Multidimensional Laplace Transforms over Quaternions, Octonions and Cayley-Dickson Algebras, Their Applications to PDE

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ABSTRACT
Multidimensional noncommutative Laplace transforms over octonions are studied. Theorems about direct and inverse transforms and other properties of the Laplace transforms over the Cayley-Dickson algebras are proved. Applications to partial differential equations including that of elliptic, parabolic and hyperbolic type are investigated. Moreover, partial differential equations of higher order with real and complex coefficients and with variable coefficients with or without boundary conditions are considered.

Keywords: Laplace Transform; Quaternion Skew Field; Octonion Algebra; Cayley-Dickson Algebra; Partial Differential Equation; Non-Commutative Integration

1. Introduction
The Laplace transform over the complex field is already classical and plays very important role in mathematics including complex analysis and differential equations [1-3]. The classical Laplace transform is used frequently for ordinary differential equations and also for partial differential equations sufficiently simple to be resolved, for example, of two variables. But it meets substantial difficulties or does not work for general partial differential equations even with constant coefficients especially for that of hyperbolic type.

To overcome these drawbacks of the classical Laplace transform in the present paper more general noncommutative multiparameter transforms over Cayley-Dickson algebras are investigated. In the preceding paper a noncommutative analog of the classical Laplace transform over the Cayley-Dickson algebras was defined and investigated [4]. This paper is devoted to its generalizations for several real parameters and also variables in the Cayley-Dickson algebras. For this the preceding results of the author on holomorphic, that is (super) differentiable functions, and meromorphic functions of the Cayley-Dickson numbers are used [5,6]. The super-differentiability of functions of Cayley-Dickson variables is stronger than the Fréchet's differentiability. In those works also a noncommutative line integration was investigated.

We remind that quaternions and operations over them had been first defined and investigated by W. R. Hamilton in 1843 [7]. Several years later on Cayley and Dickson had introduced generalizations of quaternions known now as the Cayley-Dickson algebras [8-11]. These algebras, especially quaternions and octonions, have found applications in physics. They were used by Maxwell, Yang and Mills while derivation of their equations, which they then have rewritten in the real form because of the insufficient development of mathematical analysis over such algebras in their time [12-14]. This is important, because noncommutative gauge fields are widely used in theoretical physics [15].

Each Cayley-Dickson algebra \( A_r \) over the real field \( R \) has \( 2^r \) generators \( \{i_0, i_1, \ldots, i_{2^r-1}\} \) such that \( i_0 = 1 \), \( i_j^2 = -1 \) for each \( j = 1, 2, \ldots, 2^r - 1 \), \( i_k i_j = -i_j i_k \) for every \( 1 \leq k < j \leq 2^r - 1 \), where \( r \geq 1 \). The algebra \( A_{r+1} \) is formed from the preceding algebra \( A_r \) with the help of the so-called doubling procedure by generator \( i_r \). In particular, \( A_1 = C \) coincides with the field of complex numbers, \( A_2 = H \) is the skew field of quaternions, \( A_3 \) is the algebra of octonions, \( A_4 \) is the algebra of sedenions. This means that a sequence of embeddings \( \cdots \hookrightarrow A_r \hookrightarrow A_{r+1} \hookrightarrow \cdots \) exists.

Generators of the Cayley-Dickson algebras have a natural physical meaning as generating operators of fermions. The skew field of quaternions is associative, and the algebra of octonions is alternative. The Cayley-Dickson algebra \( A_r \) is power associative, that is, \( z^{n+m} = z^n z^m \) for each \( n, m \in N \) and \( z \in A_r \). It is non-associative and non-alternative for each \( r \geq 4 \). A
conjugation \( z^* = \bar{z} \) of Cayley-Dickson numbers \( z \in A_r \) is associated with the norm \( |z|^2 = zz^* = z^*z \).

The octonion algebra has the multiplicative norm and is the division algebra. Cayley-Dickson algebras \( A_r \) with \( r \geq 4 \) are not division algebras and have not multiplicative norms. The conjugate of any Cayley-Dickson number \( z \) is given by the formula:

\[
(M1) \quad z^* := \xi^* - \eta^l.
\]

The multiplication in \( A_{r+1} \) is defined by the following equation:

\[
(M2) \quad (\xi + \eta l)(\gamma + \delta l) = (\xi \gamma - \delta \eta) + (\delta \xi + \eta \gamma)l
\]

for each \( \xi, \eta, \gamma, \delta \in A_r, z := \xi + \eta l \in A_{r+1}, \xi^* = \gamma + \delta l \in A_{r+1} \).

At the beginning of this article a multiparameter noncommutative transform is defined. Then new types of the direct and inverse noncommutative multiparameter transforms over the general Cayley-Dickson algebras are investigated, particularly, also over the quaternion skew field and the algebra of octonions. The transforms are considered in \( A_r \) spherical and \( A_r \) Cartesian coordinates. At the same time specific features of the noncommutative multiparameter transforms are elucidated, for example, related with the fact that in the Cayley-Dickson algebra \( A_r \) there are \( 2^r - 1 \) imaginary generators \( \{i_1, \ldots, i_{2^r-1}\} \) apart from one in the field of complex numbers such that the imaginary space in \( A_r \) has the dimension \( 2^r - 1 \). Theorems about properties of images and originals in conjunction with the operations of linear combinations, differentiation, integration, shift and homothety are proved. An extension of the noncommutative multiparameter transforms for generalized functions is given. Formulas for noncommutative transforms of products and convolutions of functions are deduced.

Thus this solves the problem of non-commutative mathematical analysis to develop the multiparameter Laplace transform over the Cayley-Dickson algebras. Moreover, an application of the noncommutative integral transforms for solutions of partial differential equations is described. It can serve as an effective means (tool) to solve partial differential equations with real or complex coefficients with or without boundary conditions and their systems of different types (see also [16]). An algorithm is described which permits to write fundamental solutions and functions of Green’s type. A moving boundary problem and partial differential equations with discontinuous coefficients are also studied with the use of the noncommutative transform.

Frequently, references within the same subsection are given without number of the subsection, apart from references where subsection are different.

All results of this paper are obtained for the first time.

2. Multidimensional Noncommutative Integral Transforms

2.1. Definitions Transforms in \( A_r \), Cartesian Coordinates

Denote by \( A_r \) the Cayley-Dickson algebra, \( 0 \leq r \), which may be, in particular, \( H = A_3 \) the quaternion skew field or \( O = A_8 \) the octonion algebra. For unification of the notation we put \( A_0 = R, \quad A_1 = C \). A function \( f : R^r \rightarrow A_r \) we call a function-original, where \( 2 \leq r, n \in N \), if it fulfills the following conditions (1-5).

1) The function \( f(t) \) is almost everywhere continuous on \( R^n \) relative to the Lebesgue measure \( \lambda_n \) on \( R^n \).

2) On each finite interval in \( R \) each function \( g_j(t_j) = f(t_1, \ldots, t_n) \) by \( t_j \) with marked all other variables may have only a finite number of points of discontinuity of the first kind, where \( t_j = (t_{j1}, \ldots, t_{jn}) \in R^n, \quad t_j \in R, \quad j = 1, \ldots, n \). Recall that a point \( u_0 \in R \) is called a point of discontinuity of the first type, if there exist finite left and right limits

\[
\lim_{u_0 \to u_0^+} g(u) := g(u_0^0 - 0) \in A_r \quad \text{and}
\lim_{u_0 \to u_0^-} g(u) := g(u_0^0 + 0) \in A_r.
\]

3) Every partial function \( g_j(t_j) = f(t_1, \ldots, t_n) \) satisfies the Hölder condition:

\[
|g_j(t_j + h_j) - g_j(t_j)| \leq h_j^{\gamma_j} |h_j|^{\delta_j}
\]

for each \( h_j, \gamma_j, \delta_j \in A_r, h_j = h_j^l \in A_{r+1}, \gamma_j + \delta_j l \in A_{r+1} \).

4) The function \( f(t) \) increases not faster than the exponential function, that is there exist constants \( C_r = \text{const} > 0, \quad v_j = (v_{j1}, \ldots, v_{jn}), \quad a_{j1}, a_{jn} \in R, \quad a_j = a_j^l \in A_r \) for every \( j = 1, \ldots, n \), such that

\[
|f(t)| \leq C_r \exp((v_j, t))
\]

for each \( t \in R^n \) with \( t_j v_j \geq 0 \) for each \( j = 1, \ldots, n \), \( q_j = (v_{j1}, \ldots, v_{jn}) \);

5) \( (x, y) := \sum_{j=1}^{n} x_j y_j \) denotes the standard scalar product in \( R^n \).

Certainly for a bounded original \( f \) it is possible to take \( a_j = a_{jn} = 0 \).

Each Cayley-Dickson number \( p \in A_r \) we write in the form

\[
p = \sum_{i=0}^{r-1} p^i j_i,
\]

where \( \{i_0, i_1, \ldots, i_{2^r-1}\} \) is the standard basis of generators of \( A_r \) so that \( i_0 = 1, \quad i_1^2 = -1 \) and \( i_j i_k = i_j i_k \) for each \( j > 0 \), \( i_j i_k = -i_k i_j \) for each \( j > 0 \) and \( k > 0 \) with \( k \neq j \), \( p_j \in R \) for each \( j \), if there exists an integral

\[
F^n(p) := F^n(p; \xi) := \int_{R^n} f(t) e^{-\langle p, t \rangle} dt
\]

then \( F^n(p) \) is called the noncommutative multiparameter (Laplace) transform at a point \( p \in A_r \) of the function-original \( f(t) \), where
\[ \zeta - \zeta_0 = \zeta h + \cdots + \zeta_{2^{-1}} \in A \] is the parameter of an initial phase, \( \zeta, \zeta_0 \in \mathbb{R} \) for each \( j = 0, 1, \ldots, \leq 2^{-1} \), \( \zeta \in A \), \( n = 2^{-1} \), \( dt = \lambda (dt) \).

8) \[ \langle p, t \rangle = p_0 (t_1 + \cdots + t_{2^{-1}}) + \sum_{j=1}^{2^{-1}} p_j t_j \],

we also put

8.1) \( u(p, t, \xi) = \langle p, t \rangle + \xi \).

For vectors \( v, w \in \mathbb{R}^n \) we shall consider a partial ordering

\[ \xi \leq \eta \Rightarrow \xi = \eta \quad \text{or} \quad \xi \neq \eta \]

8.1) \[ M(p, t, \xi) = (p_0, s_1 + \xi) \left[ i_1 \cos (p_{11} + \xi_{21}) + i_2 \sin (p_{12} + \xi_{22}) \cos (p_{13} + \xi_{23}) + \cdots + i_{2^{-1}} \sin (p_{2^{2^{-1}}} + \xi_{2^{2^{-1}}}) \right] \]

for the general Cayley-Dickson algebra with \( 2 \leq r < \infty \).

2.1) \( s_j = s_j(n+1) = t_1 + \cdots + t_n \) for each \( j = 1, \ldots, n \), \( n = 2^{-r} - 1 \), so that \( s_1 = t_1 + \cdots + t_n \), \( s_n = t_n \). More generally, let

2) \( u(p, t, \xi) = u(p, t, \xi) = p_0 s_1 + w(p, t) + \xi \), where \( w(p, t) \) is a locally analytic function, \( \mathbb{R}(w(p, t)) = 0 \) for each \( p \in A \) and \( t \in \mathbb{R}^{2^{-1}} \), \( \mathbb{R}(z) := (z + \bar{z})/2 \), \( \bar{z} = z^* \) denotes the conjugated number for \( z \in A \).

Then the more general non-commutative multiplicative transform over \( A \) is defined by the formula:

4) \[ F_n(u(p, t, \xi)) := \int_{\mathbb{R}^n} f(t) \exp(-u(p, t, \xi)) dt \]

for each Cayley-Dickson numbers \( p \in A \) whenever this integral exists as the principal value of either Riemann or Lebesgue integral, \( n = 2^{-r} - 1 \). This non-commutative multiplicative transform is in \( A \) spherical coordinates, when \( u(p, t, \xi) \) is given by Formulas (1,2).

At the same time the components \( p_j \) of the number \( p \) and \( \xi_j \) for \( \xi \) in \( u(p, t, \xi) \) we write in the \( p \)- and \( \xi \)-representations respectively such that

5) \[ \eta_j = -h_j (2^{-1})^{-1} \left[ \left( -h + \sum_{i=1}^{2^{j-1}} i (h_i^*) \right) \right] / 2 \]

for each \( j = 1, 2, \ldots, 2^{-1} \).

1) \[ \exp(i_1 (p_0 s_1 + \xi_1) \exp(-i_2 (p_{12} + \xi_2) \exp(-i_3 (p_{13} + \xi_3)))) \]

\[ \exp(-i_4 (p_{14} + \xi_{24} + \cdots + i_{2^{2^{-1}}} (p_{2^{2^{-1}}} + \xi_{2^{2^{-1}}})) \exp(-i_{2^{2^{-1}}} (p_{2^{2^{-1}}} + \xi_{2^{2^{-1}}}))) \]

Consider \( i_{2^{2^{-1}}} \) the generator of the doubling procedure of the Cayley-Dickson algebra \( A_{2^{2^{-1}}} \) from the Cayley-Dickson algebra \( A \), such that \( i_{2^{2^{-1}}} = i_{2^{2^{-1}}} \) for each \( j = 0, \ldots, 2^{-1} \).

9) \[ v < w \text{ if and only if } v_j \leq w_j \text{ for each } j = 1, \ldots, n \text{ and a } k \text{ exists so that } v_k < w_k, \]

\[ 1 \leq k \leq n. \]

2.2. Transforms in \( A \), Spherical Coordinates

Now we consider also the non-linear function \( u = u(p, t, \xi) \) taking into account non commutativity of the Cayley-Dickson algebra \( A \). Put

1) \[ u(p, t) := u(p, t, \xi) := p_0 s_1 + M(p, t) + \xi_0, \]

where

6) \[ h_0 = \left( h + (2^{-1})^{-1} \left[ -h + \sum_{i=1}^{2^{j-1}} i (h_i^*) \right] \right) / 2, \]

where \( 2 \leq r \leq N, h = h_0 + \cdots + h_{2^{j-1}} \), \( h_j \in A \), \( h_j \in R \) for each \( j \), \( i_0 = i_1 = \cdots = i_{2^{j-1}} = 1 \) for each \( k = 0, i_0 = 1, h \in A \). Denote \( \mathbb{F}_n(u(p, \xi)) \) in more details by \( \mathbb{F}_n(f, u, p; \xi) \).

Henceforth, the functions \( u(p, t, \xi) \) given by \( 1(8,8.1) \) or \( 1(2,2.1) \) are used, if another form \( 3 \) is not specified.

If for \( u(p, t, \xi) \) concrete formulas are not mentioned, it will be undermined, that the function \( u(p, t, \xi) \) is given in \( A \) spherical coordinates by Expressions 1,2,2.1). If in Formulas 1(7) or \( 1(2,2.1) \) the integral is not by all, but only by \( J_{1(k)}, \ldots, J_{r(k)} \) variables, where \( 1 \leq k < n, 1 \leq j(1) \leq \cdots \leq j(k) \leq n \), we denote a non-commutative transform by \( \mathbb{F}_n^{J_{1(k)}, \ldots, J_{r(k)}}(f, u, p; \xi) \), \( \mathbb{F}_n^{J_{1(k)}, \ldots, J_{r(k)}}(f, u, p; \xi) \).

Henceforth, we take \( \zeta_j = 0 \) and \( t_n = 0 \) and \( p_n = 0 \) for each \( 1 \leq m \neq 0 \) if something other is not specified.

2.3. Remark

The spherical \( A \) coordinates appear naturally from the following consideration of iterated exponents:
where \( t = (t_1, \ldots, t_n) \), \( n = n(r + 1) = 2^{r+1} - 1 \),
\( s_j = s_j(n(r + 1); t) \) for each \( j = 1, \ldots, n(r + 1) \), since
\[
s_n(n(r + 1); t) = t_n + \cdots + t_{n(r+1)}
\]
\[
= s_n(n(r); t) + s_{n(r+1)}(n(r+1); t)
\]
for each \( m = 1, \ldots, 2^r - 1 \).

An image function can be written in the form

3) \( F^n_{ru}(p; \zeta) := \sum_{j=0}^{r-1} i_{ru}^{n} F^n_{ru}(p; \zeta) \),
where a function \( f \) is decomposed in the form

3.1) \( f(t) = \sum_{m=0}^{r-1} i_{fu}^{m} f(t) \),
\( f_j : R^n \to R \) for each \( j = 1, \ldots, 2^{r-1} \), \( F^n_{ru}(p; \zeta) \)
de-notes the image of the function-original \( f_j \).

If an automorphism of the Cayley-Dickson algebra \( A_r \) is taken instead of the standard generators \( \{i_0, \ldots, i_{2^{r-1}}\} \) new generators \( \{N_0, \ldots, N_{2^{r-1}}\} \) are used,
this provides also \( M(p, r; \zeta) = M_N(p, r; \zeta) \) relative to new basic generators, where \( 2 \leq r \in N \).
In this more general case we denote by \( sF^n_{ru}(p; \zeta) \) an image for an original \( f(t) \), or in more details we denote it by
\( sF^n_{ru}(f; u; p; \zeta) \).

Formulas (7) and (4) define the right multiparameter transform. Symmetrically is defined a left multi-
parameter transform. They are related by conjugation and up to a sign of basic generators. For real valued originals they certainly coincide. Henceforward, only the right multiparameter transform is investigated.

2) \[
\int_{u} f(t) \exp(-u(p, t; \zeta)) dt \leq \int_{u} \cdots \int_{u} C_v \exp \left\{ -v_i \left( w - a_{i_1} \right), \ldots, -v_n \left( w - a_{i_n} \right) y_n - \zeta _o \right\} dy_1 \cdots dy_n
\]
\[
= C_v e^{-50} \prod_{j=1}^{n} y_j \left( w - a_j \right)^{-1} ,
\]
where \( w = Re(p) \), since \( \left| e^z \right| = \exp(Re(z)) \) for each \( z \in A_r \) in view of Corollary 3.3 [6]. While an integral,

3) \[
\left| \int_{u} f(t) \left( \frac{\partial \exp(-u(p, t; \zeta))}{\partial \bar{p}} \right) dt \right|
\]
\[
\leq \int_{u} \cdots \int_{u} C_v \left| h_0(v_i y_i + \cdots + v_n y_n), h_1(v_i y_i + \cdots + v_n y_n), \ldots, h_{n-1}(v_{n-1} y_{n-1} + v_n y_n), h_n v_n y_n \right|
\]
\[
\exp \left\{ -v_i \left( w - a_{i_1} \right), \ldots, -v_n \left( w - a_{i_n} \right) y_n - \zeta _o \right\} dy_1 \cdots dy_n
\]
for each \( h \in A_r \), since each \( z \in A_r \) can be written in the form \( z = z \exp(M) \), where \( \left| e^z \right| = \left| e^z \right| \in [0, \infty) \subset R \),
\( M \in A_r \), \( Re(M) = (M + M)^{1/2} = 0 \) in accordance with Proposition 3.2 [6]. In view of Equations (5,6):

6) \[
\left| \int_{u} f(t) \left( \frac{\partial \exp(-u(p, t; \zeta))}{\partial \zeta} \right) dt \right|
\]
\[
\leq \left| C_v \int_{u} \cdots \int_{u} C_v \exp \left\{ -v_i \left( w - a_{i_1} \right), \ldots, -v_n \left( w - a_{i_n} \right) y_n - \zeta _o \right\} dy_1 \cdots dy_n = \left| C_v e^{-50} \prod_{j=1}^{n} y_j \left( w - a_j \right)^{-1}
\right|
\]

Particularly, if \( p = (p_0, p_1, 0, \ldots, 0) \) and
\( t = (t_0, 0, \ldots, 0) \), then the multiparameter non-commu-

2.4. Theorem

If an original \( f(t) \) satisfies Conditions 1(4-1) and
\( a_i < a_{i+1} \), then its image \( F^n(f, u; p; \zeta) \) is \( A_r \)-holo-
morphic (that is locally analytic) by \( p \) in the domain \( \{ z \in A_r : a_i < Re(z) < a_{i+1} \} \), as well as by \( \zeta \in A_r \), for every \( 0 \leq n \leq 2^r - 1 \), the function \( u(p, t; \zeta) \) is

Proof. At first consider the characteristic functions \( \chi_{U_r}(t) \), where \( \chi_{U_r}(t) = 1 \) for each \( t \in U \),
while \( \chi_{U_r}(t) = 0 \) for every \( t \not\in U_r \).
where \( U_r := \{ t \in R^n : \forall j \in \{1, \ldots, n\} \} \) is the domain in the

Euclidean space \( R^n \) for any \( v \) from § 1. Therefore,

1) \( F^n_{ru}(p; \zeta) := \sum_{n=0}^{n-1} \int_{U_r} f(t) \exp(-u(p, t; \zeta)) dt, \)
\( \lambda_c(U_r \cap U_w) = 0 \) for each \( v \neq w \). Each integral
\( \int_{U_r} f(t) \exp(-u(p, t; \zeta)) dt \) is absolutely convergent for each \( p \in A_r \) with the real part \( a_i < Re(p) < a_{i+1} \), since
it is majorized by the converging integral

4) \( \frac{\partial}{\partial \bar{p}} \left( \int_{u} f(t) \exp(-u(p, t; \zeta)) dt \right) = 0 \) and
5) \( \frac{\partial}{\partial \zeta} \left( \int_{u} f(t) \exp(-u(p, t; \zeta)) dt \right) = 0 \), while

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for each \( h \in A \). In view of convergence of integrals given above (1-6) the multparameter non-commutative transform \( F^u_n (p; \zeta) \) is (super)differentiable by \( p \) and \( \zeta \), moreover, \( \partial F^u_n (p; \zeta)/\partial p = 0 \) and \( \partial F^u_n (p; \zeta)/\partial \zeta = 0 \) in the considered \( (p, \zeta) \)-representation. In accordance with [5,6] a function \( g(p) \) is locally analytic by \( p \) in an open domain \( U \) in the Cayley-Dickson algebra \( A \), \( 2 \leq r \), if and only if it is (super)differentiable by \( p \), in another words \( A \)-holomorphic. Thus, \( F^u_n (p; \zeta) \) is \( A \)-holomorphic by \( p \in A \) with \( a_i < \text{Re}(p) < a_{i+1} \) and \( \zeta \in A \), due to Theorem 2.6 [4].

**Corollary**

Let suppositions of Theorem 4 be satisfied. Then the image \( F^u_n (f, u; p; \zeta) \) with \( u = u(p, t; \zeta) \) given by (2.1) has the following periodicity properties:

1) For each \( f = 1, \ldots, n \) and \( \beta \in 2\pi \mathbb{Z} \);

2) For each \( f = 1, \ldots, n \) so that \( \zeta^\beta = e^{2\pi i} \) and \( \zeta_j^\beta = e^{2\pi i j} \), \( j \neq i \), where \( 0 \leq j \leq n \) and \( s = j + 1 \), while either \( p^\beta_j = -p^\beta_j \) and \( p^\beta_j = p^\beta_j \) for each \( l \neq j \) with \( \kappa = 2 \) or \( \kappa = 2 \) and \( f(t) \) is an even function with \( \kappa = 2 \) by the \( s_j = (t_j + \cdots + t_0) \) variable or an odd function by \( s_j = (t_j + \cdots + t_0) \) with \( \kappa = 1 \); 3) \( F^u_n (f, u; p; \zeta + \pi i) = -F^u_n (f, u; p; \zeta) \).

**Proof.** In accordance with Theorem 4 the image \( F^u_n (f, u; p; \zeta) \) exists for each \( p \in W_f := \{ \zeta \in A : a_i < \text{Re}(\zeta) < a_{i+1} \} \) and \( \zeta \in A \), where \( 1 \leq r \). Then from the \( 2\pi \)-periodicity of sine and cosine functions the first statement follows. From \( \sin(-\theta) = -\sin(\theta) \), \( \cos(\phi) = \cos(-\phi) \), \( \sin(-\theta) = -\sin(\theta) \), \( \cos(\phi) = -\cos(\phi) \) we get that \( \sin(p_j s_j + \zeta_j^\beta) = \sin(-p_j s_j + \zeta_j^\beta) \),

\[
\sin(p_j s_j + \zeta_j^\beta) = \sin(-p_j s_j + \zeta_j^\beta)(-\cos(p_j s_j + \zeta_j^\beta))
\]

and

\[
\sin(p_j s_j + \zeta_j^\beta) = \sin(-p_j s_j + \zeta_j^\beta)(-\cos(p_j s_j + \zeta_j^\beta))
\]

for each \( t \in A^r \). On the other hand, either \( p^\beta_j = -p^\beta_j \) and \( p^\beta_j = p^\beta_j \) for each \( l \neq j \geq 1 \) with \( \kappa = 2 \) or \( \kappa = 2 \) and

\[
f(t_1, \ldots, s_{j-1}, s_j, s_{j-1}, \ldots, t_n) = (-1)^{k_f} f(t_1, \ldots, s_{j-1}, s_j, s_{j-1}, \ldots, t_n)
\]

is an even with \( \kappa = 2 \) or odd with \( \kappa = 1 \) function by the \( s_j = (t_j + \cdots + t_0) \) variable for each \( t = (t_1, \ldots, t_n) \in A^n \), where \( t_i = s_j - s_{j-1} \) for \( j = 1, \ldots, n, s_{j+1} = s_{j+1} + n(t) = 0 \). From this and Formulas (2,1,2,4) the second and the third statements of this corollary follow.

