Minimal Surfaces and Gauss Curvature of Conoid in Finsler Spaces with \((\alpha, \beta)\)-Metrics*

Dinghe Xie, Qun He
Department of Mathematics, Tongji University, Shanghai, China
Email: x_dinghe8707@126.com

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ABSTRACT
In this paper, minimal submanifolds in Finsler spaces with \((\alpha, \beta)\)-metrics are studied. Especially, helicoids are also minimal in \((\alpha, \beta)\)-Minkowski spaces. Then the minimal surfaces of conoid in Finsler spaces with \((\alpha, \beta)\)-metrics are given. Last, the Gauss curvature of the conoid in the 3-dimension Randers-Minkowski space is studied.

Keywords: Isometrical Immersion; Minimal Submanifold; \((\alpha, \beta)\)-Metric; Conoid Surface; Gauss Curvature

1. Introduction
In recent decades, geometry of submanifolds in Finsler geometry has been rapidly developed. By using the Busemann-Hausdorff volume form, Z. Shen [1] introduced the notions of mean curvature and normal curvature for Finsler submanifolds. Being based on it, Bernstein type theorem of minimal rotated surfaces in Randers-Minkowski space was considered in [2]. Later, Q. He and Y. B. Shen used another important volume form, \(i.e.,\) Holmes-Thompson volume form, to introduce notions of another mean curvature and the second fundamental form [3]. Thus, Q. He and Y. B. Shen constructed the corresponding Bernstein type theorem in a general Minkowski space [4].

The theory of minimal surfaces in Euclidean space has developed into a rich branch of differential geometry. A lot of minimal surfaces have been found in Euclidean space. Minkowski space is an analogue of Euclidean space in Finsler geometry. A natural problem is to study minimal surfaces with Busemann-Hausdorff or Holmes-Thompson volume forms. M. Souza and K. Tenenblat first studied the minimal surfaces of rotation in Randers-Minkowski spaces, and used an ODE to characterize the BH-minimal rotated surfaces in [5]. Later, the nontrivial HT-minimal rotated hypersurfaces in quadratic \((\alpha, \beta)\)-Minkowski space are studied [6]. N. Cui and Y. B. Shen used another method to give minimal rotational hypersurface in quadratic Minkowski \((\alpha, \beta)\)-space [7]. However, these examples only consider the special \((\alpha, \beta)\)-metrics either Randers or quadratic. Therefore, what is the case with the general \((\alpha, \beta)\)-metric?

The main purpose of this paper is to study the conoid in \((\alpha, \beta)\)-space. It includes minimal submanifolds in Finsler spaces with general \((\alpha, \beta)\)-metric \((F = \alpha \phi \left( \frac{\beta}{\alpha} \right))\) and the Gauss curvature in Randers-Minkowski 3-space. We present the equations that characterize the minimal hypersurfaces in general \((\alpha, \beta)\)-Minkowski space. We prove that the conoid in Minkowski 3-space with metric \(F = \alpha \phi \left( \frac{\beta}{\alpha} \right)\) is minimal if and only if it is a helicoid or a plane under some conditions. Finally, similar to [7], we give the Gauss curvature of conoid in Randers-Minkowski 3-space and point out that the Gauss curvature is not always nonpositive on minimal surfaces.

2. Preliminaries
Let \(M\) be an \(n\)-dimensional smooth manifold. A Finsler metric on \(M\) is a function \(F: TM \to [0, \infty)\) satisfying the following properties: 1) \(F\) is smooth on \(TM \setminus \{0\}\); 2) \(F(x, \lambda y) = \lambda F(x, y)\) for all \(\lambda > 0\); 3) The induced quadratic form \(g\) is positively definite, where

\[
g := g(y)(x, y)dx^i \otimes dx^j,
\]
\[
g_{ij} := \frac{1}{2} \left( \frac{\partial^2 F}{\partial y^i \partial y^j} \right).
\]

Here and from now on, \(\left[ F \right]_i^j\), \(\left[ F \right]_{ij}\), \(\frac{\partial F}{\partial y^i}\), \(\frac{\partial^2 F}{\partial y^i \partial y^j}\), and we shall use the following convention of index ranges unless otherwise stated:

\(1 \leq i, j, \ldots \leq n; \quad 1 \leq \alpha, \beta, \ldots \leq m (> n).\)

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The projection $\pi: TM \to M$ gives rise to the pull-back bundle $\pi^* TM$ and its dual $\pi^* T^* M$, which sits over $TM \setminus \{0\}$. We shall work on $TM \setminus \{0\}$ and rigidly use only objects that are invariant under positive rescaling in $v$, so that one may view them as objects on the projective sphere bundle $SM$ using homogeneous coordinates.

