varepsilon))) = f\mathcal{F}^{-1}((t_r - \varepsilon, t_r + \varepsilon)) \neq \emptyset \) for any \( H \in \mathcal{D}^{-\varepsilon} \). For each \( A_x \) in \( X^W \), let \( V_x = \{ \cap_{s \in H} \mathcal{F}^{-1}((s_t - \varepsilon, s_t + \varepsilon)) \cap_{c \in H} \mathcal{F}^{-1}((s_t - \varepsilon, s_t + \varepsilon)) \neq \emptyset \} \) be the open \( C^*_0 \)-filter base on \( X \) that generates \( A_x \). Let \( z_0 \) be the \( w \)-point in \( Z \) induced by \( A_x \), then \( \mathcal{F}(z_0) = \mathcal{F}(z) = f^*(A_x) \). Thus (a): \( [A_x \in \mathcal{F}^{-1}((t_r - \varepsilon, t_r + \varepsilon))] \iff (b): [\mathcal{F}(z_0) = \mathcal{F}(z) = \mathcal{F}(A_x) = s_t \in (t_r - \varepsilon, t_r + \varepsilon)] \). Since \( \mathcal{H}(A_x) = z_0 \), so (b) iff (c): \( [\mathcal{H}(A_x) = z_0 \in \mathcal{F}^{-1}((t_r - \varepsilon, t_r + \varepsilon))] \) and (c) iff (d): \( [A_x \in \mathcal{H}^{-1}(\mathcal{F}^{-1}((t_r - \varepsilon, t_r + \varepsilon)))) \]. So, \( \mathcal{H} \) is continuous. Since \( \mathcal{H} \) is one-one, onto and \( Z, X^W \) are compact Hausdorff, by Theorem 17.14 in the Ref. [1, p. 123], \( \mathcal{H} \) is a homeomorphism. For that \( \mathcal{H}(k(x)) = h(x) \) is obvious from the definitions of \( k \) and \( h \). □

**Corollary 5.28** Let \((\beta X, h)\) be the Stone-Čech compactification of a Tychonoff space \( X \), \( D = \{ f \in C(\beta X) \} \) and \( \mathcal{H}_p: X^W \to \beta X \) is defined similarly to \( \mathcal{H} \) as above. Then \((\beta X, h)\) is homeomorphic to \((X^W, k)\) such that \( \mathcal{H}_p(k(x)) = h(x) \).

**Corollary 5.29** Let \((\gamma X, h)\) be the Wallman compactification of a normal \( T_1 \)-space \( X \), \( \mathcal{D} = \{ f \in C(\gamma X) \} \) and \( \mathcal{H}_p: X^W \to \gamma X \) is defined similarly to \( \mathcal{H} \) as above. Then \((\gamma X, h)\) is homeomorphic to \((X^W, k)\) such that \( \mathcal{H}_p(k(x)) = h(x) \).

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