**2.5. Remark**

For a subset \( U \) in \( A \) we put

\[
\pi_{s,p} (U) := \{ u \in U : z = \sum_{k \in b} w_k, u = w(t + w_p) \}
\]

for each \( s \neq p \in b \), where

\[
t := \sum_{k \in b} w_k v, v = w_p = 0, w, v \in R \forall v \in b \}
\]

where \( b := \{ i_0, i_1, \ldots, i_{2n-1} \} \) is the family of standard generators of the Cayley-Dickson algebra \( A \). That is, geometrically \( \pi_{s,p} (U) \) means the projection on the complex plane \( C_{sp} \) of the intersection \( U \) with the plane \( \pi_{s,p} \), \( \pi_{s,p} := \{ ad + kp : a, b \in R \} \), since \( sp \in b = b \{ 1 \} \). Recall that in § 2.5-7 [6] for each continuous function \( f : U \to A \) it was defined the operator \( f \) by each variable \( z \in A \). For the non-commutative integral transformations consider, for example, the left algorithm of calculations of integrals.

A Hausdorff topological space \( X \) is said to be \( n \)-connected for \( n \geq 0 \) if each continuous map \( f : S^k \to X \) from the \( k \)-dimensional real unit sphere into \( X \) has a continuous extension over \( R^{k+1} \) for each \( k \leq n \) (see also [17]). A 1-connected space is also said to be simply connected.

It is supposed further, that a domain \( U \) in \( A \), has the property that \( U \) is \( (2^r - 1) \)-connected; \( \pi_{s,p} \) is simply connected in \( C \) for each \( k = 0, 1, \ldots, 2^r - 1 \), \( s = is_k, p = is_{k+1}, t \in A_{s,p} \), and \( u \in C_{s,p} \), for which there exists \( z = u + t \in U \).

**2.6. Theorem**

If a function \( f(t) \) is an original (see Definition 1), such that \( \Delta F^u_n (p; \zeta) \) is its multparameter non-commutative transform, where the functions \( f \) and \( F^u_n \) are written in the forms given by (3,3,3,1), \( f (R^n) \subset A \), over the Cayley-Dickson algebra \( A \), where \( 1 \leq r \leq N \), \( 2^r - 1 \leq n \leq 2^r - 1 \).

Then at each point \( t \), where \( f(t) \) satisfies the Hölder condition the equality is accomplished:

\[
(\mathcal{F}^u_1) \frac{\Delta F^u_n (a + p; \zeta) \exp(u(a + p, t; \zeta))}{u(t, t; \zeta),}
\]

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where either \( u(p,t;\zeta) = (p,t) + \zeta \) or
\( u(p,t;\zeta) = p s + M_{\zeta} (p,R;\zeta) + \zeta \) (see § 1 and 2),
the integrals are taken along the straight lines
\[ p(t_j) = N_j t_j, t_j \in A, \quad t_j \in R \text{ for each } j = 1, \ldots, n; \]
a_1 < Re(p) = a < a_1, and this integral is understood in the sense of the principal value,
\[ t = (t_1, \ldots, t_n) \in R^n, \]
\[ dp = \left( \cdots \left( d[p_{N_1}] d[p_{N_2}] \right) \right) \cdots d[p_{N_n}]. \]

**Proof.** In Integral (1) an integrand \( \eta(p) dp \) certainly corresponds to the iterated integral as
\( \left( \cdots (\eta(p) d[p_{N_1}] \cdots ) d[p_{N_n}] \right), \)
where \( p = p_{N_1} + \cdots + p_{N_n}, \quad p_1, \ldots, p_n \in R \). Using Decomposition 3(3,1) of a function \( f \) it is sufficient to consider the inverse transformation of the real valued function \( f_\eta \), which we denote for simplicity by \( f \). We put
\[ \eta F_{u}(p;\zeta) = \left. f_\eta \exp (-u(p,t;\zeta)) \right| dt. \]
If \( \eta \) is a holomorphic function of the Cayley-Dickson variable, then locally in a simply connected domain \( U \) in each ball \( B(A_z, z_0, R) \) with the center at \( z_0 \) of radius \( R > 0 \) contained in the interior \( Int(U) \) of the domain \( U \) there is a unique \( \eta(a + z) \), where the integral depends only on an initial \( z_0 \) and a final \( z \) points of a rectifiable path in \( B(A, z_0, R), \quad a \in R \) (see also Theorem 2.14 [4]). Therefore, along the straight line \( N_R \) the restriction of the antiderivative has the form
\[ \int_{z_0}^{z} \eta(a + z) dz = \int_{0}^{z} \eta(a + N_j t_j) \cdot N_j dt_j, \]
where \( \partial \eta(a + z)/\partial t_j = (\partial \eta(a + z)/\partial z) \cdot N_j \) for the 
\[ g_\eta(t) = \left( \int_{0}^{z} \eta(a + N_j t_j) \cdot N_j dt_j \right) \]
for each positive value of the parameter \( 0 < b < \infty \). With the help of generators of the Cayley-Dickson algebra \( A \),
\[ 4) \quad g_\eta(t) = \left( \int_{0}^{z} \eta(a + N_j t_j) \cdot N_j dt_j \right) \]
and the Fubini Theorem for real valued components of the function the integral can be written in the form:

since the integral
\[ \int_{0}^{z} f(t) \exp \left\{ -u_N(a + p,t;\zeta) \right\} dt \]
for any marked \( 0 < \delta \leq a_1 - a_1 \) is uniformly converging relative to \( p \) in the domain
\( a_1 + \delta \leq Re(p) \leq a_1 - \delta \) in \( A \), (see also Proposition 2.18 [4]). If take marked \( t_j \) for each \( k \neq j \) and
\( S = N_j \) for some \( j \geq 1 \) in Lemma 2.17 [4] considering the variable \( t_j \), then with a suitable \( (R - \text{linear}) \) automorphism of the Cayley-Dickson algebra \( A \), an expression for \( v(M(p,t;\zeta)) \) simplifies like in the complex case with \( C_k := R \oplus RK \) for a purely imagin-
The latter identity can be applied to either

\[ S^{s} = M_{k;1}(p_{k,1}N_{k,1} + \cdots + p_{n,1}N_{n,1}; \zeta_{k,1}N_{k,1} + \cdots + \zeta_{n,1}N_{n,1}) \]

and

\[ N^{s} = M_{k;1}(p_{k,1}N_{k,1} + \cdots + p_{n,1}N_{n,1}; \zeta_{k,1}N_{k,1} + \cdots + \zeta_{n,1}N_{n,1}), \]

or

\[ S^{s} = (p_{k,1}t_{k,1} + \zeta_{k,1})N_{k,1} + \cdots + (p_{n,1}t_{n,1} + \zeta_{n,1})N_{n,1} \]

and

\[ N^{s} = (p_{k,1}t_{k,1} + \zeta_{k,1})N_{k,1} + \cdots + (p_{n,1}t_{n,1} + \zeta_{n,1})N_{n,1}, \]

where

7) \[ M_{k;1}(p_{k,1}N_{k,1} + \cdots + p_{n,1}N_{n,1}; \zeta_{k,1}N_{k,1} + \cdots + \zeta_{n,1}N_{n,1}), \]

\[ = (p_{k,1}s_{k,1} + \zeta_{k,1}) + N_{k}\left[\sin(p_{k,1}s_{k,1} + \zeta_{k,1})\cdots\sin(p_{n,1}s_{n,1} + \zeta_{n,1})\right] \]

8) \[ s_{j,k,1} = s_{j,k,1}(n,t) = t_{k,1} + \cdots + t_{n} = s_{k,j}(n,t) \]

for each \( j = 1, \cdots, n-1; \) \( s_{n-k,k,1} = s_{n-k,k,1}(n,t) = t_{n}. \) We take the limit of \( g_{b}(i) \) when \( b \) tends to the infinity. Evidently,

9) \[ u_{j,1}(p_{0} + p_{1}N_{1} + \cdots + p_{n}N_{n}; \zeta_{0} + \zeta_{1}N_{1} + \cdots + \zeta_{n}N_{n}) \]

\[ = \zeta_{0} + p_{0}s_{1,1} + M_{j}(p_{1}N_{1} + \cdots + p_{n}N_{n}; \zeta_{0} + \zeta_{1}N_{1} + \cdots + \zeta_{n}N_{n}) \]

for \( u_{N} \) given by 2(1,2,2,1), where \( M_{j} \) is prescribed by (7), \( s_{k,j} = s_{k,j}(n,\tau); \)

10) \[ u_{j,1}(p_{0} + p_{1}N_{1} + \cdots + p_{n}N_{n}; \zeta_{0} + \zeta_{1}N_{1} + \cdots + \zeta_{n}N_{n}) = \zeta_{0} + p_{0}s_{1,1} + \sum_{k=0}^{n}(p_{k}s_{k,1} + \zeta_{k})N_{k} \]

for \( u = u_{N} \) given by 1(8,8,1). For \( j > 1 \) the parameter \( \zeta_{0} \) for \( u = u_{N} \) given by 1(8,8,1) or 2(1,2,2,1) can be taken equal to zero.

When \( t_{1}, \cdots, t_{j-1}, t_{j+1}, \cdots, t_{n} \) and \( p_{1}, \cdots, p_{j-1}, p_{j+1}, \cdots, p_{n} \) variables are marked, we take the parameter

\[ \zeta' := \zeta' \left( p_{1}N_{1} + \cdots + p_{n}N_{n}; \zeta_{0} + \zeta_{1}N_{1} + \cdots + \zeta_{n}N_{n} \right) \]

\[ = \left( \zeta_{0} + \zeta_{1}N_{1} + \cdots + \zeta_{n}N_{n} \right) + (a + p_{0})s_{j,1} + p_{j,1}s_{j+1,1} + \cdots + p_{n}s_{n,1} \]

for \( u(p, \tau; \zeta) \) given by Formulas 2(1,2,2,1) or

\[ \zeta' := \zeta' \left( p_{1}N_{1} + \cdots + p_{n}N_{n}; \zeta_{0} + \zeta_{1}N_{1} + \cdots + \zeta_{n}N_{n} \right) \]

\[ = \left( \zeta_{0} + \zeta_{1}N_{1} + \cdots + \zeta_{n}N_{n} \right) + (a + p_{0})s_{j,1} + p_{j,1}s_{j+1,1} + \cdots + p_{n}s_{n,1} \]

for \( u(p, \tau; \zeta) \) described in 1(8,8,1). Then the integral operator

\[ \lim_{\varepsilon \to 0} \left[ (2\pi N_{j})^{-1} \int_{0}^{\infty} d\tau \int_{N_{j,\varepsilon}}^{N_{j,\varepsilon}} (dp_{1}N_{j}) \right] \]

(see also Formula (4) above) applied to the function

\[ f(t_{1}, \cdots, t_{j-1}, t_{j+1}, \cdots, t_{n}) \exp \left\{ -u_{N,j} \left( a + p_{0} + p_{1}N_{1} + \cdots + p_{n}N_{n}; \zeta_{0} + \zeta_{1}N_{1} + \cdots + \zeta_{n}N_{n} \right) \right\} \]

\[ \exp \left\{ u_{N,j} \left( a + p_{0} + p_{1}N_{1} + \cdots + p_{n}N_{n}; \zeta_{0} + \zeta_{1}N_{1} + \cdots + \zeta_{n}N_{n} \right) \right\} \]

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with the parameter \( \zeta \) instead of \( \zeta \) treated by Theorems 2.19 and 3.15 [4] gives the inversion formula cor-
responding to the real variable \( t \) for \( f(t) \) and to the
Cayley-Dickson variable \( p_j N_0 + p_j N_j \) restricted on the
complex plane \( C_{N_j} = R \oplus R N_j \), since
\[
d(t_j + c) = d t_j \quad \text{for each (real constant) } c. \]
After integrations with \( j = 1, \ldots, k \) with the help of Formulas (6-
10) and (3.1) we get the following:
\[
11) \lim_{b \to 0} g_b(t) = Re \left[ (2\pi N_a)^{-1} \int_0^t \int_{N_a} \int_{N_a} \right] \left( \int_0^t \int_{N_a} \int_{N_a} \right) f(t_1, \ldots, t_k, s_1, \ldots, s_k) \exp \left[ \sum_{j=1}^k (a_j + p_j N_0 + \cdots + p_j N_k, t_1, \ldots, t_k) \left( \zeta_j + \ldots + \zeta_k N_j \right) \right] \prod_{j=1}^k \left( \tau_j + \ldots + \tau_k N_j \right) dp_1 \cdots dp_k.
\]
Moreover, \( Re(f_q) = f_q \) for each \( q \) and in (11) the function \( f = f_q \) stands for some marked \( q \) in accordance with De-
compositions 3(3,3.1) and the beginning of this proof.

Mention, that the algebra \( alg_R \left( N_j, N_k, N_l \right) \) over the
field with three generators \( N_j, N_k \) and \( N_l \) is alternative. The product \( N_j N_k \) of two generators is also
the Cayley-Dickson variable \( (1,0) \) with the de-
finitive number \( m = m(k,l) \) and the sign multiplier
\( (-1)^{jk}\), where \( \zeta(k,l) \in \{0,1\} \). On the other hand,
\[
\text{Corollary}
\]
If the conditions of Theorem 6 are satisfied, then
\[
1) f(t) = (2\pi)^n \int_{\pi} \int_{\pi} \int_{\pi} \int_{\pi} \int_{\pi} \int_{\pi} \int_{\pi} \int_{\pi} \int_{\pi} (a + p + \zeta) \exp \left[ u(a + p + \zeta) \right] dp_1 \cdots dp_n = \left( \mathcal{F}^n \right)^{-1} \left( v \mathcal{F}^n \right) (a + p + \zeta, u, \tau, \zeta).
\]

Proof. Each algebra \( alg_R \left( N_j, N_k, N_l \right) \) is alternative. 
Therefore, in accordance with § 6 and Formulas
\[
2) N_j N_k \int_{N_j} \int_{N_k} f(t) \exp \left[ -u \left( a + p + \zeta \right) \right] \exp \left[ u \left( a + p + \zeta \right) \right] dp_1 \cdots dp_n = \left[ n \mathcal{F}^n \right] (a + p + \zeta, u, \tau, \zeta).
\]
for each \( j = 1, \ldots, n \), since the real field is the center of
the Cayley-Dickson algebra \( A \), while the functions
sin and cos are analytic with real expansion coef-
ficients. Thus
\[
3) g_b(t) = (2\pi)^n \left[ \int_0^t \int_{t_1}^{t_b} \int_{t_1}^{t_b} \cdots \int_{t_1}^{t_b} \right] f(t) \exp \left[ -u \left( a + p + \zeta \right) \right] \exp \left[ u \left( a + p + \zeta \right) \right] dp_1 \cdots dp_n.
\]
hence taking the limit with \( b \) tending to the infinity im-
plies, that the non-commutative iterated (multiple) inte-
gral in Formula (61) reduces to the principal value of the
usual integral by real variables \( (t_1, \ldots, t_n) \) and
\((p_1, \ldots, p_n)\) 6.1(1).

2.7. Theorem

An original \( f(t) \) with \( f(R^n) \subset A \) over the Cay-
ley-Dickson algebra \( A \), with \( 1 \leq r \in N \) is completely de-
defined by its image \( \mathcal{F}^n \left( p; \zeta \right) \) up to values at points
of discontinuity, where the function \( u(a + p + \tau; \zeta) \) is given
by \( 1(8,8.1) \) or \( 2(1-4) \) for each non-commutative integral
given by the left algorithm we get

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almost everywhere on \( \mathbb{R}^n \) continuous.

### 2.8. Theorem

Suppose that a function \( \mathcal{F}^n_u(p, \zeta) \) is analytic by the variable \( p \in A \), in a domain
\[
N_{\zeta} := \{ p \in A : a_j < \Re(p) < a_{j+1} \}, \quad \text{where} \quad 2 \leq r \in \mathbb{N},
\]
for each \( n \); \( n \in \mathbb{N} \) and \( 1 \leq r \leq 2^{n-1} \), \( \mathcal{F}^n_u(p, \zeta) \) is analytic by \( p \) in the domain \( N_{\zeta} \).

Let \( N_{\zeta} \) be holomorphic by \( p \) in the domain \( N_{\zeta} \).

Then \( \mathcal{F}^n_u(p, \zeta) \) be holomorphic by \( p \) in the domain \( N_{\zeta} \).

Moreover, for each \( a > a_i \) and \( b < a_i \), there exist constants \( C_a > 0 \), \( C_b > 0 \) and \( \epsilon_a > 0 \) and \( \epsilon_b > 0 \) such that
\[
1) \quad \left| \mathcal{F}^n_u(p, \zeta) \right| \leq C_a \exp(-\epsilon_a |p|) \quad \text{for each} \quad p \in A,
\]
with \( \Re(p) \geq a \).

2) \quad \left| \mathcal{F}^n_u(p, \zeta) \right| \leq C_b \exp(-\epsilon_b |p|) \quad \text{for each} \quad p \in A,
\]
with \( \Re(p) \leq b \), the integral,

3) \quad \int_{N_{\zeta}}^{\infty} \cdots \int_{N_{\zeta}}^{\infty} \mathcal{F}^n_u(w + p; \zeta) \, dp \quad \text{converges absolutely for} \quad k = 0 \quad \text{and} \quad k = 1 \quad \text{and each} \quad a_i < w < a_{i-1}.

Then \( \mathcal{F}^n_u(w + p; \zeta) \) is the image of the function,

\[
\hat{g}(z) \rightarrow 0 \quad \text{while} \quad |z| \quad \text{tends to the infinity, since} \quad |\zeta| \quad \text{is a finite number (see Lemma 2.23 in [4]).}
\]

We apply this to the integrand in Formula (4), since \( \mathcal{F}^n_u(w + p; \zeta) \) is locally analytic by \( p \) in accordance with Theorem 4 and Conditions (1,2) are satisfied.

Then the integral operator \( \left( 2\pi N_{\zeta} \right)^{-1} \int_{N_{\zeta}}^{\infty} \mathcal{F}^n_u(w + p; \zeta) \, dp \) on the \( j \)-th step with the help of Theorems 2.22 and 3.16 [4] gives the inversion formula corresponding to the real parameter \( t_j \) and to the Cayley-Dickson variable \( p_{j} N_{j} + p_{j} N_{j} \) which is restricted on the complex plane \( C_{N_{j}} = R \oplus R N_{j} \) (see also Formulas 6(4,11) above). Therefore, an application of this procedure by \( j = 1, 2, \ldots, n \) as in § 6 implies Formula (4) of this theorem. Thus there exist originals \( f^0 \) and \( f^1 \) for \( \mathcal{F}^n_u(p, \zeta) \) and \( \mathcal{F}^n_u(p, \zeta) \) with a choice of \( w \in R \) in the common domain \( a_j < \Re(p) < a_{j+1} \). Then \( f = f^0 + f^1 \) is the original for \( \mathcal{F}^n_u(p, \zeta) \) due to the distributivity of the multiplication in the Cayley-Dickson algebra \( A \), leading to the additivity of the considered integral operator in Formula (4).

### Corollary

Let the conditions of Theorem 8 be satisfied, then

1) \( f(t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathcal{F}^n_u(w + p; \zeta) \exp \{ u(w + p, t; \zeta) \} \, dp \cdots dp_n = \left( F^* \right)^{-1} \left( \mathcal{F}^n_u(w + p; \zeta), u, t; \zeta \right). \)

### Proof

In accordance with § 6 and 6.1 each non-commutative integral given by the left algorithm reduces to the principal value of the usual integral by the corresponding real variable:

2) \( (2\pi)^{-n} \int_{N_{\zeta}} \mathcal{F}^n_u(w + p; \zeta) \exp \{ u(w + p, t; \zeta) \} \, d(p, N_{j}) = (2\pi)^{-1} \int_{N_{\zeta}} \mathcal{F}^n_u(w + p; \zeta) \exp \{ u(w + p, t; \zeta) \} \, dp \).
for each \( j = 1, \cdots, n \). Thus Formula 8(4) with the non-commutative iterated (multiple) integral reduces to Formula 8.1(1) with the principal value of the usual integral by real variables \( (p_1, \cdots, p_n) \).

2.9. \textbf{Note}

In Theorem 8 Conditions (1,2) can be replaced on

1) \( \lim_{\rho \to \infty} \sup \rho \| \hat{F}(p) \| = 0 \),

where \( C_{R(n)} := \{ z \in R : |z| = R(n), a_i < Re(z) < a_i \} \) is a sequence of intersections of spheres with a domain \( W \), where \( R(n) < R(n+1) \) for each \( n \), \( \lim_{n \to \infty} R(n) = \infty \).

Indeed, this condition leads to the accomplishment of the A, analog of the Jordan Lemma for each \( =1, \cdots, n \) with \( \phi \) be an image of \( \Phi(t) \) with \( \phi \) spherical coordinates or \( \Phi(t) \) and \( \phi \) exist, then the integral

\[
\int_{\phi} \beta \phi(t) \exp(-u(p, t; \zeta)) dt = \int_{\phi} \alpha \phi(t) \exp(-u(p, t; \zeta)) dt + \int_{\phi} \beta \phi(t) \exp(-u(p, t; \zeta)) dt
\]

converges in the domain

\[
W_f \cap W_g = \{ p \in A : \max(a_i(f), a_i(g)) < Re(p) < \min(a_i(f), a_i(g)) \}.
\]

We have \( t \in R^n, 2^{-1} \leq n \leq 2^1 - 1 \), while \( R \) is the center of the Cayley-Dickson algebra \( A \). The quaternion skew field \( H \) is associative. Thus, under the imposed conditions the constants \( \alpha, \beta \) can be carried out outside integrals.

2.11. \textbf{Theorem}

Let \( \alpha = \text{const} > 0 \), let also \( F^a(p; \zeta) \) be an image of an original function \( f(t) \) with either \( u = \{p(t) + \zeta \} \) or \( u \) given by Formulas 2(1,2) over the Cayley-Dickson algebra \( A \) with \( 2 \leq r < \infty, 2^{-1} \leq n \leq 2^1 - 1 \). Then an image \( F^a(p; \alpha; \zeta)/\alpha^n \) of the function \( f(\alpha \tau) \) exists.

\textbf{Proof.} Since \( p_j s_j + \zeta_j = p_j(s_j/\alpha) + \alpha s_j + \zeta_j = (p_j/\alpha) s_j' + \zeta_j' \) for each \( j = 1, \cdots, n \), where \( s_j \alpha = s_j' \), \( s_j = s_j(n; t) \),

\[
1) \quad F^a \left( \frac{\partial f(t)}{\partial t} \right) \chi_{U_{t,1}} (t, u; p; \zeta) = -F^{a+1}(f(t) \chi_{U_{t,1}} (t', u(t'; \zeta); p; \zeta) + \sum_{k=1}^K p_k \exp \int_0^t \frac{\partial f(t)}{\partial t} \chi_{U_{t,1}} (t, u; p; \zeta)
\]

in the \( A \)-spherical coordinates or

3) \( f(t) \chi_{U_{t,1}} (t) \) we put

\[
W_f = \{ p \in A : \alpha \phi(t) < Re(p) \}\),

that is \( a_4 = \infty \). Cases may be, when either the left hyperplane \( Re(p) = \alpha \) or the right hyperplane \( Re(p) = a_4 \) is (or both are) included in \( W_f \). It may also happen that a domain reduces to the hyperplane \( W_f = \{ p : Re(p) = a_i = a_i \} \).

2.10. \textbf{Proposition}

If images \( F^a(p; \zeta) \) and \( G^a(p; \zeta) \) of functions \( f(t) \) and \( g(t) \) exist in domains \( W_f \) and \( W_g \) with values in \( A \), where the function \( u(p, t; \zeta) \) is given by 1(8,8.1) or 2(1,2,2).1, then for each \( \alpha, \beta \in A \), in the case \( A = H \); as well as \( f \) and \( g \) with values in \( R \) and each \( \alpha, \beta \in A \) or \( f \) and \( g \) with values in \( A \) and each \( \alpha, \beta \in R \) in the case of \( A \), with \( r \geq 3 \); the function

\[
\alpha F^a(p; \alpha; \zeta) + \beta G^a(p; \alpha; \zeta)
\]

is the image of the function \( f(t) + \beta g(t) \) in a domain \( W_f \cap W_g \).

\textbf{Proof.} Since the transforms \( F^a(p; \alpha; \zeta) \) and \( G^a(p; \alpha; \zeta) \) exist, then the integral

\[
\int_{\phi} \beta \phi(t) \exp(-u(p, \zeta)) dt = \int_{\phi} \beta \phi(t) \exp(-u(p, \zeta)) dt + \int_{\phi} \beta \phi(t) \exp(-u(p, \zeta)) dt
\]

due to the fact that the real filed \( R \) is the center \( Z(A) \) of the Cayley-Dickson algebra \( A \).

2.12. \textbf{Theorem}

Let \( f(t) \) be a function-original on the domain \( U_{1,1} \) such that \( \partial f(t)/\partial t \) also for \( k = j - 1 \) and \( k = j \) satisfies Conditions 1(1-4). Suppose that \( \{u(t; \zeta)\} \) is given by 2(1,2,2.1) or 1(8,8.1) over the Cayley-Dickson algebra \( A \) with \( 2 \leq r < \infty, 2^{-1} \leq n \leq 2^1 - 1 \). Then

\[
\int_{\phi} f(t) \exp(-u(p, t; \zeta)) dt = \int_{\phi} f(t) \exp(-u(p, t; \zeta)) dt + \int_{\phi} f(t) \exp(-u(p, t; \zeta)) dt
\]

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1.1) \( F^n \left( (\frac{\partial f}{\partial t_j}) \chi_{(t_j , \ldots , t_n)}(t), u; p; \xi) \right) = -F^{n-1}\left( f(t) \chi_{(t_j , \ldots , t_n)}(t'), u(p, t'; \xi); p; \xi \right) \)

\[ + \left[ p_0 + p_i S_{ij} \right] F^n \left( f(t) \chi_{(t_j , \ldots , t_n)}(t), u; p; \xi \right) \]

in the \( A \) Cartesian coordinates in a domain \( W = \left\{ p \in A : \max \left( a_i(f), a_i(\frac{\partial f}{\partial t_j}) \right) < \text{Re}(p) \right\}, \) where \( t^j := (t_1, \ldots , t_j, \ldots , t_n : t_j = 0), \quad S_{ij} = -\partial / \partial \xi_k \) for each \( k \geq 1 \).