In $\pi^* T^* M$ there is a global section $\omega = [F]$, $dx'$, called the Hilbert form, whose dual is $l = l' = \frac{\partial}{\partial x'}$.

$l' = y' / F$, called the distinguished field. The volume element $dV_{SM}$ of $SM$ with respect to the Riemannian metric $\hat{g}$, the pull-back of the Sasaki metric on $TM \setminus \{0\}$, can be expressed as

$$dV_{SM} = \Omega d\tau \wedge dx,$$

where

$$\Omega := \det \left( \frac{g_{ij}}{F} \right), \quad dx = dx^1 \wedge \cdots \wedge dx^n, \quad \Omega = \frac{1}{c_{n+1}} \int_{S_n \mathcal{M}} \Omega d\tau,$$

and

$$d\tau := \sum_{i=1}^{n} (-1)^{i-1} y^i dy^1 \wedge \cdots \wedge dy^{i-1} \wedge dy^{i+1} \wedge \cdots \wedge dy^n.$$

The volume form of a Finsler $n$-manifold $(M, F)$ is defined by

$$dV_{M} := \sigma(x) dx, \quad \sigma(x) := \frac{1}{c_{n+1}} \int_{S_n \mathcal{M}} \Omega d\tau,$$

where $c_{n+1}$ denotes the volume of the unit Euclidean $(n - 1)$-sphere $S^{n-1}$, $S_n \mathcal{M} = \{y||y||_T \in T_x \mathcal{M}\}$.

Let $(M, F)$ and $(\tilde{M}, \tilde{F})$ be Finsler manifolds, and $f : M \to \tilde{M}$ be an immersion. If $F(x, y) = \tilde{F}(f(x), df(y))$ for all $(x, y) \in TM \setminus \{0\}$, then $f$ is called an isometric immersion. It is clear that

$$g_{xy}(x, y) = \hat{g}_{\alpha \beta}(\tilde{x}, \tilde{y}) f_\alpha f_\beta,$$

for the isometric immersion $f : (M, F) \to (\tilde{M}, \tilde{F})$, where $\tilde{x}^\alpha = f^\alpha(x)$, $\tilde{y}^\alpha = f'_\alpha(y)$, $f_\alpha = \partial f^\alpha / \partial x^\alpha$. Let $\left(\pi^* TM\right)^\perp$ be the orthogonal complement of $\pi^* TM$ in $\pi^* (f^{-1}\tilde{M})$ with respect to $\hat{g}$, and set

$$h^\alpha = f_\alpha^\gamma y^\gamma - f_\gamma f_\alpha G^\gamma + \tilde{G}^\alpha,$$

$$h_\alpha = \hat{g}_{\alpha \beta} h^\beta, \quad h = h^\alpha \frac{\partial}{\partial x^\alpha},$$

where $f_\alpha^\gamma = \partial f^\alpha / \partial x^\gamma$, $G^\gamma$ and $\tilde{G}^\alpha$ are the geodesic coefficients of $F$ and $\tilde{F}$ respectively. We can see that $h \in \left(\pi^* TM\right)^\perp$ (see (1.14) in [3]), which is called the normal curvature. Recall that for an isometric immersion $f : (M, F) \to (\tilde{M}, \tilde{F})$, we have (see formulae (2.14) and (3.14) of Chapter V in [8])

$$G^\alpha = \phi^\alpha_{\nu} \left( f^\nu_{\mu} y^\nu y^\mu + \tilde{G}^\mu \right),$$

where $\phi^\alpha_{\nu} = f^\alpha_{\nu} \hat{g}_{\nu \beta} \tilde{g}_{\alpha \beta}$. From (2.7), it follows that

$$h^\alpha = p^\alpha_{\nu} \left( f^\nu_{\mu} y^\nu y^\mu + \tilde{G}^\mu \right),$$

where $p^\alpha_{\nu} := \delta^\alpha_{\nu} - f^\alpha_{\nu}\phi_{\nu}$. Set

$$\mu = \frac{1}{c_{n+1}} \left( \int_{S_n \mathcal{M}} \frac{h_n}{F^2} \Omega d\tau \right) d\hat{\alpha},$$

which is called the mean curvature form of $f$. An isometric immersion $f : (M, F) \to (\tilde{M}, \tilde{F})$ is called a minimal immersion if any compact domain of $M$ is the critical point of its volume functional with respect to any variation vector field. Then $f$ is minimal if and only if $\mu = 0$.

3. Minimal Hypersurfaces of $(\alpha, \beta)$-Spaces

Here and from now on, we consider general $(\alpha, \beta)$-metric.

Let $F = \alpha \phi(s)$, $s = \beta$, where $\phi(s)$ is a positive $C^\infty$ function on $(-b_n, b_n)$,

$$\alpha = \sqrt{a_0(y) y^\nu y^\nu}, \quad \beta = b_1(x) y^\nu,$$

$$\|\beta\|_b = \sqrt{a_0 b_0} = b(0 < b < b_0).$$

If $\phi(s) = 1 + s$, then $F$ is a Randers metric. If $\alpha$ is an Euclidean metric and $\beta$ is parallel with respect to $\alpha$, $F$ is a locally Minkowski metric and $(M, F)$ is called an $(\alpha, \beta)$-Minkowski metric. By [9], $F$ is a Finsler metric if and only if $\phi(s)$ satisfies

$$\phi(s) - s \phi'(s) + (b^2 - s^2) \phi^2(s) > 0, \quad |s| \leq b < b_0.$$

Let

$$A = \det (a_\nu), \quad g = \det (g_\nu), \quad \Omega = \frac{g}{F^2}. \quad \Omega = \frac{g}{F^2}.$$

It has been proved ([9]) that

$$g = \phi(s)^n H(s) A,$$

where

$$H(s) = \phi(s - s \phi')^{-n/2} \left[ \phi(s - s \phi') + (b^2 - s^2) \phi^2(s) \right].$$

In the following part, we will discuss minimal hypersurfaces in Minkowski space with $(\alpha, \beta)$-metric. Let $f : (M, F) \to (\tilde{M}, \tilde{F})$ be an isometric immersion,

$$\tilde{F} = \tilde{\alpha} \phi\left( \tilde{\beta} \right),$$

where...
\[
\tilde{\alpha} = \sqrt{\tilde{a}_{ij} y^i y^j}, \quad \tilde{\beta} = \tilde{b}_i y^i.
\]

Since \( f \) is an isometric immersion, we get
\[
F = f^* \tilde{F} = \alpha \phi \left( \frac{\beta}{\alpha} \right),
\]
where
\[
\alpha = \sqrt{a_{ij} y^i y^j}, \quad a_{ij} = \tilde{a}_{ij} f^* f^j,
\]
\[
\beta = b_i y^i, \quad b_i = \tilde{b}_i f_i^*.
\]

Note that \( (M, F) \) is a hypersurface of \( (\tilde{M}, \tilde{F}) \), let \( n = n^a \tilde{e}_a \) be the unit normal vector field of \( f(M) \) with respect to \( \tilde{\alpha} \) and \( n = \tilde{n}^a \tilde{e}_a \) be the unit normal vector field of \( M \) with respect to \( \tilde{g} \), respectively. That is
\[
\sum a^a f^* = 0, \quad \tilde{g} = \tilde{\alpha} n^a f^* = 0,
\]
\[
\tilde{\alpha}(n,n) = \tilde{\alpha}_a n^a n^b = 1, \quad \tilde{g}(n,n) = \tilde{g}_a n^a n^b = 1.
\]

There exist a function \( \lambda(x, y) \) on \( SM \), such that
\[
\tilde{g}_a n^a \tilde{\alpha} = \lambda \tilde{\alpha}_a n^b,
\]
where \( \lambda = \tilde{g}(n,n) = (\tilde{a}(n,n))^{-1} \). Then
\[
\tilde{n}^a = \lambda \tilde{g}^a n^b.
\]