**Proof.** Certainly,

2) \( \frac{\partial f}{\partial t_j} = \frac{\partial f}{\partial t_i} \) and

2.1) \( \frac{\partial f}{\partial t_j} = \sum_{i \in \xi} \frac{\partial f}{\partial (t_i)} \frac{\partial (t_i)}{\partial t_j} = \sum_{i \in \xi} \frac{\partial f}{\partial (t_i)} \frac{\partial s_i}{\partial t_j} \)

for each \( j = 2, \ldots , n, \) since \( t_j = s_j - s_{j+1}, \quad t_1 = s_1 - s_2, \) where \( s_j = s_j(n, t), \quad s_{n+1} = 0 \) for each \( l \geq 1 \). From Formulas 30(6,7) [4] we have the equality in the \( A \) spherical coordinates:

3) \( \frac{\partial \exp(-u(p, t; \xi))}{\partial s_j} = -p_0 \delta_{s_j} \exp(-u(p, t; \xi)) - p_i S_{ij} \exp(-u(p, t; \xi)) \),

since \( \exp(-u(p, t; \xi)) = \exp(-p_0 s_i - \xi_0) \exp(-M(p, t; \xi)), \)

\( \frac{\partial \exp(-p_0 s_i - \xi_0)}{\partial s_j} = -p_0 \delta_{s_j} \exp(-p_0 s_i - \xi_0) \),

\( \frac{\partial}{\partial s_j} \left[ \cos(p_j s_j + \xi_j) - \sin(p_j s_j + \xi_j) i_j \right] = \frac{\partial}{\partial s_j} \left[ \cos(p_j s_j + \xi_j) - \sin(p_j s_j + \xi_j) - \pi/2 \right] \)

\( = -p_j \left[ \cos(p_j s_j + \xi_j - \pi/2) - \sin(p_j s_j + \xi_j - \pi/2) i_j \right] = -p_j S_{ij} \left[ \cos(p_j s_j + \xi_j) - \sin(p_j s_j + \xi_j) i_j \right] \),

since \( s_j \) and \( s_k \) are real independent variables for each \( k \neq j \), where \( \delta_{s,j} = 0 \) for \( j \neq k \), while \( \delta_{s,j} = 1 \),

3.1) \( S_{ij} \left[ \cos(p_j s_j + \xi_j) - \sin(p_j s_j + \xi_j) i_j \right] = -\frac{\partial}{\partial s_j} \left[ \cos(p_j s_j + \xi_j) - \sin(p_j s_j + \xi_j) i_j \right] \),

In the \( A \) Cartesian coordinates we take \( t_j \) instead of \( s_j \) in (3.1). If \( \phi(z) \) is a differentiable function by \( z \) for each \( j, \quad \phi : A \rightarrow A, \quad z_j = p_j + t_j + \xi_j, \) then

3.2) \( \frac{\partial \exp(-\phi(z))}{\partial t_j} = -q \left[ \frac{\partial \exp(z)}{\partial z} \right] \left( \frac{\partial \phi(z)}{\partial z} \right) p_j \)

\( = -q \left[ \sum_{k=1}^{n} \sum_{t=1}^{n-1} \left( \frac{\partial \phi(z)}{\partial z} \right) \left( \frac{\partial \phi(z)}{\partial z} \right) \right] \left( \frac{n-1}{n!} \right) \)

\( = -q \left[ (\partial \exp(-\phi(z)))/\partial s_j \right] = -p_j S_{wj} \exp(-\phi(z)), \)

where either \( q = 1 \) or \( q = -1 \), since \( \partial s_j / \partial \xi_j = 1. \)

That is

3.3) \( S_{ij} \exp(-i_j (\phi_j + \xi_j)) = 0 \) for each \( j \neq k \geq 1 \) and

any positive number \( x > 0 \).

3.4) \( S_{ij} \exp(-i_j (\phi_j + \xi_j)) = \exp(-i_j (\phi_j + \xi_j - x\pi/2)) \)

for each non-negative real number \( x \geq 0, \quad \phi_j \) and \( \xi_j \in R, \) where \( S_{ij} = S_{ij}(\xi_j), \) the zero power \( S_{ij} = I \) is the unit operator;

3.5) \( S_{wj} e^{-(p_i + \xi_i)} = e^{-(p_i + \xi_i)} \left[ i_k \delta_{j,k} \cos(p_j s_j + \xi_j) + (1 - \delta_{j,1}) i_{j-1} \sin(p_j s_j + \xi_j) \cdots \cos(p_j s_j + \xi_j) \right] \)

\[ + \left[ \sum_{k=2}^{n} k \sin(p_j s_j + \xi_j) \cdots \cos(p_k s_k + \xi_k) \right] + i_{j-1} \sin(p_j s_j + \xi_j) \cdots \sin(p_j s_j + \xi_j) \cdots \sin(p_j s_j + \xi_j) \]

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in the $A_i$ spherical coordinates, where either $q = 1$ or $q = -1$ and

$$T_j^i \zeta (\xi_j) := \zeta (\xi_j - x \pi /2)$$

for any function $\zeta (\xi_j)$ and any real number $x \in \mathbb{R}$, where $j \geq 1$. Then in accordance with Formula (3.2) we have:

$$3.7 \quad S_{\nu, j} \exp (-u (p, t; \zeta)) =$$

$$\left[ \sum_{n=1}^{\infty} \sum_{l=-\infty}^{\infty} \left( \left( \frac{\partial}{\partial t} \right)^n \left( \frac{\partial}{\partial \zeta} \right)^n \frac{u (p, t; \zeta)}{n!} \right) \right]_{\xi = \lambda (p, t; \zeta)}$$

for $u (p, t; \zeta)$ given by Formulas 1(8,8.1) in the $A_i$ Cartesian coordinates, where either $q = 1$ or $q = -1$. The integration by parts theorem (Theorem 2 in § II.2.6 on p. 228 [18]) states: if $a < b$ and two functions $f$ and $g$ are Riemann integrable on the segment $[a, b]$, $F(x) = A + \int_a^b f(t) \, dt$ and $G(x) = B + \int_a^b g(t) \, dt$, where $A$ and $B$ are two real constants, then

$$\int_a^b F(x) \, g(x) \, dx = F(x) \, G(x) \bigg|_a^b - \int_a^b f(x) \, G(x) \, dx.$$ 

Therefore, the integration by parts gives

$$\int_0^\infty \left( \frac{\partial f(t)}{\partial t} \right) \exp (-u (p, t; \zeta)) \, dt$$

$$= f(t) \exp (-u (p, t; \zeta)) \bigg|_{t=0}^{t=\infty}$$

$$- \int_0^\infty \left( \frac{\partial f(t)}{\partial \zeta} \right) \exp (-u (p, t; \zeta)) \, dt.$$

Using the change of variables $t \mapsto s$ with the Jacobian $\frac{\partial (t_1, \ldots, t_n)}{\partial (s_1, \ldots, s_n)}$ and applying the Fubini’s theorem componentwise to $f_j$, we infer:

$$5 \quad \int_{t_{1,1}^t} \cdots \int_{t_{1,1}^t} \left( \frac{\partial f(t)}{\partial \zeta} \right) \exp (-u (p, t; \zeta)) \, dt = \int_{t_{1,1}^t} \cdots \int_{t_{1,1}^t} \left( \frac{\partial f(t)}{\partial \zeta} \right) \exp (-u (p, t; \zeta)) \, ds$$

in the $A_i$ spherical coordinates, or

$$5.1 \quad \int_{t_{1,1}^t} \cdots \int_{t_{1,1}^t} \left( \frac{\partial f(t)}{\partial \zeta} \right) \exp (-u (p, t; \zeta)) \, dt$$

$$= \left[ \int_0^\infty \cdots \int_0^\infty \left( \frac{\partial f(t)}{\partial \zeta} \right) \exp (-u (p, t; \zeta)) \, ds \right]^{n \infty}$$

in the $A_i$ Cartesian coordinates, since $\frac{\partial \exp (-u (p, t; \zeta))}{\partial \zeta} = -p \exp (-u (p, t; \zeta))$ for each $1 \leq j \leq n$. This gives Formula (1), where

$$6 \quad F^{n-1, t} \left( f(t) \right) \chi_{t_{1,1}^t}, u (p, t' \zeta); p; \zeta) = \int_0^\infty \cdots \int_0^\infty f(t) \exp (-u (p, t' \zeta)) \, dt$$

is the non-commutative transform by $t' = \left(t_1, \ldots, t_{j-1}, 0, t_{j+1}, \ldots, t_n\right)$.

**Remark**

Shift operators of the form $\zeta (x + \phi) = \exp(\phi d / dx) \zeta (x)$ in real variables are also frequently used in the class of infinite differentiable functions with converging Taylor series expansion in the corresponding domain.

It is possible to use also the following convention. One can put

$$\cos (\phi + \zeta_j) = \cos (\phi + \zeta_j) \cos (\psi_2) \cdots \cos (\psi_{2-1}), \ldots,$$

$$\sin (\phi + \zeta_j) \cdots \cos (\phi + \zeta_j)$$

$$= \sin (\phi + \zeta_j) \cdots \cos (\phi + \zeta_j) \cos (\psi_{11}, \ldots) \cdots \cos (\psi_{2-1}),$$

where $\psi_j = 0$ for each $j \geq 1$, $2 \leq k < 2 - 1$, so that $T^j_j \cos (\phi + \zeta_j) = 0$ for each $j > 1$ and $l \geq 1$. Then $T^j_j \sin (\phi + \zeta_j) \cdots \cos (\phi + \zeta_j) = 0$ for each $j > k$ and $l \geq 1$, where $T^j_j \zeta = T^j_{j+1} (T^j_j \zeta)$ is the iterated composition for $l > 1, l \in N$. Then $T^j_j e^{-u (p, t; \zeta)}$ gives with such convention the same result as $S^j_j e^{-u (p, t; \zeta)}$, so one can use the symbolic notation $T^j_j e^{-u (p, t; \zeta)} = e^{-u (p, t; \zeta) - j \psi_{j+1}}$. But to avoid misunderstanding we shall use $S^j_j$ and $T^j_j$ in the sense of Formulas 12.3.1-3.7.

It is worth to mention that instead of 12(3.7) also the formulas

1) $\exp (p_1 i_1, \ldots, p_n i_n) = \cos (\phi) + M \sin (\phi)$ with $\phi := \frac{\phi}{p := \left[ p_1^2 \cdots p_n^2 \right]}$ and $M = (p_1 i_1, \ldots, p_n i_n) / \phi$ for $\phi \neq 0$, $e^0 = 1$;

2) $\frac{\partial \exp (p_1 i_1, \ldots, p_n i_n)}{\partial p_i} = \left[ -\sin (\phi) + M \cos (\phi) \right] / \phi$.

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and $\partial (p, t, \zeta) / \partial \zeta = 1$ can be used.

2.13. Theorem

Let $f(t)$ be a function-original. Suppose that

1) \( \frac{\partial F^n}{\partial p} (f(t), u; p; \zeta) \cdot h = -F^n \left( f(t) s_1, u; p; \zeta \right) h_0 - S_{n_1} F^n \left( f(t) s_1, u; p; \zeta \right) h_1 - \cdots - S_{n_n} F^n \left( f(t) s_n, u; p; \zeta \right) h_n \)

in the $A_r$ spherical coordinates, or

1.1) \( \frac{\partial F^n}{\partial p} (f(t), u; p; \zeta) \cdot h = -F^n \left( f(t) s_1, u; p; \zeta \right) h_0 - S_{n_1} F^n \left( f(t) s_1, u; p; \zeta \right) h_1 - \cdots - S_{n_n} F^n \left( f(t) s_n, u; p; \zeta \right) h_n \)

in the $A_r$ Cartesian coordinates for each

\( h = h_0 + \cdots + h_n \in A_r \), where $h_0, \ldots, h_n \in \mathbb{R}, 2^{-n} \leq n \leq 2^{-1} - 1, p \in W_f \).

**Proof.** The inequalities $a_i(f) < \text{Re}(p) < a_i(f)$ are equivalent to the inequalities $a_i(f) < \text{Re}(p) < a_i(f)$, since $\lim_{n \to \infty} \exp(-b) = 0$ for each $b > 0$. An image

\( F^n \left( f(t), u; p; \zeta \right) \) is a holomorphic function by $p$ for $a_i(f) < \text{Re}(p) < a_i(f)$ by Theorem 4, also

\( \int_0^\infty e^{-ct} t^s dt < \infty \) for each $c > 0$ and $n = 0, 1, 2, \ldots$.

Thus it is possible to differentiate under the sign of the integral:

2) \( \frac{\partial}{\partial p} \left( \int_{\mathcal{U}_c} f(t)\exp(-u(p, t, \zeta)) dt \right) = \int_{\mathcal{U}_c} f(t) \left( \frac{\partial}{\partial p} \left( \exp(-u(p, t, \zeta)) \right) \right) dt \).

Due to Formulas 12(3,3,2) we get:

3) \( \frac{\partial}{\partial p} \left( \int_{\mathcal{U}_c} f(t)\exp(-u(p, t, \zeta)) dt \right) = \int_{\mathcal{U}_c} f(t) \left( \frac{\partial}{\partial p} \left( \exp(-u(p, t, \zeta)) \right) \right) dt \).

in the $A_r$ spherical coordinates, or

4) \( \frac{\partial}{\partial p} \left( \int_{\mathcal{U}_c} f(t)\exp(-u(p, t, \zeta)) dt \right) = \int_{\mathcal{U}_c} f(t) \left( \frac{\partial}{\partial p} \left( \exp(-u(p, t, \zeta)) \right) \right) dt \).

Thus from Formulas (2,3) we deduce Formula (1).

2.14. Theorem

If $f(t)$ is a function-original, then

1) \( F^n \left( f(t - \tau), u; p; \zeta + \tau \right) = F^n \left( f(t), u; p; \zeta \right) \)

for either

i) $u(p, t, \zeta) = p_0 s_1 + M(p, t, \zeta) + \zeta_0$ or

ii) $u(p, t, \zeta) = \zeta$ over $A_r$ with $2 \leq r < \infty$ in a domain $p \in W_f$, where $\tau \in \mathbb{R}^n, 2^{-1} \leq n < 2^{-1} - 1$.

2) \( (p, \tau) = p_0 s_1 + p_0 s_2 + \cdots + p_0 s_{n_1} \)

with $s_j = s_j (n; \tau)$ for each $j$ in the first (i) and $(p, \tau) = (p, \tau)$ in the second (ii) case (see also Formulas 1(8), 2(1,2.2.1)).

**Proof.** For $p$ in the domain $\text{Re}(p) > a_i$ the identities are satisfied:

3) \( F^n \left( f(t - \tau), u; p; \zeta \right) = \int_{\mathcal{U}_c} f(t - \tau) e^{u(p, \tau)} dt \)

\( = \int_{\mathcal{U}_c} f(t) e^{u(p, \tau) - \zeta} d\zeta = F^n \left( f(t - \tau), u; p; \zeta \right) \).

due to Formulas 1(7,8) and 2(1,2.2.1.4), since

\( p_0 s_1 (n; \tau) + \zeta_0 = p_0 s_2 (n; \tau) + \zeta_0 + p_0 s_3 (n; \tau) \) and

$p_{2,1}, \zeta = p_2 \zeta_1 + (\zeta + p_{2,1})$ and

$p_{j,1}, \zeta = p_{j,1} (n; \tau) + (\zeta + p_{j,1} (n; \tau))$ for each $j = 1, \ldots, 2^{-1} - 1$, where $\tau = \zeta + \tau$. Symmetrically we get (2) for $U_{n_1}$ instead of $U_{n_1 - 1}$. Naturally, that the multiparameter non-commutative Laplace integral for an original $f$ can be considered as the sum of $2^n$ integrals by the sub-domains $U_{n_1}$:

4) \( \int_{\mathcal{U}_c} f(t) \exp(-u(p, t, \zeta)) dt \)

\( = \sum_{v \in \{-1, 1\}^n} \int_{\mathcal{U}_c} f(t) \exp(-u(p, t, \zeta)) d\zeta \).

The summation by all possible $v \in \{-1, 1\}^n$ gives Formula (1).
2.15. Note
In view of the definition of the non-commutative transform $F^n$ and $u(p,t;\zeta)$ and Theorem 14 the term $\zeta_i + \cdots + \zeta_j$ is a natural interpretation as the initial phase of a retardation.

2.16. Theorem
If $f(t)$ is a function-original with values in $A_\varepsilon$ for $t
\begin{align*}
&2) \quad F^n \left( e^{\mathcal{H}(t_i+t_{i+1})} f(t) \chi_{\mathcal{C}_n}(t), u; p; \zeta \right) = \int_{t_i}^{t_{i+1}} \left( f(t) e^{\mathcal{H}(t) \mathcal{H}(t_{i+1})} \right) \exp \left( -u(p,t;\zeta) \right) dt \\
&= \int_{t_i}^{t_{i+1}} \left( f(t) \chi_{\mathcal{C}_n}(t), u; p - b; \zeta \right)
\end{align*}
converges. Applying Decomposition 14(4) we deduce Formula 1.

2.17. Theorem
Let a function $f(t)$ be a real valued original, $F(p;\zeta) = F^n \left( f(t), u; p; \zeta \right)$, where the function $u(p,t;\zeta)$ is given by 1(8,8.1) or 2(1,2,2.1). Let also $G(p;\zeta)$ and $q(p) be locally analytic functions such that
\begin{align*}
1) \quad F^n \left( g(t,\tau); u; p; \zeta \right) = G(p;\zeta) \exp \left( -u(q(p),\tau;\zeta) \right)
\end{align*}
for $u = (p,t) + \zeta$ or $u = p_0(t_i + \cdots + t_n) + M(p,t;\zeta) + \zeta_0$, then
\begin{align*}
2) \quad F^n \left( \int_{t_i}^{t_{i+1}} g(t,\tau) f(\tau) d\tau; u; p; \zeta \right) = G(p;\zeta) F(q(p);\zeta)
\end{align*}
for each $p \in W_g$ and $q(p) \in W_f$, where $2 \leq r < \infty, 2^{r-1} \leq n \leq 2^r - 1$.
\begin{proof}
If $p \in W_g$ and $q(p) \in W_f$, then in view of the Fubini’s theorem and the theorem conditions a change of an integration order gives the equalities:
\begin{align*}
\int_{t_i}^{t_{i+1}} \left( \int_{t_i}^{t_{i+1}} f(t) dt \right) \exp \left( -u(p,t;\zeta) \right) dt
\end{align*}
for $u = p_0 + M(p,t;\zeta) + \zeta_0$, given by 2(1,2,2.1), then

2.18. Theorem
If a function $f(t) \chi_{\mathcal{C}_n}(t)$ is original together with its derivative $\partial^n f(t) \chi_{\mathcal{C}_n}(t) / \partial t_1 \cdots \partial t_n$ or $\partial^n f(t) \chi_{\mathcal{C}_n}(t) / \partial t_1 \cdots \partial t_n$, where $F^n(p;\zeta)$ is an image function of $f(t) \chi_{\mathcal{C}_n}(t)$ over the Cayley-Dickson algebra $A_{\varepsilon}$ with $2 \leq r \in N, 2^{r-1} \leq n \leq 2^r - 1$, for $u = p_0 s_1 + M(p,t;\zeta) + \zeta_0$ given by 2(1,2,2.1), then
\begin{align*}
\left| \arg(p) \right| < \pi/2 - \delta
\end{align*}
for some $0 < \delta < \pi/2, 1 \leq j \leq 2^r - 1$, $p^{(i)} = \sum_{j=0}^{n} p_j, (l) = (l_1, \cdots, l_m)$. If the restriction
Cartesian coordinates, since

\[ f(t) \big|_{j_1=0,\ldots,j_m=0} = \lim_{\varepsilon \to 0} f(t) \big|_{j_1=0,\ldots,j_m=0} = \varepsilon \big|_{j_1=\varepsilon,\ldots,j_m=\varepsilon} \]

exists for all \( 1 \leq j_1 < \cdots < j_m \leq n \), then

2) \( \lim_{p \to 0} \left[ \left(p_0 + p_1 S_{j_1} \right) \left(p_0 + p_2 S_{j_2} \right) \cdots \left(p_0 + \sum_{m=0}^{n-1} (-1)^m S_{j_m} \right) \right] F_n^n(p;\zeta) + \sum_{m=0}^{n-1} (-1)^m \]

in the \( A_e \) spherical coordinates or

2.1) \( \lim_{p \to 0} \left[ \left(p_0 + p_1 S_{j_1} \right) \left(p_0 + p_2 S_{j_2} \right) \cdots \left(p_0 + \sum_{m=0}^{n-1} (-1)^m S_{j_m} \right) \right] F_n^n(p;\zeta) + \sum_{m=0}^{n-1} (-1)^m \]

in the \( A_e \) Cartesian coordinates, where \( p \to 0 \) inside the same angle.

3) \( F^n \left( \left( \partial f(t)/\partial s_j \right) \chi_{U_{i-1}}(t),u(p,t;\zeta); \zeta \right) = \left[ p_0 \delta_{i,j} + p_1 S_{j_1} \right] F_n^n \left( f(t) \chi_{U_{i-1}}(t),u(p,t;\zeta),p;\zeta \right) \]

\[ - F_{n-1}^{n-1} \left( \left( f(t) \chi_{U_{i-1}}(t),u(p,t;\zeta);\zeta \right) \right) \]

for \( u = u(p,t;\zeta) = p_0 \delta_{i,j} + M(p,t;\zeta) + \zeta_0 \) in the \( A_e \) spherical coordinates, or

3.1) \( F^n \left( \left( \partial f(t)/\partial t_i \right) \chi_{U_{i-1}}(t),u(p,t;\zeta); \zeta \right) = \left[ p_0 + p_1 S_{j_1} \right] F_n^n \left( f(t) \chi_{U_{i-1}}(t),u(p,t;\zeta),p;\zeta \right) \]

\[ - F_{n-1}^{n-1} \left( \left( f(t) \chi_{U_{i-1}}(t),u(p,t;\zeta);\zeta \right) \right) \]

in the \( A_e \) Cartesian coordinates, since

3.2) \( \partial f(t(s))/\partial s_j = - \partial f(t)/\partial t_1 + \partial f(t)/\partial t_j \)

for each \( j \geq 2 \), \( \partial f(t(s))/\partial s_1 = \partial f(t)/\partial t_1 \),

where \( p = p_0 + p_1 + \cdots + p_{2^{i-1}} \in A_e \),

\[ p_0,\ldots,p_{2^{i-1}} \in R, \left\{ t_0,\ldots,t_{2^{i-1}} \right\} \]

are the generators of the Cayley-Dickson algebra \( A_e \), \( S_{j_i} = 0 \) for each \( i \geq 1 \), the zero power \( S_0^i = I \) is the unit operator. For short we write \( f \) instead of \( f \chi_{U_{i-1}}(t) \). Thus the limit exists:

4) \( F_{n-1}^{n-1} \left( \left( f(t) \right),u(p,t;\zeta);\zeta \right) \)

\[ \lim_{t_j \to 0} \int_0^t dr_1 \cdots \int_0^t dr_j \cdots \int_0^t dr_{j+1} \cdots \int_0^t (dr_i) \]

\[ f(t) \exp\left(-u(p,t;\zeta)\right) \]

Mention, that

\[ \left( \left( f(t) \right) \right)^{\prime\prime} = \left( 0,\ldots,0,t_1,\ldots,t_n : t_j = 0 \right) \]

for every \( 1 \leq j \leq n \), since \( t_j = s_j - s_{j+1} \) for each \( 1 \leq k \leq n \). We apply these formulas (3.4) by induction \( j = 1,\ldots,n \), \( 2^{i-1} \leq j \leq 2^i - 1 \), to \( \partial^n f(t)/\partial s_1 \cdots \partial s_n \), \( \cdots \), \( \partial^n f(t)/\partial s_1 \cdots \partial s_n \), \( \cdots \), \( \partial^n f(t)/\partial s_n \) instead of \( \partial f(t)/\partial s_j \).

From Note 8 [4] it follows, that in the \( A_e \) spherical coordinates

\[ \lim_{p \to 0} F_n^n \left( \left( \partial^n f(t)/\partial s_1 \cdots \partial s_n \right) \chi_{U_{i-1}}(t),u(p,t;\zeta);\zeta \right) = 0, \]

also in the \( A_e \) Cartesian coordinates

\[ \lim_{p \to 0} F_n^n \left( \left( \partial^n f(t)/\partial t_1 \cdots \partial t_n \right) \chi_{U_{i-1}}(t),u(p,t;\zeta);\zeta \right) = 0, \]

which gives the first statement of this theorem, since

\[ u(p,0,\zeta) = u(0,t;\zeta) = u(0,0,\zeta) \]

and
If the limit \( f(i^{(l)}) \) exists, where \( f^{(l)} := (t_1, \ldots, t_{j}, \ldots, t_n) : t_j = \infty \), then

\[ \lim_{t_j \to +\infty} f(t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n) = f(i^{(l)}) \in Y. \]

Therefore, for each \( h_1, \ldots, h_k \in R^u \) and every transposition \( \sigma : \{1, \ldots, k\} \to \{1, \ldots, k\} \), \( \sigma \) is an element of the symmetric group \( S_k \), \( t \in R^v \). For convenience one puts \( f^{(0)} = f \). In particular,

\[ f^{(i)}(t) := f^{(i)}(t) \in Y \]

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A generalized function \( g' \) prescribed by the equation

\[
g' \in \left[ D \left( R^r, L_q, (R^s, Y) \right) \right],
\]

A generalized function \( g \) is called a derivative \( g' \), of a generalized function \( g \), where \( f' \in D \left( R^r, L_q, (R^s, Y) \right) \).