From above, we know that \( f \) is minimal if and only if
\[
n^a \int_{\tilde{S}} \frac{h_a}{F^2} \tilde{\alpha} \tilde{d} = 0.
\]

From (3.3) and (3.4), and in a similar way as in [5], we can get
\[
h_a = \tilde{g}_a h^a = \tilde{g}_a \left( \frac{f^* \tilde{\alpha} y^i y^j + \tilde{G}^\beta \tilde{n}^i \tilde{n}^j} {\tilde{\alpha} y^i y^j + \tilde{G}^\beta \tilde{n}^i \tilde{n}^j} \right),
\]
\[
= \lambda^2 \left[ \frac{f^i y^j + \tilde{G}^\beta \tilde{n}^i \tilde{n}^j} {\tilde{\alpha} y^i y^j + \tilde{G}^\beta \tilde{n}^i \tilde{n}^j} \right],
\]
\[
g = A \lambda^2 \tilde{\alpha} = A \lambda^2 \tilde{\alpha} \frac{\phi^{\alpha+1} \tilde{H} \tilde{A} = \frac{\phi^{\alpha+1} \tilde{H} \tilde{A}} {\lambda^2}}.
\]

Then (3.5) is equivalent to
\[
n^a a^a \int_{\tilde{S}} \phi \left( \phi - s \phi \right)^{-1} \left[ \phi - s \phi \phi + \left( \tilde{b}^2 - s^2 \right) \phi^* \right] \tilde{d} = 0.
\]

If \( \tilde{F} = \tilde{\alpha} \phi \left( \frac{\beta}{\alpha} \right) \) is an \( (\alpha, \beta) \)-Minkowski metric, then \( \tilde{G}^\beta = 0 \). In Minkowski-\( (\alpha, \beta) \) space, \( f \) is minimal if and only if
\[
f^i \left( \phi \phi - s \phi \phi + \left( \tilde{b}^2 - s^2 \right) \phi^* \right) \tilde{d} = 0.
\]

**Theorem 1** Let \( (M, F) \) be a hypersurface of \( (\tilde{M}, \tilde{F}) \), and \( \tilde{F} = \tilde{\alpha} \phi \left( \frac{\beta}{\alpha} \right) \) be an \( (\alpha, \beta) \)-Minkowski metric. Then \( f : (M, F) \to (\tilde{M}, \tilde{F}) \) is a minimal immersion if and only if
\[
f^i \left[ \phi \left( \phi \right) \tilde{d} + \phi \beta \phi \phi + \left( \tilde{b}^2 - s^2 \right) \phi^* \right] = 0,
\]
where
\[
\tilde{\alpha} = \sqrt{\tilde{y}^i + \tilde{y}^j + \tilde{y}^k}, \quad \tilde{\beta} = \tilde{b}^i \tilde{y}^i.
\]

Note that \( \tilde{\alpha} \) and \( \tilde{\beta} \) are constants. Let \( \tilde{F} = \tilde{\alpha} \phi \left( \frac{\beta}{\alpha} \right) \), where \( \tilde{h}(v) \) is an unknown function. Then
\[
\left( f^i \right)_{\alpha+3} = \left( \begin{array}{c} \cos v \\ \sin v \\ 0 \\ -u \sin v \\ u \cos v \\ \tilde{h}' \end{array} \right),
\]
\[
\left( \tilde{y}^i \tilde{y}^j \tilde{y}^k \right) = \left( \begin{array}{c} \cos v \\ \sin v \\ 0 \\ -u \sin v \\ u \cos v \\ \tilde{h}' \end{array} \right),
\]
\[
= \left( y^i \cos v - u \sin v \right) \left( y^j \sin v + u \cos v \right) \tilde{h}'^2.
\]

Assume that \( y^3 = \cos \theta, y^2 = \left( 1 - 4 \sin^2 \theta \right) \sin \theta \), \( \theta \in [0, 2\pi] \), then
\[
\tilde{\alpha} = \sqrt{\tilde{y}^i + \tilde{y}^j + \tilde{y}^k}, \quad \tilde{\beta} = \tilde{b}^i \tilde{y}^i.
\]