Another space \( B := B \left( R^s, Y \right) \) of test functions consists of all \( R \)-differentiable functions \( f : R^s \rightarrow Y \) such that the limit \( \lim_{m \rightarrow -\infty} \left| f^{(m)} (t) \right| = 0 \) exists for each \( m = 0, 1, 2, \ldots \) and each \( 0 < R < +\infty \), where \( B \left( R, z, R \right) := \{ y \in Z : |y| < R \} \) denotes a ball with center at \( z \) and radius \( R \) in a metric space \( Z \) with a metric \( \rho \). The family of all \( R \)-linear and \( A \)-additive functionals on \( B \) is denoted by \( B' \).

In particular we can take \( X = A^\beta, \ Y = A^\alpha \) with \( 1 \leq \alpha, \beta \in Z \). Analogously spaces \( D \left( U, Y \right), \ D \left( U, Y \right) \) and \( D \left( U, Y \right) \) are defined for domains \( U \) in \( R^s \), for example, \( U = U_1 \) (see also § 1).

A generalized function \( f \in B' \) is called a generalized original, if there exist real numbers \( a_i < a_{i+1} \) such that for each \( a_i < w_1, w_2, \ldots, w_n < a_{i+1} \) the generalized function

\[
\left[ f \exp(-q_i, t), \exp \left[ -\left[ u \left( p, t; \zeta \right) - q, t \right] \right] \right] \chi_{U_v}
\]

for each \( b \in R^s \) such that

\[
a_i < b_1 + b_2 < Re(p) < b_{i+1} < a_{i+1}
\]

for each \( j = 1, \ldots, n \), because \( \exp \left( -b_i, t \right) \in R \). At the same time the real field \( R \) is the center of the Cayley-

\[
1) \quad F^n \left( \delta^{(s)}(\tau - t), u, p ; \zeta \right) = \sum_{n \in \{-1,1 \}^n} \left[ \delta^{(s)}(\tau - t), \exp \left( -q_i, t \right), \exp \left[ -\left[ u \left( p, t; \zeta \right) - q, t \right] \right] \right] \chi_{U_v}
\]

\[
= (-1)^j \frac{1}{t_i} \exp \left[ -\left[ u \left( p, t; \zeta \right) \right] \right] \chi_{U_v}
\]

since it is possible to take \( -\infty < a_i < 0 < a_{i+1} < \infty \) and \( w_k = 0 \) for each \( k \in \{-1,1,-2,\ldots, n, n \} \), where \( t \in R^s \) is the parameter, \( \tau := \frac{1}{t_0} \frac{1}{t_1} \ldots \frac{1}{t_n} \). In particular, for \( j = 0 \) we have

\[
2) \quad F^n \left( \delta(\tau - t), u, p ; \zeta \right) = \exp \left[ -u \left( p, t; \zeta \right) \right].
\]

In the general case:

\[
3) \quad F^n \left( \delta(\tau - t), \hat{x}^h, \ldots \hat{x}^k ; u, p ; \zeta \right) = \sum_{0 \leq k_1 \leq k} \left( \frac{j_i}{k_i} \right) \hat{p}^{h-k_1} \left( p_1 S_{q_1} \right) \hat{p}_2 \left( p_2 S_{c_2} \right) \ldots \left( p_n S_{a_n} \right) \exp \left( -\zeta_0 - M \left( p, 0; \zeta \right) \right)
\]

in the \( A \) spherical coordinates, or
in the $A_i$ Cartesian coordinates, where 
\[ j_1 + \cdots + j_n = |J|, \quad k_1, \ldots, k_n \text{ are nonnegative integers}, \]
\[ 2^{n-1} \leq n \leq 2^{n-1} - 1, \quad (l_m) := l! [l!(l-m)!] \text{ denotes the binomial coefficient}, \]
\[ 0! = 1, 1! = 1, 2! = 2; \]
\[ l = 1, 2, \ldots, l \text{ for each } l \geq 3, \quad s_j = s_j(n,t). \]

The transform $F^n(f)$ of any generalized function $f$ is the holomorphic function by $p \in W_f$ and by $\zeta \in A_i$, since the right side of Equation 19(5) is holomorphic by $p$ in $W_f$ and by $\zeta$ in view of Theorem 4. Equation 19(5) implies, that Theorems 11-13 are accomplished also for generalized functions.

For $a_i = a_{i-1}$ the region of convergence reduces to the vertical hyperplane in $A_i$, over $R$. For $a_{i-1} < a_i$ there is no any common domain of convergence and $f(t)$ can not be transformed.

### 2.21. Theorem

If $f(t)$ is an original function on $R^n$, $F^n(p;\zeta)$ is its image, $\partial^{\beta} f(t)/\partial s^\beta_1 \cdots \partial s^\beta_n$ or $\partial^{\beta} f(t)/\partial t^\beta_1 \cdots \partial t^\beta_n$ is an original, $|J| = j_1 + \cdots + j_n, \quad 0 \leq j_1, \ldots, j_n \in Z$, $2^{n-1} \leq n \leq 2^n - 1$; then

\[ 1) \quad F^n(\partial^{\beta} f(t)/\partial s^\beta_1 \cdots \partial s^\beta_n, u; p; \zeta) = \sum_{0 \leq j \leq j} \left( \frac{j}{k_1} \right) p^{j_1} \left( p_2 S_{n_1} \right) \left( p_2 S_{n_2} \right) \cdots \left( p_2 S_{n_n} \right) F^n(f(t), u; p; \zeta) \]

for $u(p,t;\zeta) = p_0 s_1 + M(p,t;\zeta) + \xi_0$ given by 2(1,2,2,1), or

\[ 1.1) \quad F^n(\partial^{\beta} f(t)/\partial t^\beta_1 \cdots \partial t^\beta_n, u; p; \zeta) = \left( p_0 + p_2 S_{n_1} \right) \left( p_0 + p_2 S_{n_2} \right) \cdots \left( p_0 + p_2 S_{n_n} \right) F^n(f(t), u; p; \zeta) \]

for $u(p,t;\zeta)$ given by 1(8,8.1) over the Cayley- Dickson algebra $A_i$ with $2 \leq r < \infty$. Domains, where Formulas (1,1.1) are true may be different from a domain of the multiparameter noncommutative transform for $f$, but they are satisfied in the domain $a_i < Re(p) < a_{i-1}$, where

$$ a_i = \min(\{ a_i(f), a_{i-1}(\partial^{m_1} f(t)/\partial s^m_1 \cdots \partial s^m_n) : |m| \leq |J|, 0 \leq m_j \leq j \forall i \}) $$

$$ a_{i-1} = \max(\{ a_{i-1}(f), a_i(\partial^{m_1} f(t)/\partial s^m_1 \cdots \partial s^m_n) : |m| \leq |K|, 0 \leq m_i \leq j_i \forall i \}), $$

if $a_i < a_{i-1}$, where $\phi_j = s_j$ or $\phi_j = t_j$ for each $j$ correspondingly.

**Proof.** To each domain $U_v$ the domain $U_{v^*}$ symmetrically corresponds. The number of different vectors $v \in \{-1,1\}^n$ is even $2^n$. Therefore, for

$$ u = p_0 + \xi_0 + M(p,t;\zeta) $$

due to Theorem 12 the equality

$$ 2) \int_{\mathbb{R^n}} \left( \partial^j f(t)/\partial s_j \right) e^{-u(p, r, v)} ds = \int_{\mathbb{R^n}} \left( \partial^j f(t)/\partial s_j \right) e^{-u(p, r, v)} dt $$

$$ = \int_{\mathbb{R^n}} \left( f(t) e^{-u(p, r, v)} \right) ds - \int_{\mathbb{R^n}} \left( f(t) e^{-u(p, r, v)} \right) ds $$

is satisfied in the $A_i$ spherical coordinates, since the absolute value of the Jacobian $|\hat{t}|/|\hat{t}(t,s_j)|$ is unit. Since for $a_i < Re(p) < a_{i-1}$ the first additive is zero, while the second convert integrals with the help of Formulas 12(2,2.1), Formula (1) follows for $k = 1$:

$$ 3) \quad F^n(\partial^j f(t)/\partial s_j, u; p; \zeta) = $$

$$ = p_0 \delta_{i,j} \cdot F^n(f(t), u; p; \zeta) + p_s F^n(f(t), u; p; \zeta) $$

To accomplish the derivation we use Theorem 14 so that

$$ \lim_{\tau \to 0} \left[ F^n(f(t), u; p; \zeta) - F^n(f(t - \tau e)), u; p; \zeta) \right] / \tau $$

$$ = \lim_{\tau \to 0} \left[ F^n(f(t), u; p; \zeta) - F^n(f(t), u; p; \zeta + \tau(p_0 + p_1 + \cdots + p_j)) \right] / \tau $$

$$ = \lim_{\tau \to 0} \int_{\mathbb{R^n}} f(t) e^{-u(p, r, v)} - e^{-u(p, r, v + \tau(p_0 + p_1 + \cdots + p_j))) ds } \tau^{-1} dt, $$

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where $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$ with 1 on the $j$-th place. If the original $\partial^{[j]} f(t)/\partial s^j \exists \mathcal{A} \mathcal{F} \mathcal{S}$, then $\partial^{[m]} f(t)/\partial s^m \exists \mathcal{A} \mathcal{F} \mathcal{S}$ is continuous for $0 \leq |m| \leq |j| - 1$ with $0 \leq m \leq j$, for each $l = 1, \ldots, n$, where $f^0 := f$.

The interchanging of $\lim_{r \to 0}$ and $\int_{|t| = \epsilon_0}$ may change a domain of convergence, but in the indicated in the theorem domain $a_0 < \Re (p) < a_1$, when it is non void, Formula (3) is valid. Applying Formula (3) in the $A_j$ spherical coordinates by induction to

$$ (\partial^{[m]} f(t) / \partial s^m)^{\mathcal{A}_j} : |m| \leq |j|, 0 \leq m \leq j, \forall l $$

with the corresponding order subordinated to $\partial^{[m]} f(t) / \partial s^m$, or in the $A_{n}$ Cartesian coordinates using Formula (12.1) for the partial derivatives

$$ (\partial^{[m]} f(t) / \partial s^m)^{\mathcal{A}_j} : |m| \leq |j|, 0 \leq m \leq j, \forall l $$

with the corresponding order subordinated to $\partial^{[m]} f(t) / \partial t^m$ we deduce Expressions (1) and (1.1) with the help of Statement 6 from § XVII.2.3 [19] about the differentiation of an improper integral by a parameter and § 2.

2.22. Remarks

For the entire Euclidean space $\mathbb{R}^n$ Theorem 21 for $\partial f(t) / \partial s^j$ gives only one or two additives on the right side of 21(1) in accordance with 21(3).

Evidently Theorems 4, 11 and Proposition 10 are accomplished for $F^{\mathcal{A}_j}(\mathbb{R}^n - \emptyset, \mathcal{D}) (f, u, p; \zeta)$ also.

Theorem 12 is satisfied for $F^{\mathcal{A}_j}(\mathbb{R}^n - \emptyset, \mathcal{D})$ and any $j \in \{1, \ldots, k\}$, so that $s_j = s_j(k; t) = t_j + \ldots + t_{k(j)}$ for each $1 \leq l \leq k$, $p_0 = 0$ and $\zeta_0 = 0$ for each $1 \leq m \not\in \{1, \ldots, k\}$ (the same convention is in 13, 14, 17, 21, see also below). For $F^{\mathcal{A}_j}(\mathbb{R}^n - \emptyset, \mathcal{D})$ in Theorem 13 Formula 13(1) it is natural to put $t_a = 0$ and $h_b = 0$ for each $1 \leq m \not\in \{1, \ldots, k\}$, so that only $(k + 1)$ additives with $h_{b_1}, h_{b_{2(j)}, \ldots, b_{k(j)}}$ on the right side generally may remain. Theorems 14 and 17 and 21 modify for $F^{\mathcal{A}_j}(\mathbb{R}^n - \emptyset, \mathcal{D})$ putting in 14(1) and 17(1,2) and 21(1) $t_j = 0$ and $t_j = 0$ respectively for each $j \not\in \{1, \ldots, k\}$.

To take into account boundary conditions for domains different from $U$, for example, for bounded domains $V$ in $\mathbb{R}^n$ we consider a bounded noncommutative multiparameter transform

$$ F^0 (f(t), u; p; \zeta) = F^0 (f(t), u; p; \zeta). $$

For it evidently Theorems 4, 6-8, 11, 13, 14, 16, 17, Proposition 10 and Corollary 4.1 are satisfied as well taking specific originals $f$ with supports in $V$.

At first take domains $W$ which are quadrants, that is canonical closed subsets affine diffeomorphic with $Q^a = \prod_{j=1}^n [a_j, b_j]$, where $-\infty < a_j < b_j < \infty$, $[a_j, b_j] := \{x \in R : a_j \leq x \leq b_j\}$ denotes the segment in $R$. This means that there exists a vector $w \in \mathbb{R}^n$ and a linear invertible mapping $C$ on $\mathbb{R}^n$ so that $C(W) = Q$. We put $t^1 := (t_1, \ldots, t_n)$, $t^2 := (t_1, \ldots, t_n : t_j = a_j)$, $t^1 := (t_1, \ldots, t_n : t_j = b_j)$. Consider $t = (t_1, \ldots, t_n) \in Q^a$.

2.23. Theorem

Let $f(t)$ be a function-original with a support by $t$ variables in $Q^a$ and zero outside $Q^a$ such that $\partial f(t) / \partial t_j$ also satisfies Conditions 1(1-4). Suppose that $u(p, t; \zeta)$ is given by 2(1,2,2,1) or 1(8,8.1) over $A$, with $2 \leq r < \infty$, $2^{\frac{1}{r}} \leq n \leq 2^{\frac{1}{r}} - 1$. Then

$$ F^a \left( \left( \frac{\partial f(t)}{\partial t_j} \right) \chi_{Q^a}(t), u; p; \zeta \right) = F^{a+1; 1,2} \left( f(t^1), u; p; \zeta \right) - F^{a+1; 1,2} \left( f(t^1), u; p; \zeta \right) + \left[ p_0 + \sum p_j S_j \right] F^a \left( f(t), \chi_{Q^a}(t), u; p; \zeta \right) $$

in the $A_{j}$ spherical coordinates, or

$$ F^{a+1; 1,2} \left( f(t^1), u; p; \zeta \right) - F^{a+1; 1,2} \left( f(t^1), u; p; \zeta \right) + \left[ p_0 + \sum p_j S_j \right] F^a \left( f(t), \chi_{Q^a}(t), u; p; \zeta \right) $$

in the $A_{j}$ Cartesian coordinates in a domain $W \subset A_j$; if $a_j = -\infty$ or $b_j = +\infty$, then the addendum with $t^1$ or $t^2$ correspondingly is zero.

Proof. Here the domain $Q^a$ is bounded and $f$ is almost everywhere continuous and satisfies Conditions 1(1-4), hence $f(t) \exp(-u(p, t; \zeta)) \in L^1 (\mathbb{R}^n, \lambda_n, A)$ for each $p \in A$, since $\exp(-u(p, t; \zeta))$ is continuous and $\text{supp}(f(t)) \subset Q^a$.

Analogously to § 12 the integration by parts gives

$$ \int_{a_j}^{b_j} \left( \frac{\partial f(t)}{\partial t_j} \right) \exp(-u(p, t; \zeta)) dt_j = f(t) \exp(-u(p, t; \zeta)) \left[ \int_{a_j}^{b_j} \left( \frac{\partial}{\partial t_j} \exp(-u(p, t; \zeta)) \right) dt_j \right], $$

where $t = (t_1, \ldots, t_n)$. Then the Fubini's theorem implies:
3) \(\int_{\Omega} \frac{\partial f}{\partial t} \exp(-u(p,t,t_\xi)) \, dt = \int_{\Omega} \cdots \int_{\Omega} \int_{\Omega} \int_{\Omega} \left[ \left( \frac{\partial f}{\partial t} \right) \exp(-u(p,t,t_\xi)) \right] \, dt' \)

\[= \left[ \int_{\Omega} \frac{\partial f}{\partial t} \exp(-u(p,t^1,\xi)) \, dt' \right] - \left[ \int_{\Omega} \frac{\partial f}{\partial t^1} \exp(-u(p,t^1,\xi)) \, dt' \right] + \left[ p_j + p_{\Sigma^j} \right] \int_{\Omega} \cdots \int_{\Omega} \int_{\Omega} f(t) \exp(-u(p,t,t_\xi)) \, dt \]

in the \( A_r \) spherical coordinates or

3.1) \(\int_{\Omega} \frac{\partial f}{\partial t} \exp(-u(p,t,t_\xi)) \, dt \)

\[= \left[ \int_{\Omega} \frac{\partial f}{\partial t} \exp(-u(p,t^1,\xi)) \, dt' \right] - \left[ \int_{\Omega} \frac{\partial f}{\partial t^1} \exp(-u(p,t^1,\xi)) \, dt' \right] + \left[ p_j + p_{\Sigma^j} \right] \int_{\Omega} \cdots \int_{\Omega} f(t) \exp(-u(p,t,t_\xi)) \, dt \]

in the \( A_r \) Cartesian coordinates, where as usually

\(t^r = \{1, \ldots, t_{j_1}, 0, t_{j_2}, \ldots, t_n\}, \quad dt^r = dt_{j_1} \cdots dt_{j_n} \).

This gives Formulas (1.1.1), where

4) \(F^{n-1/j,k} \left( f(t^{j,k}) \mathcal{X}_{\xi} \right) (p^{t,t^1}; \xi; p; \xi) \)

\[= \int_{\Omega} \cdots \int_{\Omega} \int_{\Omega} f(t^{j,k}) \exp(-u(p,t^{j,k}; \xi)) \, dt^{j,k} \]

is the non-commutative transform by \( t^{j,k} \),

\(2^{r-1} \leq n \leq 2^{r-1} - 1, \quad dt^{j,k} \) is the Lebesgue volume element

on \( R^{n-1} \).

1) \(\lim_{p \to 0^+} \left[ \left[ p_0 + p_{\Sigma^j} \right] p_2 S_2 \cdots p_n S_n \right] \)

\[= \sum_{1 \leq j_1 < \ldots < j_m \leq n} \left[ p_0 \delta_{j_1,j_2} + p_{j_1} S_{j_2} \right] p_{j_2} \cdots p_{j_m} F^{n-m}_u \left( p^{(j_m)}; \xi \right) = (-1)^{n-m} f(0) e^{-\alpha(0,\xi)} \]

in the \( A_r \) spherical coordinates, or

1.1) \(\lim_{p \to 0^+} \left[ \left[ p_0 + p_{\Sigma^j} \right] \cdots \left[ p_0 + p_{\Sigma^j} \right] \right] \)

\[= \sum_{1 \leq j_1 < \ldots < j_m \leq n} \left[ p_0 + p_{j_1} S_{j_2} \right] \cdots \left[ p_0 + p_{j_m} S_{j_m} \right] F^{m-n}_u \left( p^{(j_m)}; \xi \right) = (-1)^{n-m} f(0) e^{-\alpha(0,\xi)} \]

in the \( A_r \) Cartesian coordinates, where

\( f(0) = \lim_{\mathcal{X}_{\xi} \to 0} f(t) \),

\( p \) tends to the infinity inside the angle

2) \(F^{n-1/j,k} \left( f(t^{j,k}) \mathcal{X}_{\xi} \right) (p^{t,t^1}; \xi; p; \xi) \)

\[= \lim_{t_{j,k} \to 0^+} \int_{\Omega} \cdots \int_{\Omega} \int_{\Omega} \left[ \left( f(t^{j,k}) \right) \exp(-u(p,t^{j,k}; \xi)) \right] \, dt \]

where \( \beta_{j_1} = a_i = 0, \quad \beta_{j_2} = b_j > 0, \quad k = 1,2 \).

Mention, that

\( \beta_{j_1} = a_i = 0, \quad \beta_{j_2} = b_j > 0, \quad k = 1,2 \).

for every \( 1 \leq j \leq n \).

Analogously to § 12 we apply

2.24. Theorem

If a function \( f(t) \mathcal{X}_{\xi} \) (t) is original together with its derivative \( \frac{\partial f(t) \mathcal{X}_{\xi} (t)}{\partial \xi_1 \cdots \partial \xi_n} \) or

\( \frac{\partial f(t) \mathcal{X}_{\xi} (t)}{\partial \xi_1 \cdots \partial \xi_n} \), where \( F^{n}_u(p; \xi) \) is an image function of \( f(t) \mathcal{X}_{\xi} \) (t) over the Cayley-Dickson algebra \( A_r \) with \( 2 \leq r \leq 4 \), \( 2^{r-1} \leq n \leq 2^{r-1} - 1 \), for the function \( u(p,t; \xi) \) given by \( 21(1.2.1) \) or \( 1(8.8.1) \), 

\( Q^c = \{1\} \in \mathcal{C} \mid 0 \leq b_j \), \( b_j > 0 \) for each \( j \), then

| Arg(p) | < \pi/2 - \delta | for some 0 < \delta < \pi/2. |

Proof. In accordance with Theorem 23 we have

Equalities 23(1.1.1). Therefore we infer that

Formula (2) by induction \( j = 1, \ldots, n \), \( 2^{r-1} \leq n \leq 2^{r-1} - 1 \), to

\( \partial^n f(t(s)) / \partial \xi_1 \cdots \partial \xi_n \cdots \partial^n f(t(s)) / \partial \xi_1 \cdots \partial \xi_n \)}
instead of $\partial f(t(s))/\partial s_j$, $s_j = s_j(n,t)$ as in § 2, or applying to the partial derivatives

$$\partial^nf(t)/\partial t_1\cdots\partial t_n$$

instead of $\partial f(t)/\partial t_i$ correspondingly. If $s_j > 0$ for some $j \geq 1$, then $s_j > 0$ for $Q^n$ and

$$\lim_{p \to +e^\infty} e^{-u(p; f^{(t)})} = 0 \text{ for such } \mu(t)$$

where

$$t = (t_1, \ldots, t_n), \ (l) = (l_1, \ldots, l_n), \ |l| = l_1 + \ldots + l_n, \ t_i^{(j)} = a_j \ \text{for } l_j = 1 \text{ and } t_i^{(j)} = b_j$$

for $l_j = 2, 1 \leq j \leq 2^n - 1$. Therefore,

$$\lim_{p \to +e^\infty} \sum_{j=1; \mu(t) \neq 0}^{j=2^n - 1} (-1)^{j} f^{(t)} e^{-u(p; f^{(t)})}$$

$$= (-1)^n f(0) e^{-u(0,0,0,\ldots)}$$

since $u(p; 0,0) = u(0,0,0,\ldots)$, where

$$f(t) = \lim_{t \to +e^\infty}{f(t)}.$$

In accordance with Note 8 [4]

$$\lim_{p \to +e^\infty} \sum_{l \to +e^\infty} \sum_{j=1}^{j=2^n - 1} (-1)^{j} f^{(t)} e^{-u(p; f^{(t)})}$$

in the $A_i$ spherical coordinates and

$$\lim_{p \to +e^\infty} \sum_{l \to +e^\infty} \sum_{j=1}^{j=2^n - 1} (-1)^{j} f^{(t)} e^{-u(p; f^{(t)})}$$

in the $A_i$ spherical coordinates, which gives the statement of the theorem.

2.25. Theorem

Suppose that $f(t) \chi_p^{(t)}(t)$ is an original function, $F^n(p; \zeta)$ is its image, $\partial^nf(t) \chi_p^{(t)}(t)/\partial t_1\cdots\partial t_n$ is an original, $\mu(t) = j_1 + \ldots + j_n, 0 \leq j_1 + \ldots + j_n \in Z$, $2^{-1} \leq n' \leq 2^{n-1}$, $-\infty < a_k < b_k < \infty$ for each $k = 1,\ldots,n$, $(l) = (l_1,\ldots,l_n), l_n \in \{0,1,2\}$, $W = A_i$, for bounded $Q^n$. Let $W = \{ p \in A_i : a_k < R(e) \}$ for $b_k = +\infty$ and some $k$ and finite $a_k$ for each $k$; $W = \{ p \in A_i : R(e) < a_k < +\infty \}$ when $a_k = -\infty$ and $b_k = +\infty$ for some $k$ and $1; \ \mu(t) = (t_1^{(j)},\ldots,t_n^{(j)})$.

We put $t_i^{(j)} = t_i$ and $a_k = 0$ for $l_i = 0$, $t_i^{(j)} = b_k$ for $l_i = 2$, $(q) = (q_1,\ldots,q_n)$, $a_1, a_2, \ldots, a_n$ the operators $a_k > -\infty$

$$T_{(m)}F(p; \zeta) := F(p; \zeta - (i,m_1 + \ldots + i,m_n)\pi/2),$$

also the operator $a_k > -\infty$

$$(SO) S_{(m)}F(p; \zeta) = S_{(m)}^{(1)} \cdots S_{(m)}^{(n)} F(p; \zeta),$$

where $(m) = (m_1,\ldots,m_n) \in \{0,\infty\}^n \subset R^n$, $S_{(m)} = S_{(m)}^{(1)} \ldots S_{(m)}^{(n)}$ for each positive number $0 < k \in R$, $S_0 = I$ is the unit operator for $(m) = 0$ (see also Formulas 12(3.1-3.7)). As usually let $e_i = (1,0,\ldots,0), \ldots, e_n = (0,\ldots,0,1)$ be the standard orthonormal basis in $R^n$ so that $(m) = m_1 e_1 + \ldots + m_n e_n$.