Note that the normal vector of the surface is
\[
= \left( \begin{array}{c} -h \sin v \\ h \cos v \\ -u \sqrt{h^2 + 2} \\ \sqrt{h^2 + 2} \\ -u \sqrt{h^2 + 2} \\ h' \end{array} \right),
\]
and
\[
\left( f^i \right)_{\alpha+3} = \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ \tilde{y}^i \tilde{y}^j \tilde{y}^k \right) = \left( \begin{array}{c} -h \sin v \\ h \cos v \\ 0 \\ -u \sin v \\ u \cos v \\ h' \end{array} \right),
\]
\[
\left( f^i \right)_{\alpha+3} = \left( \begin{array}{c} -h \sin v \\ h \cos v \\ -u \sin v \\ h' \end{array} \right).
\]

Set
\[
W^i = \int_{\tilde{S}} \phi \left( \phi \right) \tilde{d} + \phi \beta \phi \phi + \left( \tilde{b}^2 - s^2 \right) \phi^* \tilde{d} = 0,
\]
Then (3.8) is equivalent to
\[
\sum_{a=1}^{3} \left(2 f^m_{\alpha} h^m W^{12} + f^m_{\alpha} n^{m} W^{22} \right) = 0. \tag{20}
\]
Since \( S_{x} \) is symmetric with respect to \( y^{1} \) and \( \tilde{\beta} \) is a function only depending on \( y^{2} \),
\[
W^{12} = \int_{S_{x}} y^{2} \left( \phi (\tilde{\beta}) - \tilde{\beta} \phi (\tilde{\beta}) \right)
\cdot \left( \phi^{2} (\tilde{\beta}) - \left( \beta^{2} - \tilde{\beta}^{2} \right) \phi (\tilde{\beta}) \right) d \tau = 0,
\]
Therefore, (3.10) becomes to
\[
\text{uh}^{*} W^{22} = 0, \quad \forall u.
\]
However, \( W^{22} = 0 \) is impossible. Recall that
\[
W^{22} = \int_{S_{x}} y^{2} \frac{\sqrt{g}}{\phi^{2} (s)} d \tau, \quad \phi (s) > 0,
\]
and \( y^{2} \) is not identically vanishing, we can obtain \( W^{22} > 0 \).
Then \( h^{*} = 0 \),
\[
h = cv + d,
\]
where \( c, d \) are arbitrary constants.

**Theorem 2** Let \( (V, \tilde{F}) \) be an \((\alpha, \beta)\)-Minkowski space, \( \tilde{F} = \tilde{a} \phi \left( \frac{\beta}{\alpha} \right), \tilde{\beta} = \tilde{b} y^{3} \), and
\[
f = \left( u \cos v, u \sin v, h (v) \right) \text{ be a conoid. Then } f \text{ is minimal if and only if } f \text{ is a helicoid or a plane.}
\]

**Remark 3.1** From theorem 2, we can affirm that a helicoid is minimal not only in Euclidean space but also in \((\alpha, \beta)\) Minkowski space, where \( \tilde{\beta} = \tilde{b} y^{3} \). This is an interesting result for minimal surfaces.