Theorem. Then

$$T_{(m)}F(p; \zeta) := F(p; \zeta - (i,m_1 + \ldots + i,m_n)\pi/2),$$

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for \( u(p,t,\zeta) \) in the \( A \) spherical coordinates or the \( A \) Cartesian coordinates over the Cayley-Dickson algebra \( A \) with \( 2 \leq r < \infty \), where

\[
R_{x_1} := p_0 + p_1 S_{x_1}, \ldots, R_{x_n} := p_0 + p_n S_{x_n}
\]
in the \( A \) spherical coordinates, while

\[
R_{x_1} = f(t) \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left( \frac{\partial^{m_j+1} f(t)}{\partial t^{m_j}} \cdots \frac{\partial^{m_{j+1}} f(t)}{\partial t^{m_{j+1}}} \cdots \frac{\partial^{m_n} f(t)}{\partial t^{m_n}} \right) e^{-\alpha(p,\zeta)} dt
\]
is satisfied for \( 0 \leq m_k \leq j_k \) for each \( k = 1, \ldots, n \) with \( |m| < |j| \). On the other hand, for \( p \in W \) additives on the right of (2) convert with the help of Formula 23(1). Each term of the form

\[
\int_{[a,b]} \left( \frac{\partial^{m_j+1} f(t)}{\partial t^{m_j}} \cdots \frac{\partial^{m_{j+1}} f(t)}{\partial t^{m_{j+1}}} \cdots \frac{\partial^{m_n} f(t)}{\partial t^{m_n}} \right) e^{-\alpha(p,\zeta)} dt
\]
can be further transformed with the help of (2) by the considered variable \( t_k \) only in the case \( t_k = 0 \).

Applying Formula (2) by induction to partial derivatives

\[
\frac{\partial^{m_j+1} f(t)}{\partial t^{m_j}} \cdots \frac{\partial^{m_{j+1}} f(t)}{\partial t^{m_{j+1}}} \cdots \frac{\partial^{m_n} f(t)}{\partial t^{m_n}}
\]
as in § 21 and using Theorem

\[
F^k \left( f(t) \chi_{U_{1,\xi}}(t), u; p; \zeta \right) = F_{k_1} R_{k_2} \cdots F_{k_n} \left( g(t), u; p; \zeta \right)
\]
is satisfied, since

\[
\frac{\partial^{m_j+1} f(t)}{\partial t^{m_j}} \cdots \frac{\partial^{m_{j+1}} f(t)}{\partial t^{m_{j+1}}} \cdots \frac{\partial^{m_n} f(t)}{\partial t^{m_n}} = \left( f \chi_{U_{1,\xi}}(t) \right)
\]
where \( j_1 = 1, \ldots, j_k = 1, j_k = 1 \). On the other hand, \( g(t) \) is equal to zero on \( \partial U_{1,\zeta} \) and outside \( U_{1,\zeta} \) in accordance with formula (1), hence all terms on the right side of Equation (3) with \( |j| > 0 \) disappear and \( \text{supp} (g(t)) \subset U_{1,\zeta} \). Thus we get Equation (2).

### 2.27. Theorem

Suppose that \( F^k \left( p; \zeta \right) \) is an image

\[
F^{\xi_{j_1,\eta_{j_1}} \cdots \xi_{j_k,\eta_{j_k}}} \left( f(t) \chi_{U_{1,\xi}}(t), u; p; \zeta \right)
\]
of an original function \( f(t) \) for \( u \) given by 2(1,2,2,1) in the half space

\[
W := \left\{ p \in A : \text{Re}(p) > a_i \right\}
\]
with \( 2 \leq r < \infty \), where

\[
p_i = 0, \ldots, p_{-1} = 0; \quad \xi_j = \pi/2, \ldots, \xi_{j_k} = \pi/2
\]
for each \( j \geq 2 \) in the \( A \) spherical coordinates or

\[
\zeta_i = 0, \ldots, \zeta_{j_k} = 0
\]
for each \( j \geq 2 \) in the \( A \) Cartesian coordinates.

1.2), \( R_{x_1} := p_0 + p_1 S_{x_1}, \ldots, R_{x_n} := p_0 + p_n S_{x_n} \) in the \( A \) Cartesian coordinates, i.e. \( R_{x_1}(p) \) are operators depending on the parameter \( p \). If \( t^{(j)} \to \infty \) for some \( 1 \leq j \leq n \), then the corresponding addendum on the right of (1) is zero.

**Proof.** In view of Theorem 23 we get the equality

\[
\text{eq}
\]

14 and Remarks 22 we deduce (1).

### 2.26. Theorem

Let \( f(t) \chi_{U_{1,\xi}}(t) \) be a function-original with values in \( A \) with \( 2 \leq r < \infty \), \( 2^{-1} \leq n \leq 2^{-1} \), \( u \) is given by 2(1,2,2,1) or 1(8,8,1),

1) \( g(t) := \int_0^t \cdots \int_0^t f(x) dx \), then

\[
F^k \left( f \chi_{U_{1,\xi}}(t), u; p; \zeta \right)
\]
\( = R_1 R_2 \cdots R_n F^k \left( g(t), u; p; \zeta \right) \)
in the domain \( \text{Re}(p) > \max(a_i,0) \), where the operators \( R_i \) are given by Formulas 25(1,1,1,2).

**Proof.** In view of Theorem 25 the equation
where \( p_i = 0, \cdots, p_{j-1} = 0 \) for each \( j \geq 2 \); 
\( \zeta_1 = \pi/2, \cdots, \zeta_{j-1} = \pi/2 \) and \( \xi_j = t_j (k; t) \) in the \( A \) spherical coordinates, while \( \zeta_1 = 0, \cdots, \zeta_{j-1} = 0 \) and \( \xi_j = t_j \) in the \( A \) Cartesian coordinates correspondingly for each \( j \geq 1 \).

**Proof.** Take a path of an integration belonging to the half space \( Re(p) \geq w \) for some constant \( w > a_i \). Then

\[
\left| \int_{t_1, \cdots, t_k} f(t) \exp(-u(p, t; \zeta)) \right| dt \\
\leq C \left| \int_{t_1, \cdots, t_k} \exp(- (p_0 - a_i)(t_1 + \cdots + t_k)) \right| dt < \infty
\]

converges, where \( C = \text{const} > 0 \), \( p_0 \geq w \). For \( t_j > 0 \) for each \( j = 1, \cdots, k \) conditions of Lemma 2.23 [4] (that is of the noncommutative analog over \( A \) of Jordan’s lemma) are satisfied. If \( t_j \to \infty \), then \( s_j \to \infty \), since all \( t_1, \cdots, t_k \) are non-negative. Up to a set \( \partial U_{l_1, \cdots, l_k} \) of \( A_k \) Lebesgue measure zero we can consider that \( t_1 > 0, \cdots, t_k > 0 \). If \( s_j \to \infty \), then also \( s_i \to \infty \). The converging integral can be written as the following limit:

\[
5) \int_{p_{j, j}^0} \int_{p_{j, j}^1} F^k(p_0 + z; \zeta) dz \\
= \lim_{r \to 0} \int_{p_{j, j}^0} \int_{p_{j, j}^1} \left( f(t) \exp(-u(p_0 + z; t; \zeta)) \right) dt dz
\]

for \( 1 \leq j \leq k \), since the integral \( \int_{S^1} F^k (w + z; \zeta) \) is absolutely converging and the limit \( \lim_{r \to 0} \exp(-\kappa |z|) = 1 \) uniformly by \( z \) on each compact subset in \( A \), where \( S \) is a purely imaginary marked Cayley-Dickson number with \( |S| = 1 \). Therefore, in the integral

\[
6) \int_{p_{j, j}^0} \int_{p_{j, j}^1} F^k(p_0 + z; \zeta) dz \\
= \int_{p_{j, j}^0} \int_{p_{j, j}^1} \left( f(t) \exp(-u(p_0 + z; t; \zeta)) \right) dt dz
\]

the order of the integration can be changed according to the formula

\[
A \left[ f(t) \right](t) = g(t)
\]

Consider a partial differential equation of the form:

1) \( A[f](t) = g(t) \), where

2) \( A[f](t) := \sum_{|a| = 0} a_j(t) \left( \partial^{\alpha} f(t) / \partial t^{k_a} \right) \)

where \( a_j \) are continuous functions, \( k_a = a_1 + \cdots + a_n \) with \( a_i \in R \) for each \( i = 0, \cdots, n \);

3) \( A[f](t) := \sum_{|a| = 0} b_j(t) \left( \partial^{\alpha} f(t) / \partial t^{k_a} \right) \)

where \( b_j(t) \) are continuous functions, \( k_a = a_1 + \cdots + a_n \) with \( a_i \in R \) for each \( i = 0, \cdots, n \), the operator \( A \) can be rewritten in terms coordinates as

\[
A[f](t) := \sum_{|a| = 0} b_j(t) \left( \partial^{\alpha} f(t(s)) / \partial x^{k_a} \right)
\]

That is, there exists \( b_j \neq 0 \) for some \( j \) with \( |a| = \alpha \) and \( b_0 = 0 \) for \( |j| > \alpha \), while a function \( \sum_{|a| = 0} b_j(t(s)) \partial^{\alpha} f(t(s)) / \partial x^{k_a} \) is not zero identically on the corresponding domain \( V \). We consider that

\[
(D1) \ U \text{ is a canonical closed subset in the Euclidean space } R^n, \text{ that is } U = cI (\text{Int}(U)) \text{, where Int}(U) \text{ denotes the interior of } U \text{ and } cI (U) \text{ denotes the closure of } U.
\]

Particularly, the entire space \( R^n \) may also be taken.
Under the linear mapping \((t_1, \ldots, t_n) \mapsto (s_1, \ldots, s_n)\) the domain \(U\) transforms onto \(V\).

We consider a manifold \(W\) satisfying the following conditions (i-\(\nu\)).

i. The manifold \(W\) is continuous and piecewise \(C^\alpha\), where \(C^\alpha\) denotes the family of \(l\) times continuously differentiable functions. This means by the definition that \(W\) as the manifold is of class \(C^\alpha \cap C^\infty\). That is \(W\) is of class \(C^\alpha\) on open subsets \(W_{a,i}\) and \(W_i \setminus \bigcup \{W_{a,j}\}\) has a codimension not less than one in \(W\).

ii. \(W = \bigcup_{i=1}^n W_i\), where \(W_i = \bigcup_{a,j} W_{a,j}\). \(W_i \cap W_j = \emptyset\) for each \(k \neq j\), \(m = \dim W\), \(\dim W_j = m - j\), \(W_{a,i} \subset \partial W_i\).

iii. Each \(W_j\) with \(j = 0, \ldots, m-1\) is an oriented \(C^\alpha\) manifold, \(W_j\) is open in \(\bigcup_{k \neq j} W_k\). An orientation of \(W_{a,i}\) is consistent with that of \(\partial W_i\) for each \(j = 0, \ldots, m-2\). For \(j > 0\) the set \(W_j\) is allowed to be void or non-void.

iv. A sequence \(W^k\) of \(C^\alpha\) orientable manifolds embedded into \(R^n\), \(\alpha \geq 1\), exists such that \(W^k\) uniformly converges to \(W\) on each compact subset in \(R^n\) relative to the metric \(\operatorname{dist}\).

For two subsets \(B\) and \(E\) in a metric space \(X\) with a metric \(\rho\) we put

\[\operatorname{dist}(B, E) := \max \{\sup_{b \in B} \operatorname{dist}\{b, E\}, \sup_{e \in E} \operatorname{dist}\{B, \{e\}\}\},\]

where \(\operatorname{dist}\{b, E\} := \inf_{b \in B} \rho(b, e)\), \(\operatorname{dist}\{B, \{e\}\} := \inf_{b \in B} \rho(b, e)\), \(b \in B\), \(e \in E\).

Generally, \(\dim W = m \leq n\). Let \((e^1(x), \ldots, e^m(x))\) be a basis in the tangent space \(T_x W^k\) at \(x \in W^k\) consistent with the orientation of \(W^k\), \(k \in N\).

We suppose that the sequence of orientation frames \((e^1(x), \ldots, e^m(x))\) of \(W^k\) at \(x\) converges to \((e^1(x), \ldots, e^m(x))\) for each \(x \in W\), \(\lim_{x \to x_0} e_i(x) = e_i(x_0)\), where \(\lim_{x \to x_0} x = x_0 \in W\), while \(e_i(x), \ldots, e_m(x)\) are linearly independent vectors in \(R^n\).

v. Let a sequence of Riemann volume elements \(\lambda_k\) on \(W^k\) (see \S\ XIII.2.1 [19]) induce a limit volume element \(\lambda\) on \(W\), that is,

\[\lambda(B \cap W) = \lim_{k \to \infty} \lambda_k(B \cap W^k)\]

for each compact canonical closed subset \(B \subset R^n\), consequently,

\[\lambda(W \setminus W_0) = 0\].

We shall consider surface integrals of the second kind, i.e. by the oriented surface \(W\) (see (iv)), where each \(W_j\), \(j = 0, \ldots, m-1\) is oriented (see also \S\ XIII.2.5 [19]).

vi. Let a vector \(w \in \text{Int}(U)\) exist so that \(U - w\) is convex in \(R^n\) and let \(\partial U\) be connected. Suppose that a boundary \(\partial U\) of \(U\) satisfies Conditions (i-v) and,

\[\text{supp}(g) \subset U\]

then \(\text{supp}(g X U_1) \subset U_1\). Therefore, any problem (1) on \(U_1\) can be considered as the restriction of the problem (1) defined on \(U\), satisfying (D1D4, i-vii). Any solution \(f\) of (1) on \(U\) with the boundary conditions on \(\partial U\) gives the solution as the restriction \(f|_{U_1}\) on \(U_1\) with the boundary conditions on \(\partial U\).

Henceforward, we suppose that the domain \(U\) satisfies Conditions (D1D4, i-vii), which are rather mild and natural. In particular, for \(Q^n\) this means that either \(a_k = -\infty\) or \(b_k = +\infty\) for each \(k\). Another example is: \(U_1\) is a ball in \(R^n\) with the center at zero,

\[U = U_1 \cup \{R^n \setminus U_{i-k}\}, \quad w_i = 0\;\text{or}\; \infty\]

is consistent with the Lebesgue measure on \(U^k\) induced from \(R^n\) for each \(k\). This induces the measure \(\lambda\) on \(\partial U\) as in (v).

Also the boundary conditions are imposed:

\[f(t)|_{U_0} = f_0(t),\]

\[\left(\partial^k f(t)/\partial s^a_1 \cdots \partial s^a_n\right)|_{U_0} = f_0(t),\]

for \(|q| \leq \alpha - 1\), where \(s = (s_1, \ldots, s_n) \in R^n\), \((q) = (q_1, \ldots, q_n)\), \(|q| = q_1 + \cdots + q_n\), \(0 \leq q_k \in Z\) for each \(k\), \(t \in \partial U\) is denoted by \(t^i\). \(f_0, \ f_0, \ f_0\) are given functions. Generally these conditions may be excessive, so one uses some of them or their linear combinations (see (5.1) below). Frequently, the boundary conditions

\[f(t)|_{U_0} = f_0(t),\]

\[\left(\partial^k f(t)/\partial v^j\right)|_{U_0} = f_j(t),\]

for \(1 \leq \alpha \leq \alpha - 1\) are also used, where \(v\) denotes a real variable along a unit external normal to the boundary \(\partial U\) at a point \(t^i \in \partial U_0\). Using partial differentiation in local coordinates on \(\partial U\) and (5) one can calculate in principle all other boundary conditions in (4) almost everywhere on \(\partial U\).

Suppose that a domain \(U_1\) and its boundary \(\partial U_1\) satisfy Conditions (D1, i-vii) and \(g_i = g X U_1\) is an original on \(R^n\) with its support in \(U_1\). Then any original \(g\) on \(R^n\) gives the original \(g X U_2\) on \(R^n\), where \(U_2 = \cap U_1\). Therefore, \(g_1 + g_2\) is the original on \(R^n\), where \(g_1\) and \(g_2\) are two originals with their supports contained in \(U_1\) and \(U_2\) correspondingly. Take new domain \(U\) satisfying Conditions (D1, i-vii) and (D2-D3):

\[D2\] \(U \supset U_1\) and \(\partial U \subset \partial U_1\);

\[D3\] if a straight line \(\xi\) containing a point \(w_i\) (see (vii)) intersects \(\partial U_1\) at two points \(y_1\) and \(y_2\), then only one point either \(y_1\) or \(y_2\) belongs to \(\partial U\), where \(w_1 \in U_1\), \(U = U_1\) and \(U = U_1\) are convex; if \(\xi\) intersects \(\partial U_1\) only at one point, then it intersects \(\partial U\) at the same point. That is,

\[D4\] any straight line \(\xi\) through the point \(w_i\) either does not intersect \(\partial U\) or intersects the boundary \(\partial U\) only at one point.

Take now \(g\) with \(\text{supp}(g) \subset U\), then \(\text{supp}(g X U_1) \subset U_1\). Therefore, any problem (1) on \(U_1\) can be considered as the restriction of the problem (1) defined on \(U\), satisfying (D1D4, i-vii). Any solution \(f\) of (1) on \(U\) with the boundary conditions on \(\partial U\) gives the solution as the restriction \(f|_{U_1}\) on \(U_1\) with the boundary conditions on \(\partial U\).
$U = U_1 \cup \{ t \in \mathbb{R}^* : t \in \varepsilon \}$ with a marked number $0 < \varepsilon < 1/2$. But subsets $\partial U_{(i)}$ in $\partial U$ can also be specified, if the boundary conditions demand it.

The complex field has the natural realization by $2 \times 2$ real matrices so that $i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $i^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. The quaternion field, as it is well-known, can be realized with the help of $2 \times 2$ complex matrices with the generators $i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $i^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let subsets in $2 \times 2$ complex matrices with the generators $i$, $J$, $K$, and $L$. When $r = 2$, $f$ and $g$ have values in $A = H$ and $2 \leq n \leq 4$ and coefficients of differential operators belong to $A_2$, then the multiparameter noncommutative transform operates with the associative case so that

$$F^n (af) = aF^n (f)$$

for each $a \in H$. The left linearity property $F^n (af) = aF^n (f)$ for any $a \in H_{J,K,L}$ is also accomplished for either operators with coefficients in $R$ or $RC = IR \oplus iR$ or $H_{J,K,L} = IR \oplus JR \oplus KR \oplus LR$ and $f$ with values in $A$, with $1 \leq n \leq 2^r - 1$ or vice versa $f$ with values in $C_i$ or $H_{J,K,L}$ and coefficients $a$ in $A$, but with $1 \leq n \leq 4$. Thus all such variants of operator coefficients $a_j$ and values of functions $f$ can be treated by the noncommutative transform. Henceforward, we suppose that these variants take place.

We suppose that $g(t)$ is an original function, that is satisfying Conditions 1(1-4). Consider at first the case of constant coefficients $a_j$ on a quadrant domain $Q^r$. Let $Q^r$ be oriented so that $a_k = -\infty$ and $b_k = +\infty$ for each $k = n - k$; either $a_k = -\infty$ or $b_k = +\infty$ for each $k > n - k$, where $0 \leq k \leq n$ is a marked integer number. If conditions of Theorem 25 are satisfied, then

$$F^n (A[f](t),u;p;\xi) = \sum_{j=0} a_j \left[ R_{n_j} (p) R_{n_2} (p) \right]^{\xi_j} \left[ R_{n_1} (p) \right]^{\xi_1} F^n (f(t) \chi_{c^n}(t),u;p;\xi)$$

for $u(p;\xi) \in \mathcal{A}_n$, spherical or $A_n$ Cartesian coordinates, where the operators $R_j (p)$ are given by Formulas 25(1.1) or 25(1.2). Here $\{ l \}$ enumerates faces $\partial Q^r_{(i)}$ in $\partial Q^r_{(i)}$ for $l = 1$, $k = 1$, so that $\partial Q^r_{(i)} = \bigcup_{l=1}^{\partial Q^r_{(i)}}$. The $\partial Q^r_{(i)}$ is a marked integer number. Therefore, Equation (6) shows that the boundary conditions are necessary:

$$\left[ \partial^{|\beta|} f(t^{(i)}) / \partial t_1^{\beta_1} \cdots \partial t_n^{\beta_n} \right]_{Q^r_{(i)}} \quad \text{for} \quad |\beta| \leq \alpha, \quad |\beta| \geq 1,$$

and

$$\left[ \partial^{|\beta|} f(t^{(i)}) / \partial t_1^{\beta_1} \cdots \partial t_n^{\beta_n} \right]_{Q^r_{(i)}} \quad \text{can be calculated if know} \quad \left[ \partial^{|\beta|} f(t^{(i)}) / \partial t_1^{\beta_1} \cdots \partial t_n^{\beta_n} \right]_{Q^r_{(i)}} \quad \text{for} \quad |\beta| = q,$$

where $\beta = (\beta_1, \ldots, \beta_n)$, $m = |b(l)|$, a number $\gamma(k)$ corresponds to $l^{(k)} > 0$, since $q_k = 0$ for $l_k = 0$ and $q_k > 0$ only for $l_k = 0$ and $k > n - k$. That is, $l^{(k)} \cdots l^{(1)}$ are coordinates in $R^n$ along unit vectors orthogonal to $\partial Q^r_{(i)}$. We choose them so that each two different quadrants may intersect only by their borders, each $U_k$ satisfies the same conditions as $U$ and

$$\lim_{\alpha \to 0} \text{dist} (U, U_k) = 0.$$

Therefore, Equation (6) can be written for more general domain $U$ also.

For $U$ instead of $Q^r$ we get a face $\partial U_{(i)}$ instead of $\partial Q^r_{(i)}$ and local coordinates $\tau_{(i)}^{(1)}, \ldots, \tau_{(i)}^{(n)}$ orthogonal to $\partial U_{(i)}$ instead of $l^{(1)} \cdots l^{(n)}$ (see Conditions (i-iii) above).

Thus the sufficient boundary conditions are:
5.1) \( \left( \partial^{\beta} f \left( t^{(j)} \right) \right) \left[ \partial \tau_{(j)}^{\alpha_{1}} \cdots \partial \tau_{(j)}^{\alpha_{m}} \right] \mid_{U(\alpha)} = \phi \left( \alpha \right) \left( t^{(j)} \right) \)

for \( |\beta|=|\alpha| \), where \( m=|h(\alpha)| \), \( |\alpha| \leq \alpha \), \( \left( t^{(j)} \right) \geq 1 \), \( \alpha_{j} \neq 0 \), \( \alpha_{q}=0 \) for \( \alpha_{q}=0 \), \( m_{q}+q_{h}+h_{k}=j_{k} \), \( h_{k}=\text{sign} \left( l_{q}, l_{h} \right) \), \( 0 \leq q_{h} \leq j_{k} \) for \( k > n-k \):

\( \phi \left( \alpha \right) \left( t^{(j)} \right) \) are known functions on \( \partial U(\alpha) \), \( t^{(j)} \in \partial U(\alpha) \).

In the half-space \( t_{\alpha} \geq 0 \) only

5.2) \( \partial f \left( t \right) / \partial \tau_{\alpha}^{(j)} \mid_{U(\alpha)} = 0 \)

are necessary for \( \beta = |\alpha| < \alpha \) and \( g \) as above.

Depending on coefficients of the operator \( A \) and the domain \( U \) some boundary conditions may be dropped, when the corresponding terms vanish in Formula (6). For example, if \( A=\partial^{2} / \partial t_{1} \partial t_{2} \), \( U=U_{12}, \ n=2 \), then \( \partial f / \partial \tau_{\alpha}^{(j)} \mid_{U(\alpha)} \) is not necessary, only the boundary condition

\[
\left\{ S_{(m)} F^{(g)} (t,u,p;\zeta) ; S_{(m)} F^{(a;h;l)} (t^{(j)}) \partial^{\phi} f \left( t^{(j)} \right) \right\} \mid_{U(\alpha)} = \phi \left( \alpha \right) \left( t^{(j)} \right) \mid_{U(\alpha)} \]

and polynomials of \( p \), where \( Z \) denotes the ring of integer numbers, where the corresponding term \( F^{(a;h;l)}(t^{(j)}) \) is zero when \( t_{\alpha}^{(j)} = \pm \infty \) for some \( j \). In the \( A \), Cartesian coordinates.

2.28.2. Example
We take the partial differential operator of the second order

\( A = \sum_{h,m=1}^{n} a_{h,m} \partial_{\tau_{h}}^{2} \partial_{\tau_{m}} \partial_{\tau_{h}} + \sum_{k=1}^{n} a_{h} \partial_{\tau_{h}} + \omega \)

where the quadratic form \( a(\tau) := \sum_{h,m=1}^{n} a_{h,m} \tau_{h} \tau_{m} \) is non-degenerate and is not always negative, because otherwise we can consider \(-A\). Suppose that \( a_{h,m} = a_{m,h} \in R \), \( a_{h} \tau_{h} \in R \) for each \( h,m = 1, \cdots, n \), \( \omega \in A_{1} \). Then we reduce this form \( a(\tau) \) by an invertible \( R \) linear operator \( C \) to the sum of squares. Thus

9) \( A = \sum_{k=1}^{n} a_{k} \partial_{\tau_{k}}^{2} + \sum_{k=1}^{n} b_{k} \partial_{\tau_{k}} + \omega \)

where \( (t_{1}, \cdots, t_{n}) = (t_{1}, \cdots, t_{n}) C \) with real \( a_{k} \) and \( \beta_{k} \) for each \( h \). If coefficients of \( A \) are constant, using a multiplier of the type \( \exp \left( \sum a_{h} \tau_{h} \right) \) it is possible to reduce this equation to the case so that if \( a_{h} \neq 0 \), then \( \beta_{h} = 0 \) (see § 3, Chapter 4 in [20]). Then we can simplify the operator with the help of a linear trans-
formulation of coordinates and consider that only $\beta_n$ may be non-zero if $a_n = 0$. For $A$ with constant coefficients as it is well-known from algebra one can choose a constant invertible real matrix $(\alpha_{n,m})_{n,m=1,\ldots,k}$ corresponding to $C$ so that $a_n = 1$ for $h \leq k_n$ and $a_n = -1$ for $h > k_n$, where $0 < k_n \leq n$. For $k_n = n$ and $\beta = 0$ the operator is elliptic, for $k_n = n - 1$ with $a_n = 0$ and $\beta_n \neq 0$ the operator is parabolic, for $0 < k_n < n$ and $\beta = 0$ the operator is hyperbolic. Then Equation (6) simplifies:

\[ F^n \left( A[f](t,u,p;\zeta) \right) = \sum_{h=1}^{k} \left[ R_{a_h} (p) \right]^2 F^n \left( f(t) \chi_{C(t)}(t),u,p;\zeta \right) \]

\[ + \sum_{h \in \{1,2,\ldots:k\}} (-1)^h \left[ F^{n-1} \left( \partial f(t^{(h)}) \chi_{C(t^{(h)})}(t^{(h)})/\partial t_h,u,p;\zeta \right) \right] \]

\[ + \beta_n \left[ F^{n-1-\alpha_h} \left( f(t^{(h)}) \chi_{C(t^{(h)})}(t^{(h)}) \right) - F^{n-1-\alpha_h} \left( f(t^{(h)}) \chi_{C(t^{(h)})}(t^{(h)}) \right) \right] \]

\[ + \left[ R_{a_h} (p) \right]^2 F^n \left( f(t) \chi_{C(t)}(t),u,p;\zeta \right) \]

\[ + \alpha F^n \left( f(t) \chi_{C(t)}(t),u,p;\zeta \right) = F^n \left( g(t),u,p;\zeta \right) \]

in the $A$, spherical or $A$, Cartesian coordinates, where $e_n = (0,\ldots,0,1,0,\ldots,0) \in \mathbb{R}^n$ with 1 on the $h$-th place, $S_n = I$ is the unit operator, the operators $R_{a_h} (p)$ are given by Formulas 25(1.1) or 25(12) respectively.

We denote by $\delta_{\epsilon}(x)$ the delta function of a continuous piecewise differentiable manifold $S$ in $\mathbb{R}^n$ satisfying conditions (vi) so that

\[ \int_S \eta(x) \delta_{\epsilon}(x) \, dx = \int_S \eta(y) \lambda_{\epsilon} \, dy \]

for a continuous integrable function $\eta(x)$ on $\mathbb{R}^n$.

\[ \beta_n \left[ F^{n-1-\alpha_h} \left( f^{(h)} \chi_{C(t^{(h)})}(t^{(h)}) \right) - F^{n-1-\alpha_h} \left( f^{(h)} \chi_{C(t^{(h)})}(t^{(h)}) \right) \right] \]

\[ = F^{n-1-\alpha_h} \left( f(t) \chi_{Q(t)}(t),u,p;\zeta \right) \]

where $\beta(t')$ is a piecewise constant function on $\partial Q'$ equal to $\beta_n$ on the corresponding faces of $Q'$ orthogonal to $e_n$ given by condition: either $t_n = a_n$ or $t_n = b_n$; $\beta(t') = 0$ is zero otherwise.

If $a_n = -\infty$ or $b_n = +\infty$, then the corresponding term disappears. If $R^n$ embed into $A$ with $2^{n-1} \leq n \leq 2^{n-1}$ as $R_1 \oplus \cdots \oplus R_k$, then this induces the corresponding embedding $\Theta$ of $Q'$ or $U$ into $A$. This permits to make further simplification:

\[ \sum_{h=1}^{k} \sum_{j \in \{1,2,\ldots:j\}} (-1)^j \left[ R_{a_h} (p) \right]^2 F^{n-1} \left( f(t^{(h)}) \chi_{C(t^{(h)})}(t^{(h)}) + F^{n-1} \left( \partial f(t^{(h)}) \chi_{C(t^{(h)})}(t^{(h)})/\partial t_h,u,p;\zeta \right) \right] \]

\[ + \left[ P(t') \chi_{Q(t)}(t'),u,p;\zeta \right] \]

where $\nu = \nu(t')$ denotes a real coordinate along an external unit normal $M(t')$ to $\Theta(\partial U)$ at $\Theta(t')$, so that $M(t')$ is a purely imaginary Cayley-Dickson number, $a(t')$ is a piecewise constant function equal to $a_h$ for the corresponding $t'$ in the face $\partial Q'^{n}_h$ with $l_j > 0$; $P(t',p) := P(t') : R_{a_h} (p)$ for $t' \in \partial Q'^{n}_h$, $h = 1,\ldots,n$, since $\sin(\nu + \pi) = -\sin(\nu)$ and

\[ \cos(q + \pi) = -\cos(q) \]

for each $q \in R$. Certainly the operator-valued function $P(t')$ has a piecewise continuous extension $P(t)$ on $Q'$. That is

\[ \int_0^1 P(\xi(t'))(t',u,p,t';\zeta) \, dt \]
for an integrable operator-valued function \( \xi(t) \) so that \( [\xi(t) f(t)] \) is an original on \( U \) whenever this integral exists. For example, when \( \xi \) is a linear combination of shift operators \( S_{a_n} \) with coefficients \( \epsilon_{a_n}(t,p) \) such that each \( \epsilon_{a_n}(t,p) \) as a function by \( t \in U \) for each \( p \in W \) and \( f(t) \) are original on \( f \) and \( g \) are generalized functions. For two quadrants \( Q_{m,j} \) and \( Q_{m,k} \) intersecting by a common face \( \Gamma \):

\[
14) \quad F^n( A(f(t),u,p;\xi)) = \left\{ \sum_{k=1}^{n} a_k R_k(p) \right\}^n F^n(f(t) \chi_U(t),u,p;\xi) + \left\{ \sum_{k=1}^{n} \left[ \beta_a R_a(p) \right] f(t) \chi_U(t),u,p;\xi \right\}
\]

where \( P(t',p') := P(t',p') := \sum_{k=1}^{n} a_k R_k(p)(\partial\nu/\partial t_k) \)

for each \( t' \in \partial U_0 \) (see also Stokes' formula in § XIII. 3.4 [19] and Formulas (14.2,14.3) below). Particularly, for the quadrant domain \( Q^n \) we have \( a(t) = a_k \) for \( t \in \partial Q^n \) with \( l_k > 0 \) and \( \beta(t) = \beta_a \) for \( t \in \partial Q^n \) with \( l_k > 0 \) and zero otherwise.

The boundary conditions are:

\[
14.1) \quad f(t) \bigg|_{\partial U_0} = \phi(t) \bigg|_{\partial U_0}, \quad (\partial f(t)/\partial \nu) \bigg|_{\partial U_0} = \phi(t) \bigg|_{\partial U_0}.
\]

The functions \( a(t) \) and \( \beta(t) \) can be calculated from \( \{a_k : h\} \) and \( \beta_a \) almost everywhere on \( \partial U \) with the help of change of variables from \( (t,\cdots,t) \) to \( (y_1,\cdots,y_n) \), where \( (y_1,\cdots,y_n) \) are local coordinates in \( \partial U_0 \) in a neighborhood of a point \( t' \in \partial U_0 \) and \( y_n = \nu \), since \( \partial U_0 \) is of class \( C^1 \). Consider the differential form

\[
\sum_{h=1}^{n}(1-h) a_h d r_1 \wedge \cdots \wedge d r_n = a d y_1 \wedge \cdots \wedge d y_{n-1}
\]

and its external product with \( d\nu = \sum_{h=1}^{n} (\partial\nu/\partial t_h) dt_h \), then

\[
14.2) \quad a(t) \bigg|_{\partial U_0} = \sum_{h=1}^{n} a_h (\partial\nu/\partial t_h) \bigg|_{\partial U_0}
\]

and

\[
14.3) \quad \beta(t) \bigg|_{\partial U_0} = \beta_a \chi_{U_0} (\partial\nu/\partial t_h) \bigg|_{\partial U_0}.
\]

This is sufficient for the calculation of \( F^n_{2} \).

2.28.3. Inversion Procedure in the A, Spherical Coordinates

When boundary conditions 28(3.1) are specified, this Equation 28(6) can be resolved relative to \( F^n(t) \chi_U(t),u,p;\xi) \), particularly, for Equations 28.2(14,14.1) also. The operators \( S_j \) and \( T_j \) of § 12 have the periodicity properties:

\[
S^{4-k}_j F(p;\xi) = S^{4-k}_j F(p;\xi) \quad \text{and}
\]

\[
T^{4-k}_j F(p;\xi) = T^{4-k}_j F(p;\xi)
\]

for each positive integer number \( k \) and \( 1 \leq j \leq 2^r - 1 \). We put

\[
6.1) \quad F_j(p;\xi) := \left\{ S^{4-k}_j - S^{4-k}_{j+1} \right\} F(p;\xi)
\]

for any \( 1 \leq j \leq 2^r - 2 \),

\[
6.2) \quad F^{r-1}_j (p;\xi) := \left\{ S^{4-k}_j \right\} F(p;\xi)
\]

Then from Formula 28(6) we get the following equations:

\[
6.3) \quad \left\{ \left[ p_0 + p T \right]^{4} \left[ p_0 + p_1 T + p_2 T \right]^{2} \cdots \left[ p_0 + p_1 T + \cdots + p_n T \right] \right\} \bigg|_{p_0 = 0 \forall b > w} F_n(p;\xi)
\]

\[
= \left\{ \left[ p_0 + p T \right]^{4} \left[ p_0 + p_1 T + p_2 T \right]^{2} \cdots \left[ p_0 + p_1 T + \cdots + p_n T \right] \right\} \bigg|_{p_0 = 0 \forall b > w}
\]

for each \( w = 1, \cdots, n \), where

\[
F(p;\xi) = F^n\left( f(t) \chi_{U}^{n}(t),u,p;\xi \right)
\]

and

\[
G(p;\xi) = F^n\left( g(t) \chi_{U}^{n}(t),u,p;\xi \right)
\]

These equations are resolved for each \( w = 1, \cdots, n \) as it
is indicated below. Taking the sum one gets the result
\[ F(p; \zeta) = F_1(p; \zeta) + \cdots + F_n(p; \zeta), \]
Since
\[ \left[ \sum_{j=1}^{2^r-2} \left( S_{ij} - S_{ij+1} \right) \right] + S_{2^r-1} e^{\omega(p; \zeta)} = S_{2^r} e^{\omega(p; \zeta)} = e^{\omega(p; \zeta)} \]
The analogous procedure is for Equation (14) with the domain \( U \) instead of \( Q^* \).
From Equation (6.3) or (14) we get the linear equation:
\[ \sum_{i} \psi(i) x_i = \phi, \]
where \( \phi \) is the known function and depends on the parameter \( \zeta \), \( \psi(i) \) are known coefficients depending on \( p, x \), \( x \) are indeterminates and may depend on \( \zeta \), \( l_i = 0, 1 \) for \( h = 1 \), so that \( x_i = x \) for \( l_i = 0, 1, 2, 3 \) for \( h > 1 \), where \( x_i = x \) for each \( h > 1 \) in accordance with Corollary 4.1, \( (1, 1, \cdots, 1) \) for \( > 1 \) for \( = 1 \).
Acting on both sides of (6.3) or (14) with the shift operators \( T_{(m)} \) (see Formula 25(SO)), where \( m_i = 0, 1 \), \( m_h = 0, 1, 2, 3 \) for each \( h > 1 \), we get from (15) a system of \( 2^{l-1} \) linear equations with the known functions \( \phi_{(m)} := T_{(m)} \phi \) instead of \( \phi \), \( \phi \):
\[ \sum_{i} \psi(i) T_{(m)} x_i = \phi_{(m)} \]
Each such shift of \( \zeta \) left coefficients \( \psi(i) \) intact and
\[ x_i = (-1)^{l_i} x_i \mod 2, l_i = l_i + m_i \mod 4 \]
for each \( h > 1 \), where \( \eta = 1 \) for \( l_i + m_i = l_i + 2 \), \( \eta = 2 \) otherwise. This system can be reduced, when a minimal additive group
\[ G := \{ i : l_i \mod 2, l_j \mod 4 \} \forall 2 \leq j \leq k \]
generated by all \( (1) \) with non-zero coefficients in Equation (15) is a proper subgroup of \( G \), \( G \) denotes the finite additive group for \( 0 < h \in Z \). Generally the obtained system is non-degenerate for \( 1 \), almost all \( p = (p_1, \cdots, p_n) \in R^{+1} \) or in \( W \), where \( \lambda_{n+1} \) denotes the Lebesgue measure on the real space \( R^{n+1} \).
We consider the non-degenerate operator \( A \) with real, complex \( C \), or quaternion \( H_{J,K,L} \) coefficients. Certainly in the real and complex cases at each point \( p \), where its determinate \( \Delta = \Delta(p) \) is non-zero, a solution can be found by the Cramer’s rule. Generally, the system can be solved by the following algorithm. We can group variables by \( l_1, l_2, \cdots, l_k \). For a given \( l_1, \cdots, l_k \) and \( l_0 = 0, 1 \) subtracting all other terms from both sides of (15) after an action of \( T_{(m)} \) with \( m_i = 0, 1 \) and marked \( m_h \) for each \( h > 1 \) we get the system of the form
\[ a_i + b_i x_i = b_i, \]
which generally has a unique solution for \( \lambda_{n+1} \) almost all \( p \):
\[ x_i = \left( \alpha \left( \alpha^2 + \beta^2 \right)^{-1} \right) b_i - \left( \beta \left( \alpha^2 + \beta^2 \right)^{-1} \right) b_i, \]
where \( b_i, b_2 \in A \), for a given set \((m_1, \cdots, m_n)\).
When \( l_i = 0 \) are specified for each \( 1 \leq h \leq k \), where \( h = h_0 \), \( 1 < h \leq k \), then the system is of the type:
\[ a_i + b_i x_i + c_i x_i + d_i x_i = b_i, \]
Thus on each step either two or four indeterminates are calculated and substituted into the initial linear algebraic system that gives new linear algebraic system with a number of indeterminates less on two or four respectively. May be pairwise resolution on each step is simpler, because the denominator of the type \( \alpha \) should be \( \lambda_{n+1} \) almost everywhere by \( p \) for \( \lambda_{n+1} \) positive (see (6), (14) above). This algorithm acts analogously to the Gauss’ algorithm. Finally the last two or even the last two indeterminates remain and they are found with the help of Formulas either (17) or (19) respectively. When for a marked \( h \) in (6) or (14) all \( l_i = 0 \mod 2 \) (remains only \( x_i \), for \( h = 1 \), or remain \( x_i \) and \( x_i \) for \( h > 1 \) or for some \( h > 1 \) all \( l_i = 0 \mod 4 \) (remains only \( x_i \)) a system of linear equations as in (13,13,13) simplifies.
Thus a solution of the type prescribed by (8) generally \( \lambda_{n+1} \) almost everywhere by \( p \in W \), where \( W \) is a domain
\[ W = \{ p \in A : a_i < Re(p) < a_i, p_j = 0 \forall j > n \} \]
transform, when it is non-void, \( 2^{-1} \leq n \leq 2^{r} - 1 \),

\[ \text{Re}(p) = p_0, \quad p = p_0 + \cdots + p_n. \]

This domain \( W \) is caused by properties of \( g \) and initial conditions on \( \partial U \) and by the domain \( U \) also. Generally \( U \) is worthwhile to choose with its interior \( \text{Int}(U) \) non-intersecting with a characteristic surface \( \phi(x_1, \ldots, x_n) = 0 \), i.e. at each point \( x \) of it the condition is satisfied

\[
(CS) \sum_{j=0}^n a_j(t(x)) (\partial^j \phi/\partial x_i) \cdots (\partial^j \phi/\partial x_n) = 0
\]

and at least one of the partial derivatives \( (\partial^j \phi/\partial x_i) \neq 0 \) is non-zero.

In particular, the boundary problem may be with the right side \( g = \xi f \) in (2.2.1.14), where \( \xi \) is a real or complex \( C \) or quaternion \( H \), or multiplier, when boundary conditions are non-trivial. In the space either

\[ \frac{\partial}{\partial W} \]

is satisfied

\[ \text{is a real or complex boundary conditions are non-trivial. In the space } \frac{\partial}{\partial W} \text{ either} \]

\[ \right \text{in (2,2,1), where } \frac{\partial}{\partial W} \text{ is non-void, } \]

\[ \text{due to Corollary 4.1. In accordance with (16,17) we get:} \]

\[
21) \quad F^n(A[f](t), u; p; \xi) = \left\{ p_0 \left( p_0^2 + 3 \left( p_n S_8 \right)^2 \right) + \sum_{j=2}^n \gamma_j \left( p_j S_j \right)^4 \right\} F^n(f(t), u; p; \xi)
\]

\[ + p_0 \left( 3 p_0^2 + \left( p_n S_8 \right)^2 \right) S_8 F^n(f(t), u; p; \xi) = F^n(g(t), u; p; \xi)
\]

\[ \text{for each } w = 1, \ldots, n, \]

\[ \text{where } \alpha_w = \alpha = \left[ p_0 \left( p_0^2 - 3 p_n^2 \right) + \sum_{j=2}^n \gamma_j p_j \right]_{p_0 = 0 \forall b \neq w}, \]

\[ \beta_w = \beta = p_0 \left( 3 p_0^2 - p_n^2 \right)_{p_0 = 0 \forall b \neq w}. \]

From Theorem 6, Corollary 6.1 and Remarks 24 we infer

\[
25) \quad F^n(A f(t), u; p; \xi) = p_0^2 \left( p_0^2 - 3 p_n^2 \right) S_8^2 F^n(f(t), u; p; \xi)
\]

\[ + 2 p_0 p_n^2 S_8 F^n(f(t), u; p; \xi) + \sum_{j=2}^n \gamma_j \left( p_j S_j \right)^4 F^n(f(t), u; p; \xi) = F^n(g(t), u; p; \xi)
\]

If on the same spaces an operator is:

\[
26) \quad A = \xi \frac{\partial}{\partial \xi} + \sum_{j=2}^n \gamma_j \frac{\partial^2}{\partial \xi^2} + \sum_{j=2}^n \gamma_j \frac{\partial^3}{\partial \xi^3}, \quad \text{where } n \geq 3, \quad \text{then (6) takes the form:}
\]

\[
27) \quad F^n(A f(t), u; p; \xi) = p_0^2 S_8^2 F^n(f(t), u; p; \xi)
\]

\[ + p_0^2 p_n^2 S_8 F^n(f(t), u; p; \xi) + \sum_{j=2}^n \gamma_j \left( p_j S_j \right)^4 F^n(f(t), u; p; \xi) = F^n(g(t), u; p; \xi).
\]

To find \( F^n(f(t), u; p; \xi) \) in (23) or (27) after an action of suitable shift operators \( T_{(0, 2, 0, \ldots, 0)} \), \( T_{(1, 0, \ldots, 0)} \) and \( T_{(2, 0, \ldots, 0)} \) we get the system of linear algebraic equations:

\[
28) \quad ax_1 + bx_3 + cx_4 = b_1, \quad bx_1 + cx_3 + ax_4 = b_2, \quad ax_2 + bx_4 + c_1 = b_3, \quad -cx_1 + bx_2 + ax_4 = b_4.
\]
with coefficients $a$, $b$, and $c$, and Cayley-Dickson numbers on the right side $h_1, \ldots, h_4 \in A$, where

$$x_1 = T_1 F_u (p; \zeta), \quad x_2 = T_2 F_u (p; \zeta), \quad x_3 = T_3 F_u (p; \zeta), \quad x_4 = T_4 F_u (p; \zeta),$$

$$b_1 = G_u (p; \zeta) = \left( F^a \left( g(t,u; p; \zeta) \right) \right)_w, \quad b_2 = T_2 G_u (p; \zeta), \quad b_3 = T_3 G_u (p; \zeta), \quad b_4 = T_4 G_u (p; \zeta).$$

Coefficients are:

$$a_w = a \left[ \sum_{j=2}^{n} \gamma_j p^j \right] \epsilon H_{j,K,L} \epsilon \in R, \quad b_w = b \left[ p^2_2 (p^2_2 - p^2_1) \right] \epsilon R, \quad c_w = c = 2 p_0 p_2 p_2^2 \epsilon R,$$

for $A$ given by (24);

$$a_w = a \left[ \sum_{j=2}^{n} \gamma_j p^j \right] \epsilon H_{j,K,L} \epsilon \in R, \quad b_w = b \left[ p^2_2 (p^2_2 - p^2_1) \right] \epsilon R, \quad c_w = c = p_1 p_2^2 \epsilon R,$$

for $A$ given by (26), $w = 1, \ldots, n$. If $a = 0$ the system reduces to two systems with two indeterminates $(x_1, x_2)$ and $(x_3, x_4)$ of the type described by (16) with solutions given by Formulas (17). It is seen that these coefficients are non-zero almost everywhere on $R^{n+1}$. Solving this system for $a \neq 0$ we get:

$$F_u (p; \zeta) = a^{-1} b_1 - \left[ (a^2 - b^2 + c^2)^{1 \over 4} + 4 b^2 c^2 \right] a^{-1} \left[ (a^2 - b^2 + c^2) (c^2 - b^2) h_1 + a b b_2 - 2 b c h_3 + a c h_4 \right] - 2 b c \left( 2 b c h_1 - a c h_2 + \left( c^2 - b^2 \right) h_3 + a h_4 \right).$$

Finally Formula (23) provides the expression for $f$ on the corresponding domain $W$ for known integrals converge. If $\gamma_j > 0$ for each $j$, then $a > 0$ for each $p^2_2 + \cdots + p^2_n > 0$.

For a partial differential equation

$$a(t_{n+1}, \ldots, t_{n+1}) Af(t_1, \ldots, t_{n+1}) + \partial f(t_1, \ldots, t_{n+1}) \epsilon J_{n+1} = g(t_1, \ldots, t_{n+1})$$

with octonion valued functions $f, g$, where $A$ is a partial differential operator by variables $t_1, \ldots, t_n$ of the type given by (2.2.1) with coefficients independent of $t_1, \ldots, t_n$, it may be simpler the following procedure. If a domain $V$ is not the entire Euclidean space $R^{n+1}$ we impose boundary conditions as above in (5.1). Make the noncommutative transform $F^{\alpha \beta \gamma \delta \eta} \epsilon$ of both sides of Equation (30), so it takes the form:

$$a(t_{n+1}) F^{\alpha \beta \gamma \delta \eta} \left( Af(t_1, \ldots, t_{n+1}), u; p; \zeta \right) + \partial F^{\alpha \beta \gamma \delta \eta} \left( f(t_1, \ldots, t_{n+1}), u; p; \zeta \right) \epsilon J_{n+1}$$

$$= F^{\alpha \beta \gamma \delta \eta} \left( g(t_1, \ldots, t_{n+1}), u; p; \zeta \right).$$

In the particular case, when

$$a(t_{n+1}) \sum_{j \in \mathbb{Z}^d} a_j \left( t_{n+1} \right) \sum_{k \in \mathbb{Z}^n} \left( \frac{j}{k} \right) S(j, j; 2^{-1} \epsilon) e^{-\alpha \left( p, \zeta \right)} = e^{-\alpha \left( p, \zeta \right)}$$

for each $t_{n+1}$, $p$, $t$ and $\zeta$, with the help of (6.8) one can deduce an expression of

$$F^a \left( p; \zeta; t_{n+1} \right) = \exp \left\{ \int_{t_\alpha}^{t_{n+1}} b(p_0, \ldots, p_n; \zeta) d\xi \right\} C_0 + \int_{t_\alpha}^{t_{n+1}} Q(p_0, \ldots, p_n; \tau) \exp \left\{ \int_{t_\alpha}^{t_{n+1}} b(p_0, \ldots, p_n; \zeta) d\xi \right\} d\tau \right\}$

through

$$G^a \left( p; \zeta; t_{n+1} \right) = F^{\alpha \beta \gamma \delta \eta} \left( g(t_1, \ldots, t_{n+1}), u; p; \zeta \right)$$

and boundary terms in the following form:

$$b(p_0, \ldots, p_n; \zeta) F^a \left( p; \zeta; t_{n+1} \right) + \partial F^a \left( p; \zeta; t_{n+1} \right) \epsilon J_{n+1} = Q(p_0, \ldots, p_n; t_{n+1})$$.
where \( b(p_0, \cdots, p_n; t_{n+1}) \) is a real mapping and 
\( Q(p_0, \cdots, p_n; t_{n+1}) \) is an octonion valued function. The latter differential equation by \( t_{n+1} \) has a solution ana-
logously to the real case, since \( t_{n+1} \) is the real va-
riable, while \( R \) is the center of the Cayley-Dickson algebra \( A_r \). Thus we infer:

\[
33) \quad F^n (p; \xi; t_{n+1}) = \exp \left\{ - \int_{t_0}^{t_{n+1}} b(p_0, \cdots, p_n; \xi) \, dt \right\} C_0 + \left[ \int_{t_0}^{t_{n+1}} Q(p_0, \cdots, p_n; \xi) \exp \left\{ - \int_{t_0}^{t_{n+1}} b(p_0, \cdots, p_n; \xi) \, dt \right\} \, dt \right] 
\]

since the octonion algebra is alternative and each equation 
\( bx = c \) with non-zero \( b \) has the unique solution 
\( x = b^{-1}c \), where \( C_0 \) is an octonion constant which can 
be specified by an initial condition. More general partial 

differential equations as (30), but with \( \delta f / \delta t_{n+1} \), \( l \geq 2 \), 
instead of \( \delta f / \delta t_{n+1} \) can be considered. Making the 
inverse transform \( \left\{ F^n_{q_{3k}, \cdots, -q_r} \right\}^{-1} \) of the right side of (33) 
one gets the particular solution \( f \).