But whether the result hold if the condition \( \tilde{\beta} = \tilde{b} y^{3} \) is not satisfied? Now we consider the following condition:
\[
\tilde{\beta} = \tilde{b}_{1} y^{1} + \tilde{b}_{2} y^{2} + \tilde{b}_{3} y^{3}
\]
\[
= \left( \tilde{b}_{1} \cos v + \tilde{b}_{2} \sin v \right) y^{1} + \left( \tilde{b}_{2} \cos v - \tilde{b}_{1} \sin v \right) y^{2},
\]
where \( \tilde{b}_{1}, \tilde{b}_{2}, \tilde{b}_{3} \) are not all zeros. To simplify the computation, we only discuss quadratic \((\alpha, \beta)\)-metric:
\[
F = \alpha + k \frac{\beta^{2}}{\alpha}. \quad \text{Set } \quad B_{i} = \tilde{b}_{i} \cos v + \tilde{b}_{2} \sin v,
\]
\[
B_{2} = \tilde{b}_{2} \cos v - \tilde{b}_{1} \sin v. \quad \text{Then (3.8) becomes an equation respect to } u:
\]
\[
C_{5} (v) u^{3} + C_{4} (v) u^{2} + C_{3} (v) u + C_{2} (v) u + C_{0} (v) = 0,
\]
where
\[
C_{5} = \frac{15}{8} B_{2}^{2} h^{*},
\]
\[
C_{4} = \frac{15}{2} B_{2} \tilde{b}_{1} \left( \tilde{b}_{2} \left( h^{*} \right)^{2} + B_{2}^{2} \right) h^{*},
\]
\[
C_{3} = \frac{15}{8} B_{2}^{3} h^{*}
\]
\[
C_{2} = \frac{15}{2} B_{2} \tilde{b}_{1} \left( \tilde{b}_{2} \left( h^{*} \right)^{2} + B_{2}^{2} \right) h^{*},
\]
\[
C_{1} = \frac{15}{8} B_{2}^{3} h^{*}
\]
\[
C_{0} = \frac{15}{2} B_{2} \tilde{b}_{1} \left( \tilde{b}_{2} \left( h^{*} \right)^{2} + B_{2}^{2} \right) h^{*},
\]
Since (3.11) is valid for any \( u \), we can obtain
\[
C_{i} = 0 (i = 0, \ldots, 5), \quad \forall v.
\]
If \( \tilde{b}_{1} \neq 0 \) or \( \tilde{b}_{2} \neq 0 \), then \( B_{1} \neq 0 \) or \( B_{2} \neq 0 \), such that \( h^{*} (v) = 0 \). Therefore, when \( \tilde{b}_{1}, \tilde{b}_{2} \) are not all zeros, \( h (v) = \text{const}. \), That is to say a minimal conoid hypersurface is a plane with respect to the given metric above.

**Theorem 3** Let \( (V, \tilde{F}) \) be an \((\alpha, \beta)\)-Minkowski space, where \( \tilde{F} = \tilde{a} + k \frac{\beta^{2}}{\alpha} \), \( \tilde{\beta} = \tilde{b} \) satisfying
\[
\beta = \beta_{1} y^{1} + \beta_{2} y^{2} + \beta_{3} y^{3} \quad (\beta_{1}, \beta_{2}, \beta_{3} \text{ are not all zeros}). \quad \text{Then a minimal conoid hypersurface in } (V, \tilde{F}) \text{ is a plane.}
\]

**4. Gauss Curvature of Conoid in Randers 3-Space**

As we all known, the Gauss curvature of a minimal surface is nonpositive everywhere in Euclidean space. Then, a natural problem arises: whether this fact holds for minimal surfaces in Minkowski-Randers 3-space? In this section, we study the Gauss curvature of conoid in Minkowski-Randers 3-space around \( x^{1} \)-axis in the direction \( \beta^{*} \), that is \( \beta^{*} = \tilde{b} y^{3} \). Consider the conoid
\[
f (u, v) = \left( u \cos v, u \sin v, h (v) \right), \quad \text{where } u > 0 \text{ and } v \in S^{1}.
\]
Let \( e_{1} = \frac{\partial}{\partial u} \), \( e_{2} = \frac{\partial}{\partial v} \). Then \( y = \xi e_{1} + \eta e_{2} \) gives a natural coordinates \((u, v, \xi, \eta)\) on its tangent bundle. In this section we shall use the convention that \( 1 \leq i, j \leq 2 \) and \( 1 \leq \alpha, \beta \leq 3 \). Besides, the notations \( \xi := u, \xi := v \) and \( y^{1} := \xi, y^{2} := \eta \) are also used.

Note that the induced 1-form \( \beta = f \tilde{\beta} \) on the surface is closed. Then the Ricci curvature tensor of \( F = f \tilde{F} \) is given by ([10], Page 118)
\[
Ric = Ric + \frac{1}{4F^2} \left( 3r_0^2 - 2Fr_{000} \right),
\]
where \(Ric\) denotes the Ricci curvature tensor of the induced Riemannian metric \(\alpha = f^* \tilde{\alpha}\), \(r_{00} = b_{0j}y^j\) and \(b_{ij}\) denote the coefficients of the covariant derivatives of \(\beta\) with respect to \(\alpha\). Then the Gauss curvature of the surface is given by
\[
K(x, y) = \frac{Ric(y)}{F^2} = \frac{1}{K + \frac{1}{4F^2} \left( 3r_0^2 - 2Fr_{000} \right)},
\]
where \(x = f(u, v)\), \(K\) denotes the Gauss curvature with respect to \(\alpha\).

Denote \(z_i^a = \frac{\partial \alpha^a}{\partial \zeta^i}\) and \(z_i^a = \frac{\partial^2 \alpha^a}{\partial \zeta^i \partial \zeta^f}\). Then
\[
\begin{align*}
(z_i^a)_{2\times 3} &= \begin{pmatrix}
\cos v & \sin v & 0 \\
-u \sin v & u \cos v & h'(v)
\end{pmatrix}, \\
(z_i^a)_{1\times 3} &= \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \\
(z_i^a)_{1\times 3} &= \begin{pmatrix} 0 & 0 & 0 \end{pmatrix},
\end{align*}
\]
\[
(z_i^a)_{1\times 3} = \begin{pmatrix} -\sin v & \cos v & 0 \end{pmatrix},
\]
\[
(z_i^a)_{1\times 3} = \begin{pmatrix} -u \cos v & -u \sin v & h^* \end{pmatrix}.
\]
Noting that the Gauss curvature is computed in Euclidean space as follows:
\[
\tilde{K} = \frac{LN - M^2}{EG - F^2},
\]
where
\[
L = z_{11}^a \cdot n^a, \quad M = z_{12}^a \cdot n^a, \quad N = z_{22}^a \cdot n^a,
\]
\[
E = z_{11}^a \cdot z_{11}^a, \quad F = z_{12}^a \cdot z_{12}^a, \quad G = z_{22}^a \cdot z_{22}^a.
\]
By direct computation, we can obtain
\[
\tilde{K} = \frac{-h^2}{(u^2 + h^2)^2}.
\]
Meanwhile, the coefficients of \(\alpha = f^* \tilde{\alpha}\) are given by
\[
(a_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & u^2 + (h^*)^2 \end{pmatrix},
\]
\[
(a^{ij}) = (a_{ij})^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & u^2 + (h^*)^2 \end{pmatrix},
\]
where \(a_{ij} = z_i^a z_j^a \delta_{ab}\). It is easy to verify that
\[
\nabla_i^a = \Gamma_{ij}^{a} z_j^a \delta_{ab} \partial_{ab}.
\]
By a direct computation, we have
\[
\Gamma_{ij}^{a} = \begin{pmatrix} 0 & 0 \\ 0 & -u \end{pmatrix},
\]
\[
\Gamma^{a}_{ij} = \begin{pmatrix} 0 & u \\ u & u^2 + (h^*)^2 \\ -h^* & u^2 + (h^*)^2 \\ -u^2 + h^2 & u^2 + h^2 \end{pmatrix}.
\]
Since \(b_i = \tilde{b}_i^a\),
\[
b_{ij} = -b_{i,j}^{a} \Gamma_{ij}^{a} = \tilde{b}_i^a \begin{pmatrix} 0 & -h^* u \\ -u & -h^* \end{pmatrix} \begin{pmatrix} u^2 + h^2 \end{pmatrix}.
\]
From \(b_{ij} = b_{ij}^{a} \Gamma_{ij}^{a} \), \(b_{00} = b_{0j}^{a} \Gamma_{0j}^{a} \), we have
\[
\begin{align*}
b_{233} &= b_{233} = \frac{-\tilde{b}_u h^*}{(u^2 + h^2)^2}, \\
b_{232} &= b_{232} = \frac{-\tilde{b}_h u^2 h^*}{(u^2 + h^2)^2}, \\
b_{233} &= b_{233} = \frac{-\tilde{b}_h u^2 h^*}{(u^2 + h^2)^2}.
\end{align*}
\]
Besides,
\[
r_{00} = b_{0j} y^j = \frac{-\tilde{b}_h u^2 h^*}{u^2 + h^2} \frac{y^2}{(y^2)^2},
\]
\[
r_{00} = b_{0j} y^j y^k = \frac{-\tilde{b}_h u^2 h^*}{u^2 + h^2} \frac{(y^2)^2}{(y^2)^2} + \frac{\tilde{b}_h y^2}{(y^2)^2} \frac{(y^2)^2}{(y^2)^2} + \frac{-\tilde{b}_h u^2 h^*}{u^2 + h^2} \frac{(y^2)^3}{(y^2)^3}.
\]
Then, from (4.2) and (4.3), we obtain the following theorem.
\[\textbf{Theorem 4}
\]
Let \(\{V^3, \tilde{F}\}\) be an Randers-Minkowski space with \(\tilde{\beta} = \tilde{b}^a \), the Gauss curvature of the conoid \(f(u, v) = \big(u \cos v, u \sin v, h(v)\big)\) at \(x = f(u, v)\) in direction of \(y = \xi e_1 + \eta e_2\) is given by
\[
K(x, y) = -\Gamma^{a} + \frac{1}{4F^2} \left[ 12 \tilde{b} \Gamma^{a} \xi^2 \eta - 8 \tilde{b} F \Gamma^{a} \xi^2 \eta + 4 \tilde{b} F \Gamma^{a} \xi^2 \eta - 3 \Gamma^{a} \eta^4 \right],
\]
where
\[ \Gamma^0 = \frac{h'}{u^2 + h'^2}, \quad \Gamma^1 = \frac{u'^2}{u^2 + h'^2}, \quad \Gamma^2 = \frac{h''}{u^2 + h'^2}, \]
\[ \Gamma^3 = \frac{u'^3h'}{u^2 + h'^2}, \quad \Gamma^4 = \frac{uh''}{u^2 + h'^2}(h'^2 - 2u'), \]
\[ \Gamma^5 = \frac{u'^h}{u^2 + h^2}(\frac{1}{u^2 + h'^2} + h^2), \quad \Gamma^6 = \frac{u'^h}{u^2 + h'^2}. \]