2.28.5. Integral Kernel

We rewrite Equation 28(6) in the form:

\[
34) \quad A_s F^n \left( f \, \chi_{q_{n+1}}, u; p; \xi \right) = F^n \left( g \, \chi_{q_{n+1}}, u; p; \xi \right) 
- \sum_{1 \leq j \leq n} a_j \sum_{1 \leq k \leq n} a_k \left( S_{n+k} \right) \left( -\mathbf{i} \right)^{n+k} 
\sum_{l=0}^{n-k} \left( S_{n+k+l} \right) \left( \partial^l f(t) \right) \left( \partial^l \eta_{q_{n+1}} \right) \left( f^{(l)}(t), u; p; \xi \right), 
\]

where

34.1) \( S_k(p) := S_k := R_{q_k}(p) \) in the \( A_s \) spherical or \( A_r \) Cartesian coordinates respectively (see also Formulas 25(1.1,1.2)), for each 
\( k = 1, \cdots, n \),

34.2) \( S^m := S^m := S_{m_1} \cdots S_{m_n} \),

35) \( A_s := \sum_{j=1}^{n} a_j S_j^l(p) \).

Then we have the integral formula:

36) \( A_s F^n \left( f \, \chi_{q_{n+1}}, u; p; \xi \right) = \int_{q_{n+1}} f(t) \left[ A_s \exp \left( -u(p, t; \xi) \right) \right] \, dt 
\]
in accordance with 1(7) and 2(4). Due to § 28.3 the 
operator \( A_s \) has the inverse operator for \( A_{s+1} \) almost

37) \( (2\pi)^n \int_{q_{n+1}} \exp \left( -u(a + p, t; \xi) \right) \exp(u(a + p, t; \xi)) \, dp_1 \cdots dp_n = \delta(t; \tau), 
\]

38) \( \left[ \delta, f \right](\tau) = \int_{q_{n+1}} f(t) \delta(t; \tau) \, dt_1 \cdots dt_n = f(\tau) \)
at each point \( \tau \in R^s \), where the original \( f(\tau) \) satisfies 
Hölder’s condition. That is, the functional \( \delta(t; \tau) \) is \( A_r \) 
linear. Thus the inversion of Equation (36) is:

39) \( \left. \int_{q_{n+1}} \left( \int_{q_{n+1}} f(t) \, \chi_{q_{n+1}}(t) \left[ A_s \exp \left( -u(p, a + t; \xi) \right) \right] \xi(p + a, t, \tau; \xi) \right) \, dt \right| \, dp_1 \cdots dp_n = f(\tau) \)

so that

40) \( A_s \exp \left( -u(p, a + t; \xi) \right) \xi(p + a, t, \tau; \xi) = (2\pi)^n \exp \left( -u(p, a + t; \xi) \right) \exp \left( -u(p + a + t; \xi) \right), 
\]

where the coefficients of \( A_s \) commute with generators 
\( i_j \) of the Cayley-Dickson algebra \( A_r \) for each \( j \). Con-
Consider at first the alternative case, i.e. over the Cayley-
Dickson algebra \( A_r \) with \( r \leq 3 \).

Let by our definition the adjoint operator \( A_s^\ast \) be 
defined by the formula

41) \( A_s^\ast \eta(p, t; \xi) = \sum_{i,j=1}^{n} a^*_i S^i \eta^*(p, t; \xi) \)

for any function \( \eta: A_r \times R^s \times A_r \rightarrow A_r \), where \( t \in R^s \), 
\( p \) and \( \xi \in A_r \), \( S^i \eta^*(p, t; \xi) := \left[ S^i \eta(p, t; \xi) \right]^* \). Any 
Cayley-Dickson number \( z \in A_r \) can be written with the 
help of the iterated exponent (see § 3) in \( A_r \) spherical 
coordinates as

42) \( z = |e| \exp(-u(0, 0, \psi)) \),

where \( |e| \geq r, \psi \in A_r, u \in A_r, Re(\psi) = 0 \). Certainly
the phase shift operator is isometrical:

\[ \mu_{k_1, k_2, \ldots, k_n} = z = r \text{exp}(\phi M), \]

for any Cayley-Dickson number \( M \) is a purely imaginary Cayley-Dickson number, \( \mu \) and \( \nu \) are associated with \( \text{Exp}(\nu) \), and \( \nu \) is a parameter, then \( \mu = \nu \) for any restriction of \( \nu \) to the Cayley-Dickson number \( M \). In view of \( \Delta \), \( \pi \) is defined by (40) and using Formulas (34.1, 34.2), we infer, that

\[ \xi(p, t, r; \zeta) = (2\pi)^n \left[ A_{\zeta} \exp(-u(\text{Im}(p), t; \text{Im}(\zeta))) \right] \]

\[ \exp(-ul(m(p), t; \text{Im}(\zeta))) \left[ \exp(u(p, t; \zeta)) \right], \]

since \( z^{-1} = z^* / |z|^2 \) for each non-zero Cayley-Dickson number \( z \in A_r \), \( v \geq 1 \), where \( \text{Im}(p) = p_{11} + \cdots + p_{n1}, p = p_{00} \cdots + p_{n1}, p_0 = \text{Re}(p) \).

Generally, for \( r \geq 4 \), Formula (45) gives the integral kernel \( \xi(p, t, r; \zeta) \) for any restriction of \( \zeta \) on the octonion subalgebra \( \text{alg}_o(N_1, N_2, N_3) \) embedded into \( A_r \). In view of \( \Delta \), \( \zeta \) is unique and is defined by (45) on each subalgebra \( \text{alg}_o(N_1, N_2, N_3) \), consequently, Formula (45) expresses \( \xi \) by all variables \( p, \zeta \in A_r \) and \( t, r \in R^r \). Applying Formulas (39,45) and 28.2(\( \Delta \)) to Equation (34), when Condition 8(3) is satisfied, we deduce, that

\[ \int_{R^r} \left[ \frac{1}{2\pi} \frac{\text{Exp}(\nu)}{\text{Exp}(\nu)} \right] \exp(-ul(m(p), t; \zeta)) \exp(u(p, t; \zeta)) \text{d}p_1 \cdots \text{d}p_n, \]

where \( \text{dim}_{\nu} \text{Exp}(\nu) = n - 1, t(\nu) \in \text{Exp}(\nu) \) in accordance with \( \Delta \), \( S^a(p) \) is given by Formulas (34.1, 34.2) above.

For simplicity the zero phase parameter \( \zeta = 0 \) in (46) can be taken. In the particular case \( Q_r = R^r \) all terms with \( \int_{R^r} \text{d}p_n \) vanish.

Terms of the form

\[ \int_{R^r} \text{d}p_n \left[ S^a(p) \right] ^2 \]

\[ \beta S^a(p) + \omega \]

and

\[ (f \chi_G)(t) = \int_{R^r} \left[ \int_{R^r} f(t') \left[ \text{Exp}(p, t') \right] \text{d}t' \right] \text{d}p_1 \cdots \text{d}p_n, \]

\[ \int_{R^r} \left[ \int_{R^r} a(t') \left[ \text{Exp}(p, t') \right] \text{d}t' \right] \text{d}p_1 \cdots \text{d}p_n, \]

For a calculation of the appearing integrals the generalized Jordan lemma (see \( \Delta \) and 24 in [4]) and residues of functions at poles corresponding to zeros

\[ \text{d}p_1 \cdots \text{d}p_n \]

can be used.

Take \( g(t) = \delta(y, t) \), where \( y \in R^r \) is a parameter, then

\[ A_{\zeta} \exp(-u(\text{Im}(p), t; \text{Im}(\zeta))) = 0 \text{ by variables} \]

\[ p_1, \ldots, p_n \text{ can be used.} \]
is the fundamental solution in the class of generalized functions, where

\[ \int_{\mathbb{R}} \Delta(y; t) \left[ \exp(-u(p + a, t, \xi)) \gamma(p + a, t, \xi) \right] \, dt \, dp_1 \cdots dp_n = E(y; \tau) \]

for each continuous function \( f(t) \) from the space \( L^p \left( \mathbb{R}^n, A_j \right) \); \( A \) is the partial differential operator as above acting by the variables \( t = (t_1, \cdots, t_n) \) (see also § 19, 20 and 33-35).

2.29. The Decomposition Theorem of Partial Differential Operators over the Cayley-Dickson Algebras

We consider a partial differential operator of order \( u \):

1) \( Af(x) = \sum_{|\alpha| = u} a_{\alpha}(x) \partial^\alpha f(x) \),

where \( \partial^\alpha f = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f(x) \), \( x = x_0 i_0 + \cdots + x_n i_n \),

\( x_j \in \mathbb{R} \) for each \( j \), \( 1 \leq n = 2^r - 1 \), \( \alpha = (\alpha_0, \cdots, \alpha_n) \),

\( |\alpha| = \alpha_0 + \cdots + \alpha_n \), \( 0 \leq \alpha_n \in \mathbb{Z} \). By the definition this means that the principal symbol

2) \( A_0 := \sum_{|\alpha| = u} a_{\alpha}(x) \partial^\alpha \)

has \( \alpha \) so that \( |\alpha| = u \) and \( a_{\alpha}(x) \in A \) is not identically zero on a domain \( U \) in \( A \). As usually \( C^k(U, A) \) denotes the space of \( k \) times continuously differentiable functions by all real variables \( x_0, \cdots, x_n \) on \( U \) with values in \( A \), while the \( x \)-differentiability corresponds to the super-differentiability by the Cayley-Dickson variable \( x \).

Speaking about locally constant or locally differentiable coefficients we shall understand that a domain \( U \) is the union of subdomains \( U_j \) satisfying conditions 28(D1,i-vii) and \( U \cap U' = \partial U \cap \partial U' \) for each \( j \neq k \). All coefficients \( a_{\alpha} \) are either constant or differentiable of the same class on each \( \text{Int}(U_j) \) with the continuous extensions on \( U_j \). More generally it is up to a \( C^k \) or \( x \)-differentiable diffeomorphism of \( U \) respectively.

If an operator \( A \) is of the odd order \( u = 2s - 1 \), then an operator \( E \) of the even order \( u + 1 = 2s \) by variables \( (t, x) \) exists so that

3) \( Eg(t, x) \big|_{t=0} = Ag(0, x) \)

for any \( g \in C^{u+1}([c, d] \times U, A_j) \), where \( t \in [c, d] \subset \mathbb{R} \), \( c \leq 0 < d \), for example, \( Eg(t, x) = \partial^i \left( tA(g(t, x)) \right) / \partial t^i \).

Therefore, it remains the case of the operator \( A \) of the even order \( u = 2s \). Take

\( z = z_0 i_0 + \cdots + z_n i_n \), \( z_j \in R \). Operators depending on a less set \( z_{\alpha} \), \( \alpha \), of variables can be considered as restrictions of operators by all variables on spaces of functions constant by variables \( z_\alpha \) with \( \alpha \in \{ 1, \cdots, n \} \).

**Theorem.** Let \( A = A_j \) be a partial differential operator of an even order \( u = 2s \) with locally constant or variable \( C^\alpha \) or \( x \)-differentiable on \( U \) coefficients \( a_\alpha(x) \in A_j \) such that it has the form

4) \( Af = c_{\alpha, x} (B_{\alpha, s}f) + \cdots + c_{\alpha, k} (B_{\alpha, k}f) \),

where each

5) \( B_{\alpha, p} = B_{\alpha, p, 0} + Q_{p-1} \)

is a partial differential operator by variables \( x_{\alpha, 1}, \cdots, x_{\alpha, p-1}, \cdots, x_{\alpha, p, 0} = 0 \), \( c_{\alpha, k}(x) \in A \) for each \( k \), its principal part

6) \( B_{\alpha, p, 0} = \sum_{|\alpha| = p} a_{\alpha, p, 0}(x) \partial^\alpha \)

is elliptic with real coefficients \( a_{\alpha, p, 0}(x) \geq 0 \), either \( 0 \leq r \leq 3 \) and \( f \in C^k(U, A_j) \), or \( r \geq 4 \) and \( f \in C^k(U, R) \). Then three partial differential operators \( Y^r \) and \( Y_i \) and \( Q \) of orders \( s \) and \( p \) with \( p \leq u - 1 \) with locally constant or variable \( C^\alpha \) or \( x \)-differentiable correspondingly on \( U \) coefficients with values in \( A \), exist, \( r \leq v \), such that

7) \( Af = Y^r (Y_i f) + Q f \).

2.30. Corollary 1

Let suppositions of Theorem 29 be satisfied. Then a change of variables locally correspondingly exists so that the principal part \( A_{0, 0} \) of \( A \) becomes with constant coefficients, when \( a_{\alpha, p, 0} > 0 \) for each \( p, \alpha \) and \( x \).

2.31. Corollary 2

If two operators \( E = E_j \) and \( A = A_{j-1} \) are related by Equation 29(3), and \( E_j \) is presented in accordance with Formulas 29(4,5), then three operators \( Y^r \), \( Y_i^r \) and \( Q \) of orders \( s \), \( s - 1 \) and \( 2s - 2 \) exist so that

1) \( A_{s-1} = Y_i^r + Q \).

2.32. Products of Operators

We consider operators of the form:

1) \( (Y^r + \beta I) f(z) = \sum_{|\alpha| = 0} \partial^\alpha f(z) \eta_\alpha + f(z) \beta(z) \)

with \( \eta_\alpha(z) \in A_j, \alpha = (\alpha_0, \cdots, \alpha_n), 0 \leq \alpha \in N \) for each \( k \), \( |\alpha| = \alpha_0 + \cdots + \alpha_n \), \( \beta I, f(z) := f(z) \beta \).
\[ \partial^\alpha f(z) := \partial^\alpha f(z)/\partial z^0 \cdots \partial z^{r-1}_{\alpha}, \quad 2 \leq r \leq v < \infty, \]
\[ \beta(z) \in A_\alpha, \quad z_0, \cdots, z_{\alpha} \in R, \quad z = z_0 + \cdots + z_{\alpha} i_{\alpha}. \]

**Proposition.** The operator \( (Y^\alpha + \beta) (Y^\beta + \beta) \) is elliptic on the space \( C^{2k}(\mathbb{R}^p, A_\alpha) \).

### 2.33. Fundamental Solutions

Let either \( Y \) be a real \( Y = A_\alpha \) or complexified \( Y = (A_\alpha) \) or quaternionified \( Y = (A_\alpha)_H \) Cayley-Dickson algebra (see § 28). Consider the space \( B(\mathbb{R}^p, Y) \) (see § 19) supplied with a topology in it is given by the countable family of semi-norms

1. \[ p_{a,k}(f) := \sup_{a \in \mathbb{R}^p, x \in R} \left| (1 + |x|^k) \partial^\alpha f(x) \right|, \]

where \( k = 0, 1, 2, \cdots; \quad \alpha = (\alpha_1, \cdots, \alpha_n), \quad 0 \leq \alpha_j \in Z. \) On this space we take the space \( B'(\mathbb{R}^p, Y) \) of all \( Y \) valued continuous generalized functions (functionals) of the form

2. \[ f = f_1 h_0 + \cdots + f_{\alpha} i_{\alpha}, \quad \text{and} \]
\[ g = g_0 h_0 + \cdots + g_{\alpha} i_{\alpha}, \]

where \( f_j \) and \( g_j \in B'(\mathbb{R}^p, Y) \), with restrictions on \( B(\mathbb{R}^p, R) \) being real or \( C_0 \) or \( H_{J,K,L} \) valued generalized functions \( f_0, \cdots, f_{\alpha}, g_0, \cdots, g_{\alpha} \) respectively. Let \( \phi = \phi_0 h_0 + \cdots + \phi_{\alpha} i_{\alpha} \), with \( \phi_0, \cdots, \phi_{\alpha} \in B(\mathbb{R}^p, Y) \), then

3. \[ \{f, \phi\} = \sum_{j,k=0}^{\alpha} [f_j, \phi_k] i_j i_k. \]

We define their convolution as

4. \[ \{f \ast g, \phi\} = \sum_{j=0}^{\alpha} [f_j, g_0] i_j \phi_k. \]

for each \( \phi \in B'(\mathbb{R}^p, Y) \). As usually

5. \[ \{f \ast g\}(x) = f(x-y) g(y) = f(y) g(x-y) \]

for all \( x, y \in \mathbb{R}^p \) due to (4), since the latter equality (5) is satisfied for each pair \( f_j \) and \( g_k \). Thus a solution of the equation

6. \[ (Y^\alpha + \beta) f = g \quad \text{in} \quad B(\mathbb{R}^p, Y) \] or in the space \( B'(\mathbb{R}^p, Y) \) is:
7. \[ f = E_{Y^\alpha + \beta} g, \]

where \( E_{Y^\alpha + \beta} \) denotes a fundamental solution of the equation

8. \[ (Y^\alpha + \beta) E_{Y^\alpha + \beta} = \delta, \quad (\delta, \phi) = \phi(0). \]

The fundamental solution of the equation

9. \[ A_\alpha V = \delta \quad \text{with} \quad A_\alpha = (Y^\alpha + \beta) (Y^\alpha + \beta) \]

using Equalities 32(2-4) can be written as the convolution

10. \[ V = \left[ V_0 + E_{Y^\alpha + \beta} E_{Y^\alpha + \beta} \right]. \]

More generally we can consider the equation

11. \[ Af = g \quad \text{with} \quad A = A_\alpha + (Y_\alpha + \beta_\alpha), \]

where \( A_\alpha = (Y^\alpha + \beta)(Y^\alpha + \beta) \), \( Y, Y_\alpha, Y_\beta \) are operators of orders \( s_\alpha \), \( s_\beta \) and \( s_\alpha \) respectively given by 32(1) with \( z \)-differentiable coefficients. For \( Y_\alpha + \beta_\alpha = 0 \) this equation was solved above. Suppose now, that the operator \( Y_\alpha + \beta_\alpha \) is non-zero.

To solve Equation (11) on a domain \( U \) one can write it as the system:

12. \[ (Y_\alpha + \beta_\alpha) f = g_\alpha, \quad (Y + \beta) g_\alpha = g - (Y_\alpha + \beta_\alpha) f. \]

Find at first a fundamental solution \( V_\alpha \) of Equation (11) for \( g = \delta \). We have:

13. \[ f = E_{Y_\alpha + \beta_\alpha} g_\alpha + E_{Y_{\alpha + \beta}} g \]

In accordance with (3-5) and 32(1) the identity is satisfied:

\[ E_{Y_{\alpha + \beta}} g \] \[ + \]

Thus (13) is equivalent to

14. \[ E_{Y_\alpha + \beta_\alpha} g_\alpha + E_{Y_{\alpha + \beta}} g \]

We consider the Fourier transform \( F \) by real variables with the generator \( i \) commuting with \( i_j \) for each \( j = 0, \cdots, 2^{r-1} - 1 \) such that

\[ (F(Fg)(y)) = \int x \epsilon^{i\langle x, y \rangle} g(x) dx_1 \cdots dx_n \]

for any \( g \in \mathcal{L}(\mathbb{R}^p, A_\alpha) \), i.e.
\[ \int \epsilon^{i\langle x, y \rangle} g(x) dx_1 \cdots dx_n < \infty, \]

where \( g : \mathbb{R}^p \to Y \) is an integrable function, \( \langle y, x \rangle = x_1 y_1 + \cdots + x_n y_n \), \( x = (x_1, \cdots, x_n) \in \mathbb{R}^p \), \( x_j \in R \) for every \( j \). The inverse Fourier transform is:

\[ (F^{-1}g)(y) = (2\pi)^{n/2} \int g(x) e^{ix_1 \cdot y_1} dx_1 \cdots dx_n \]

For a generalized function \( f \) from the space \( B'(\mathbb{R}^p, Y) \) its Fourier transform is defined by the formula

\[ (F^{-1}f)(\phi) = (f, F\phi), \quad (F^{-1}f, \phi) = (f, F^{-1}\phi). \]

In view of (2-5) the Fourier transform of (14) gives:

15. \[ F\left[ E_{Y_\alpha + \beta_\alpha} g_\alpha \right] F(g_\alpha) \]

for \( g = \delta \). With generators \( i_{l_1} \cdots i_{l_{2^{r-1} - 1}} \) the latter equation gives the linear system of \( 2^{r-2} \) equations over the real field, or \( 2^{r-2} \) equations when \( Y = (A_\alpha)_H \). From it \( F(g_\alpha) \) and using the inverse transform \( F^{-1} \) a generalized function \( g_\alpha \) can be found, since \( F(g_\alpha) = F(g_\alpha, g_\alpha) i_{l_1} + \cdots + F(g_\alpha, g_\alpha) i_{l_{2^{r-1} - 1}} \)

and

\[ F^{-1}g = F^{-1}(g_\alpha, g_\alpha) i_{l_1} + \cdots + F^{-1}(g_\alpha, g_\alpha) i_{l_{2^{r-1} - 1}} \] (see also the Fourier transform of real and complex generalized func-
tions in $[1,21]$. Then

16) $V_d = E_{t_1+\cdots+t_n} * g_1$ and $f = V_d * g$ gives the
solution of (11), where $g_1$ was calculated from (15).

Let $\pi_j : (A_j)_h \to (A_j)$ be the $R$-linear projection
operator defined as the sum of projection operators
$\pi_j + \cdots + \pi_{j+1}$, where $\pi_j : (A_j)_h \to H_i$, where $j = 0,\ldots,2^n - 1$. Indeed, Formulas (2,5,6) have
the natural extension on $(A_i)_h$, since the generators $J$
and $L$ commute with $i_j$ for each $j$.

Finally, the restriction from the domain in $A_i$ onto
the initial domain of real variables in the real shadow and
the extraction of $\pi_j * f \in A_j$ with the help of Formulas
(2,5,6) gives the reduction of a solution from $A_i$ to $A_j$.

Theorems 29, Proposition 32 and Corollaries 30, 31
together with formulas of this section provide the algorithm
for subsequent resolution of partial differential equations
for $s_1 = -1,\ldots,2^n - 1$. We suppose $n \geq 2$. A special
parts of operators $A_j$ on the final step are with constant coefficients. A residue term $Q$ of the first order can be integrated along a path using a non-commutative line integration over the Cayley-Dickson algebra $[5,6]$.

### 2.34. Multiparameter Transforms of Generalized Functions

If $\phi \in B(R^n, Y)$ and $g \in B(R^n, Y)$ (see § 19 and
33) we put

1) $\sum_{j=0}^{2^n - 1} F^n(g, \phi)i_j$

or shortly

2) $\sum_{j=0}^{2^n - 1} \left[ g, F^n(\phi) \right]i_j$

If the support $\text{supp}(g)$ of $g$ is contained in a domain $U$, then it is sufficient to take a base function $\phi$ with the restriction $\phi, \phi \in B(U, Y)$ and any $s \phi^{(n)}, U, \in C^n$.

### 2.35. Examples

Let

1) $A_f(t) = \sum_{j=0}^{2^n - 1} \left( \delta_j f(t) \right) c_j$

be the operator with constant coefficients $c_j \in A_i$, $|c_j| = 1$, by the variables $t_1, \ldots, t_n$, $n \geq 2$. We suppose that $c_j$ are such that the minimal subalgebra $\text{alg}_R(c_j, c_k)$ containing $c_j$ and $c_k$ is alternative for each $j$ and $k$ and $\left| \cdots (c_1^2 c_2^2) \cdots c_n^2 \right| = 1$. Since

2) $\delta f(t) \delta t_j = \sum_{j=0}^{2^n - 1} \left( \delta f(t) \delta s_j \right) c_j$

the operator $A$ takes the form

3) $A_f = \sum_{j=0}^{2^n - 1} \left( \delta^2 f(t) \delta s_j \right) c_j$

where $s_j = t_1, \ldots, t_n$ for each $j$. Therefore, by Theorem 12 and Formulas 25 $(SO)$ and 28(6) we get:

4) $F^n(A_f u; p, \zeta) = \sum_{j=0}^{2^n - 1} \left( \delta^2 f(t) \delta s_j \right) c_j$

for $u(p, t, \zeta)$ either in $A_i$, spherical or $A_j$, Cartesian coordinates with the corresponding operators $R_j(p)$
(see also Formulas 25(1,1,1,2)).

On the other hand,

5) $F^n(\delta; u; p, \zeta) = e^{-a(p, p, \zeta)}$

in accordance with Formula 20(2). The delta function
$\delta(t)$ is invariant relative to any invertible linear operator
$C : R^n \to R^n$ with the determinant $\text{det}(C) = 1$, since

$\int e^{\delta(Cx)} \phi(x) \text{dx} = \int e^{\delta(x)} \phi(C^{-1} y) \text{det}(C) \text{dy} = \phi(C^{-1} 0) = \phi(0)$.

Thus

3) $F^n(A_f u; p, \zeta) = F^n(A_f u; p, \zeta)$

for any Fundamental solution $f$, where

$Cg(t) = g(Ct), \quad A_f = \delta$.

If $C : R^n \to R^n$ is an invertible linear operator and $C' = C \zeta$, $p = C^{-1} q$ and $\zeta = C^{-1} C'$. In the multiparameter noncommutative transform $F^n$ there are the corresponding variables $(t_1, p, \zeta)$. This is accomplished in particular for the operator

$C(t_1, \ldots, t_n) = \left( s_1, \ldots, s_n \right)$. The operator $C^{-1}$ transforms the right side of Formula (4), when it is written in the $A_j$, spherical coordinates, into

$\sum_{j=0}^{2^n - 1} \left( \left( p_0 + q s_1 \right) \right)^2 F_n \left( q, \zeta \right) c_j$. The Cayley-Dickson number $q = q_0 + q_1 i + \cdots + q_n i_n$ can be written as $q = q_0 + q_1 M$, where $|M| = 1, M$ is a purely imaginary Cayley-Dickson number, $q_0 U, R$, $q_1 M = q_1 i_1 + \cdots + q_n i_n$, since $q_0 = \text{Re}(q)$. After a suitable automorphism $\theta : A_i \to A_j$ we can take

$\theta(q) = q_0 + q_1 i_1$, so that $\theta(0) = x$ for any real number.

The functions $\sum_{j=0}^{2^n - 1} \left( \left( p_0 + q s_1 \right) \right)^2 F_n \left( q, \zeta \right) c_j$ and $\sum_{j=0}^{2^n - 1} \left( \left( p_0 + q s_1 \right) \right)^2 F_n \left( q, \zeta \right) c_j$ are even by each variable $q_i$ and $p_j$ respectively.

Therefore, we deduce in accordance with (5) and 2(3,4) and Corollary 6.1 with parameters $p_0 = 0$ and $\zeta = 0$ and $c_j \in \{-1, 1\}$ for each $j$.

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6) \( (F^n)^{-1} \)
\[
\left( \frac{1}{1 \left[ \sum_{j=1}^{n} \left\{ \frac{p_j^2}{2} S_{ij} \right\} c_j \right]} u; y, \zeta \right)
= -g, e^{\beta (1; \beta (0; 0))}
\]
in the \( A_j \) spherical coordinates, where
\( g = 1 / \left[ \sum_{j=1}^{n} f_j c_j \right] \), or

6.1) \( (F^n)^{-1} \left( \frac{1}{1 \left[ \sum_{j=1}^{n} \left\{ \frac{p_j^2}{2} S_{ij} \right\} c_j \right]} u; y, \zeta \right) \)

6.1) \( (F^n)^{-1} \left( \frac{1}{1 \left[ \sum_{j=1}^{n} \left\{ \frac{p_j^2}{2} S_{ij} \right\} c_j \right]} u; y, \zeta \right) \)
in the \( A_j \) Cartesian coordinates, where
\( g = 1 / \left[ \sum_{j=1}^{n} f_j c_j \right] \), \( N = y / |y| \) for \( y \neq 0 \), \( N = i \) for

7) \( (F^n)^{-1} \left( \frac{1}{1 \left[ \sum_{j=1}^{n} \left\{ \frac{p_j^2}{2} S_{ij} \right\} c_j \right]} u; y, \zeta \right) = (2\pi)^n \int_{\mathbb{R}^n} \exp \left( i (p_1 y_1 + \cdots + p_n y_n) \right) \left( \frac{1}{1 \left[ \sum_{j=1}^{n} p_j^2 \right]} \right) u; y, \zeta \)
in the \( A_j \) spherical coordinates and

7.1) \( (F^n)^{-1} \left( \frac{1}{1 \left[ \sum_{j=1}^{n} \left\{ \frac{p_j^2}{2} S_{ij} \right\} c_j \right]} u; y, \zeta \right) = (2\pi)^n \int_{\mathbb{R}^n} \exp \left( i (p_1 y_1 + \cdots + p_n y_n) \right) \left( \frac{1}{1 \left[ \sum_{j=1}^{n} p_j^2 \right]} \right) u; y, \zeta \)
in the \( A_j \) Cartesian coordinates, since for any even function its cosine Fourier transform coincides with the Fourier transform.

The inverse Fourier transform
\( (F^{-1} g)(x) = (2\pi)^n \left( Fg \right)(-x) =: \Psi^n \) of the functions
\( g = 1 / \left( \sum_{j=1}^{n} p_j^2 \right) \) for \( n \geq 3 \) and \( P \left( \frac{1}{1 \left[ \sum_{j=1}^{n} p_j^2 \right]} \right) \) for
\( n = 2 \) in the class of the generalized functions is known (see [21] and § 9.7.11 [21]) and gives

8) \( \Psi^n \left( z_1, \cdots, z_n \right) = C_n \left( \sum_{j=1}^{n} \frac{1}{z_j^2} \right) \)
for \( 3 \leq n \), where \( C_n = -1/ \left[ \left( n - 2 \right) \sigma_n \right] \),
\( \sigma_n = 4 \pi^{n/2} / \Gamma \left( \left( n/2 \right) - 1 \right) \) denotes the surface of the unit sphere in \( R^n \), \( \Gamma \left( x \right) \) denotes Euler’s gamma-function, while

9) \( \Psi_2 \left( z_1, z_2 \right) = C_2 \ln \left( \sum_{j=1}^{2} \frac{1}{z_j^2} \right) \)
for \( n = 2 \), where \( C_2 = 1 / (4\pi) \).

Thus the technique of § 2 over the Cayley-Dickson algebra has permitted to get the solution of the Laplace operator.

For the function

10) \( P \left( x \right) = \sum_{k=1}^{n} \frac{1}{x_k^2} - \sum_{j=k+1}^{n} x_j^2 \)
with \( 1 \leq k_1 < n \) the generalized functions \( \left( P \left( x \right) + i0 \right)^{1/2} \) and \( \left( P \left( x \right) - i0 \right)^{1/2} \) are defined for any \( \lambda \in C = R \oplus iR \) (see Chapter 3 in [21]). The function \( P^\lambda \) has the cone surface \( P \left( z_1, \cdots, z_n \right) = 0 \) of zeros, so that for the correct definition of generalized functions corresponding to \( P^\lambda \) the generalized functions

11) \( (P \left( x \right) + c i0)^{1/2} = \lim_{\epsilon \to 0^+} \left( P \left( x \right) + c i\epsilon \right)^{1/2} \)

\( \exp \left( i2\arg \left( P \left( x \right) + c i\epsilon \right) \right) \)
with either \( c = -1 \) or \( c = 1 \) were introduced. Therefore, the identity

12) \( F \left( \Psi_{k_1, \cdots, k_n} \right) \left( x \right) = \left( \sum_{j=1}^{n} x_j^2 - \sum_{j=k_1+1}^{n} x_j^2 \right) \left[ F \left( \Psi_{k_1, \cdots, k_n} \right) \left( x \right) \right]^2 \)

or

13) \( F \left( \Psi \right) = -1 / \left( P \left( x \right) + c i0 \right) \)
follows, where \( c = -1 \) or \( c = 1 \).

The inverse Fourier transform in the class of the generalized functions is:

14) \( F^{-1} \left( \left( P \left( x \right) + c i0 \right)^{1/2} \right) \left( z_1, \cdots, z_n \right) = \exp \left( -pi \left( n - k_1 \right) / 2 \right) 2^{k_1 - 1 / 2} \pi^{n/2} \Gamma \left( \lambda + n/2 \right) \)
\( \left( Q \left( z_1, \cdots, z_n \right) - c i0 \right)^{-\lambda / 2} / \left( \Gamma \left( -\lambda \right) \right)^{1/2} \)
for each \( \lambda \in C \) and \( n \geq 3 \) (see § IV.2.6 [21]), where \( D = \det \left( g_{ij} \right) \) denotes a discriminant of the quadratic form \( P \left( x \right) = \sum_{i,j=1}^{n} g_{ij} x_i x_j \), while
Consider \( \lambda \in R \), the generalized function

\[
\left( P(x)^2 + \varepsilon^2 \right)^{\lambda/2} \exp \left( i \lambda \text{Arg} \left( P(x) + l\varepsilon \right) \right)
\]

is non-degenerate and for it the Fourier transform is defined. The limit \( \lim_{\varepsilon \to 0, \varepsilon > 0} \) gives by our definition the Fourier transform of \( \left( P(x) + c\varepsilon \right)^{\lambda/2} \). Since

\[
c_j \left( \beta_j + \sum_{j=\ell \in \mathbb{K}} n_j c_j \beta_j \right) = \sum_{j=\ell \in \mathbb{K}} n_j c_j \beta_j
\]

for all \( \beta_j \in R \) and any \( 1 \leq j \leq n \) in accordance with the conditions imposed on \( c_j \) at the beginning of this section and \( iN_j = N_j i \) for each \( j \), the Fourier transform with the generator \( i \) can be accomplished subseqently by each variable using Identity (19). The transform \( x_j \leftrightarrow \left( c_j \right)^{1/2} x_j \) is diagonal and

\[
\left[ \cdots \left( c_j^{1/2} c_{j+1}^{1/2} \right) \cdots \right] c_n^{1/2} = 1,
\]

so we can put \( |D| = 1 \). Each Cayley-Dickson number can be presented in the polar form

\[
z = |z| e^{i\phi} \quad , \quad \phi \in R \quad , \quad |\phi| \leq \pi, \quad M \quad \text{is a pure imaginary Cayley-Dickson number} \quad |M| = 1,
\]

\[
\text{Arg}(z) = (\phi + 2\pi k)M \quad \text{has the countable number of values,} \quad k \in Z \quad (\text{see § 3 in [5,6]). Therefore, we choose the branch} \quad |z|^{1/2} = |z|^{1/2} \exp \left( (\text{Arg}z)/2 \right), \quad |x|^{1/2} > 0 \quad \text{for} \quad z \neq 0, \quad \text{with} \quad |\text{Arg}(z)| \leq \pi, \quad \text{Arg}(M) = M\pi/2 \quad \text{for each purely imaginary} \quad M \quad \text{with} \quad |M| = 1.
\]

We treat the iterated integral as in § 6, i.e. with the same order of brackets. Taking initially \( c_j \in R \) and considering the complex analytic extension of formulas given above in each complex plane \( \mathbb{R} \ominus \mathbb{N} \), by \( c_j \) for each \( j \) by induction from 1 to \( n \), when \( c_j \) is not real in the operator \( A \), \( \text{Im}(c_j) \in \mathbb{R}N_j \), we get the fundamental solutions for \( A \) with the form

\[
\left( P(x) + c\varepsilon \right)^{\lambda/2} \quad \text{instead of} \quad \left( P(x) + c\varepsilon \right)^{\lambda/2}
\]

with multipliers

\[
\left[ \cdots \left( c_j^{1/2} c_{j+1}^{1/2} \right) \cdots \right] c_n^{1/2}
\]

instead of

\[
\exp \left( i\lambda \text{Arg} \left( P(x) + c\varepsilon \right) \right)
\]

as above and putting \( |D| = 1 \). Thus

\[
\left[ \cdots \left( c_j^{1/2} c_{j+1}^{1/2} \right) \cdots \right] c_n^{1/2} = 1.
\]

2.36. Noncommutative Transforms of Products and Convolutions of Functions in the \( A \), Spherical Coordinates

For any Cayley-Dickson number \( z = z_{0\ell} + \cdots + z_{n\ell} i \), we consider projections

\[
\theta_j(z) = z_j, \quad z_j \in R \quad \text{or} \quad C_j \quad \text{or} \quad H_{j,k,\ell},
\]

\[
j = 0, \cdots, 2^{r-1}, \quad \theta_j(z) = \pi_j(z)\bar{c}_j
\]

given by Formulas 2(5,6) and 33(17). We define the following operators

\[
Q(y) = \sum_{j,k=0}^n g^{j,k} x_j x_k
\]
is the dual quadratic form so that

\[
\sum_{j=0}^n g^{j,k} x_j x_k = \delta^{jl} \quad \text{for all} \quad j,l; \quad \delta^{jl} = 1 \quad \text{for} \quad j = l \quad \text{and} \quad \delta^{jl} = 0 \quad \text{for} \quad j \neq l.
\]

In the particular case of \( n = 2 \) the inverse Fourier transform is given by the formula:

\[
15) \quad F^{-1}\left[ \left( P(x) + ci\varepsilon \right)^{\lambda/2} \right](z_1, z_2)
\]

\[
= -4^{-1} |D|^{1/2} \exp \left( -\pi\varepsilon (n - k_i) i/2 \right)
\]

\[
\ln \left( Q(z_1, \ldots, z_n) - ci\varepsilon \right).
\]

Making the inverse Fourier transform \( F^{-1} \) of the function \(-1/\left( P(x) + ci\varepsilon \right)\) in this particular case of \( \lambda = -1 \) we get two complex conjugated fundamental solutions

\[16) \quad \Psi_{k_1, \ldots, k_n} (z_1, \ldots, z_n)
\]

\[
= -\exp \left( \pi\varepsilon (n - k_i) i/2 \right) \Gamma \left( (n/2) - 1 \right)
\]

\[
\left( Q(z_1, \ldots, z_n) + ci\varepsilon \right)^{1/(n/2)} \left( 4\pi^{n/2} \right)
\]

for \( 3 \leq n \) and \( 1 \leq k_i < n \), while

\[17) \quad \Psi_{1,1} (z_1, z_2)
\]

\[
= 4^{-1} \exp \left( \pi\varepsilon (n - k_i) i/2 \right) \ln \left( Q(z_1, z_2) + ci\varepsilon \right)
\]

for \( n = 2 \), where either \( c = 1 \) or \( c = -1 \).

Generally for the operator \( A \) given by Formula (1) we get

\[P(x) = g_{\alpha\beta} x_\alpha x_\beta \]

where

\[
P_{\alpha\beta} (x) = \sum_{j=0}^n x_j^2 \text{Re}(c_j)
\]

and \( P(x) = \sum_{j=0}^n x_j^2 \text{Im}(c_j) \) are the real and imaginary parts of \( P \), \( \text{Im}(x) = x - \text{Re}(x) \) for any Cayley-Dickson number \( x \). Take \( l = 1 \) and consider the form \( P(x) + c\varepsilon \) with \( \varepsilon \neq 0 \) and either \( c = 1 \) or \( c = -1 \), then \( P(x) + c\varepsilon \neq 0 \) for each \( x \in R^c \). We put

\[18) \quad (P(x) + c\varepsilon)^{\lambda/2} = \lim_{\varepsilon \to 0, \varepsilon > 0} \left( P(x)^2 + \varepsilon^2 \right)^{\lambda/2}
\]

\[
\exp \left( i\lambda \text{Arg} \left( P(x) + l\varepsilon \right) \right).
\]

for \( 3 \leq n \), while

\[19) \quad \Psi (z_1, \ldots, z_n)
\]

\[
= -\Gamma \left( (n/2) - 1 \right) \left( P^*(z_1, \ldots, z_n) - c\varepsilon \right)^{1/(n/2)} \left[ \cdots \left( c_j^{1/2} c_{j+1}^{1/2} \right) \cdots \right] c_n^{1/2} \left( 4\pi^{n/2} \right)
\]

since \( c_j = c_j^{-1} \) for \( j \), \( y_j q_j = y_j \left( c_j^{1/2} \right) q_j c_j^{1/2} \), while

\[
\left[ \cdots \left( d_j c_j^{1/2} q_j c_j^{1/2} \right) \cdots \right] d_j c_j^{1/2} q_j = d_j \cdots d_n \left[ \cdots \left( c_j^{1/2} c_{j+1}^{1/2} \right) \cdots \right] c_n^{1/2} = 1.
\]
2) $R_{n,j}(F^n(p;\zeta)) = F^n(p_0,1^{(n)}p_1,\ldots,(1)^{n}\zeta_{j+1-\delta_{j,n}}p_{j+1-\delta_{j,n}},$
\hspace{1cm}$p_{j+2-\delta_{j,n}},\ldots,p_{n+1}(1)^{n}\zeta_{1}+\alpha\alpha_{1}/2,\ldots,(1)^{n}\zeta_{j+1-\delta_{j,n}}\zeta_{j+1-\delta_{j,n}}+\pi\alpha_{j+1-\delta_{j,n}}/2,\zeta_{j+2-\delta_{j,n}},\ldots,\zeta_{n})$

on images $F^n_{\alpha}$, $2n^{s-1} \leq n \leq 2s^{r-1}$, $j = 0,\ldots,n$. For $\alpha_j$ and $\beta_j \in \{0,1\}$ their sum $\alpha_j + \beta_j$ is considered by $(mod \ 2)$, i.e. in the ring $Z_2 = Z/2Z$, for two vectors $\alpha$ and $\beta \in \{0,1\}^{s-1}$ their sum is considered componentwise in $Z_2$. Let

3) $F^n(f;u;p;\zeta) = \sum_{j=0}^{u} \sum_{i=0}^{v-1}(\theta_i(\zeta)(f;u;p;\zeta))_j \nu_i$,

4) $F^n(fg;u;p;\zeta) = \sum_{j=0}^{u} \sum_{\alpha,\beta \in \{0,1\}^{s-1}} (\theta_i(\zeta)(f;u;p;\zeta))_j \nu_i$, also $F^n(p;\zeta) := \sum_{j=0}^{u} \theta_i(F^n(\theta_j(f;u;p;\zeta))_j)$

for an original $f$, where $u(p;\zeta)$ is given by Formulas 2(1,2.2,1). If $f$ is real or $C_i$ or $H_{1,1,1}$-valued, then $F^n = \theta_i(F^n)$.

Theorem. If $f$ and $g$ are two originals, then

5) $f(t)g(t) = \sum_{j=0}^{u} \sum_{\alpha,\beta \in \{0,1\}^{s-1}} \theta_i(\zeta)(f(t)) \theta_j(\zeta)(g(t))_j$.

The functions $\theta_i(f)$ and $\theta_j(g)$ are real or $C_i$ or $H_{1,1,1}$ valued respectively. The non-commutative transform of $fg$ is:

$$F^n(\theta_i(f)) = \int_{\mathbb{R}^n} f(t)g(t) \exp(-u(p;\zeta)) \, dt = \left[ \int_{\mathbb{R}^n} f(t)g(t) e^{-\rho_0 i} \cos(p_0 s_1 + \zeta_1) \nu_0 \right] dt$$

On the other hand,

$$\int_{\mathbb{R}^n} \left( \prod_{j=1}^{n-1} \int_{\mathbb{R}^n} f_j(t) \, dt \right) e^{-\rho_0 i} \sum_{j=1}^{n-1} \int_{\mathbb{R}^n} \left( \prod_{j=1}^{n-1} \int_{\mathbb{R}^n} f_j(t) \, dt \right) e^{-\rho_0 i} \sum_{j=1}^{n-1} \int_{\mathbb{R}^n} f_j(t) \, dt$$

where $k = 1,\ldots,n$. Therefore, using Euler’s formula $e^{i\phi} = \cos(\phi) + i\sin(\phi)$ and the trigonometric formulas
\[ \cos(\phi + \psi) = \cos(\phi)\cos(\psi) - \sin(\phi)\sin(\psi), \]
\[ \sin(\phi + \psi) = \sin(\phi)\cos(\psi) + \cos(\phi)\sin(\psi) \]
for all $\phi, \psi \in \mathbb{R}$, and Formulas (6,7), we deduce expressions for $\theta_j(F^n(\theta_i(f)))$. We get the integration by $q_0,\ldots,q_n$, which gives convolutions by the $p_0,\ldots,p_n$ variables. Here $q_0 \in R$ and $\eta \in A_1$ are any marked numbers. Thus from Formulas (5-7) and 2(1,2,2,1,4) we deduce Formula (4).

Moreover, one certainly has

8) $\int_{\mathbb{R}^n} (f* g)(t) \, dt = \left[ \int_{\mathbb{R}^n} f(t) e^{-\rho_0 i} \sum_{j=1}^{n-1} \int_{\mathbb{R}^n} f_j(t) \, dt \right] \left[ \int_{\mathbb{R}^n} g(t) e^{-\rho_0 i} \sum_{j=1}^{n-1} \int_{\mathbb{R}^n} g_j(t) \, dt \right]$. 

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for each $1 \leq k \leq n$, $\gamma_j \in \{-1, 1\}$, since $s_j(t) = s_j(t-\tau) + s_j(\tau)$ for all $j = 1, \ldots, n$ and $t, \tau \in \mathbb{R}^s$. Thus from Relations (6.8) and 2(1,2,1,4 and Euler’s formula one deduces expressions for $\theta \{ F^m'(f^g)\}$ and Formula (4.1).

### 2.37. Moving Boundary Problem

Let us consider a boundary problem

1) $Af = g$ in the half-space $t_n \geq \phi(t_n)$, where

2) $F^n\left(\sum_{|\alpha|\leq m} b_{\alpha} \partial_{\alpha} \chi_{\gamma_{j_0} \geq 0} f(t); p; \zeta\right) = \sum_{|\alpha|\leq m, 0 < \alpha_0, \sum_{\alpha_0}=1} b_{\alpha} \left(\delta_{\alpha, \alpha_0} - 1\right)
\left(p_0 + S_n p_1\right)^{\alpha_1} \left(p_0 + p_2 S_2\right)^{\alpha_2} \cdots \left(p_0 + p_n S_n\right)^{\alpha_n} F^{n-1,*}\left(\delta_{\alpha_0}^w(y) u\left(p, (y); \zeta\right)\right) + \sum_{|\alpha|\leq m} a_{\alpha} \left(\delta_{\alpha, \alpha_0} - 1\right)
\left(p_0 + S_n p_1\right)^{\alpha_1} \left(p_0 + p_2 S_2\right)^{\alpha_2} \cdots \left(p_0 + p_n S_n\right)^{\alpha_n} F^n\left(\chi_{\gamma_{j_0} \geq 0} (y) w(y); p; \zeta\right) = G^n\left(p; \zeta\right)$

in the $A_\gamma$, spherical coordinates and

2.1) $F^n\left(\sum_{|\alpha|\leq m} a_{\alpha} \partial_{\alpha} \chi_{\gamma_{j_0} \geq 0} f(t); p; \zeta\right) = \sum_{|\alpha|\leq m, 0 < \alpha_0, \sum_{\alpha_0}=1} a_{\alpha} \left(\delta_{\alpha, \alpha_0} - 1\right)
\left(p_0 + S_n p_1\right)^{\alpha_1} \left(p_0 + p_2 S_2\right)^{\alpha_2} \cdots \left(p_0 + p_n S_n\right)^{\alpha_n} F^{n-1,*}\left(\delta_{\alpha_0}^w(y) u\left(p, (y); \zeta\right)\right) + \sum_{|\alpha|\leq m} a_{\alpha} \left(\delta_{\alpha, \alpha_0} - 1\right)
\left(p_0 + S_n p_1\right)^{\alpha_1} \left(p_0 + p_2 S_2\right)^{\alpha_2} \cdots \left(p_0 + p_n S_n\right)^{\alpha_n} F^n\left(\chi_{\gamma_{j_0} \geq 0} (y) w(y); p; \zeta\right) = G^n\left(p; \zeta\right)$

in the $A_\gamma$, Cartesian coordinates, where $w(y) = f(t(y))(dy_n/dy_n)$.

Expressing $F^n\left(\chi_{\gamma_{j_0} \geq 0} (y) w(y); p; \zeta\right)$ through $G^n\left(p; \zeta\right)$ and the boundary terms $F^{n-1,*}\left(\delta_{\alpha_0}^w(y) u\left(p, (y); \zeta\right)\right)$ as in § 8.3 and making the inverse transform 8(4) or 8.1(1), or using the integral kernel $\zeta$ as in § 8.5, one gets a solution $w(y)$ or $f(t) = w(y(t))(dy_n(t)/dt_n)$ (See reference [21-30]).

### 2.38. Partial Differential Equations with Discontinuous Coefficients

Consider a domain $U$ and its subdomains $U \supset U_1 \supset \cdots \supset U_k$ satisfying Conditions 28(D1,D4,j-vii) so that coefficients of an operator $A$ (see 28(2)) are constant on $\text{Int}(U_k)$ and on $V_i = U \backslash \text{Int}(U_i)$, $V_1 = U \backslash \text{Int}(U_1)$, \ldots, $V_k = U \backslash \text{Int}(U_k)$ and are allowed to be discontinuous at the common borders $\partial V_j \cap \partial U_k$ for each $j = 1, \ldots, k$. Each function $f\chi_{U_j}$ is an original on $U$ or a generalized function with the support $\text{supp}\{f\chi_{U_j}\} \subset U_j$ if $f$ is an original or a generalized function on $U$. Choose operators $A^j$ with constant coefficients on $U_j$ and $A^j|_{\text{Int}(U_j)} = 0$, where $U^0 = U$, so that $A^0 = A^k$, \ldots, $A^1 = A^k + \cdots + A^4$, \ldots,

$\phi(0) = 0$ and $\phi(t_0) < t_0$ for each $0 \leq t_0 \in R$. Suppose that the function $t_n - \phi(t_n) = \psi(t_n)$ is differentiable and bijective. For example, if $0 < v < 1$ and $\phi(t_j) = vt_j$, then the boundary is moving with the speed $v$. Make the change of variables $y_n = \psi(t_n)$, $y_1 = t_1, \ldots, y_{n-1} = t_{n-1}$, then $t_n = y_n^{-1}(y_n)$ and $dt_n = ds_n = (dy_n/dy_n)dy_n$ and due to Theorem 25 we infer that $t_n = y_n^{-1}(y_n)$. Theorem 8 and 28.1).

When the corresponding condition 8(3) is satisfied (see 28(1,2)) a solution $w(y)$ exists, then the boundary is moving with the speed $v$. Make the change of variables $y_n = \psi(t_n)$, $y_1 = t_1, \ldots, y_{n-1} = t_{n-1}$, then $t_n = y_n^{-1}(y_n)$ and due to Theorem 25 we infer that $t_n = y_n^{-1}(y_n)$.

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corresponding derivatives \( \left( \partial^\beta f^{\mu_1\cdots\mu_j} / \partial \nu^\beta \right) \big|_{\nu^\beta \in \nu} \) for 

some \( \beta < \text{ord}(\lambda^\alpha) \) in accordance with the operator \( \lambda^\alpha \) and the boundary conditions \( 2 \partial f^{j} / \partial \nu^{j} \) on \( U \), having found \( f^{j} \) for each \( j = 0, \ldots, k \) one gets the solution \( f = f^{0} + \cdots + f^{k} \) on \( U \) of Problem (1) with the boundary conditions \( 2 \partial f^{j} / \partial \nu^{j} \) on \( U \).

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