Note that a helicoid is minimal if and only if it is a conoid with respect to \((\alpha, \beta)\)-metrics (where \(\beta = \beta y^3\)). Let \( h(v) = cv + d \) \((c \text{ is a constant})\), then the Gauss curvature of this surface is given by

\[ K(x, y) = -\Pi^0 + \frac{1}{F^3}[3\beta^2\Pi^1\xi^2 \eta^2 - \beta F\Pi^2\eta(2\xi^2 + \eta^2)], \]

(25)

where

\[ \Pi^0 = \frac{c^2}{u^2 + c^2}, \quad \Pi^1 = \frac{c^2u'^2}{(u^2 + c^2)^2}, \quad \Pi^2 = \frac{cu^2}{(u^2 + c^2)^2}. \]

However, for a given point \( x = f(u, v) \), in which directions of \( T, S \), \( K(x, y) > 0 \), \( K(x, y) = 0 \), \( K(x, y) < 0 \)?

1) If \( \eta = 0 \), then \( K(x, y) = 0 \) for any \( c \neq 0 \);

2) If \( \xi < 0 \), since

\[ F = \alpha + \beta = \sqrt{(u'^2 + c^2)\eta + \beta\eta}, \]

Equation (4.4) becomes

\[ K(x, y) = -\frac{1}{(u^2 + c^2)^2} \left[ c^2 + \frac{2cbu^2}{\sqrt{u^2 + c^2} \eta + \beta\eta} \right]^2. \]

If \( c > 0 \), let \( \eta < 0 \), then

\[ K(x, y) = -\frac{1}{(u^2 + c^2)^2} \left[ c^2 + \frac{2b^2u^2}{(cb - \sqrt{u^2 + c^2})^3} \right]. \]

we can also make \( c^2 + \frac{2cbu^2}{(cb - \sqrt{u^2 + c^2})^3} < 0 \), then

\[ K(x, y) < 0 \); Otherwise, let \( \eta > 0 \), then

\[ K(x, y) > 0. \]

In sum, the Gauss curvature is not nonpositive anywhere.

REFERENCES


