Uniformly Stable Positive Monotonic Solution of a Nonlocal Cauchy Problem

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Received October 9, 2011; revised December 7, 2011; accepted December 30, 2011

ABSTRACT
In this paper, we study the existence of a uniformly stable positive monotonic solution for the nonlocal Cauchy problem

\[ x'(t) = f(t, x(t)), t \in [0, T] \quad \text{with the nonlocal condition} \quad \sum_{j=1}^{m} b_j x(\eta_j) = x_i, \quad \text{where} \quad \eta_j \in (0, a) \subset [0, T]. \]

Keywords: Nonlocal Cauchy Problem; Local and Global Existence Nondecreasing Positive Solution; Continuous Dependence; Lyapunov Uniformly Stability

1. Introduction
Problems with non-local conditions have been extensively studied by several authors in the last two decades. The reader is referred to (see [1-14] and [15-18]) and references therein.

Here we are concerned with the nonlocal Cauchy problem

\[ x'(t) = f(t, x(t)), t \in [0, T], \]
\[ \sum_{j=1}^{m} b_j x(\eta_j) = x_i, \quad \eta_j \in (0, a) \subset [0, T], \quad \text{and} \quad \left( \sum_{j=1}^{m} b_j \right) \neq 0. \]

Let \( X \) be the class of all continuous functions defined on \( [0, T], T < \infty \) with the norm \( \| x \| = \sup_{t \in [0, T]} | x(t) |, x \in X. \)

Let \( Y \) be the class of all continuous functions defined on \( [\tau_0, T], T < \infty \) with the equivalent norm \( \| x \| = \sup_{t \in [\tau_0, T]} e^{|x(t)|} | x(t) |, x \in Y, \)

where \( \tau_0 = \max \{ \eta_j, j = 1, 2, \ldots, m \} \), and \( N \) is positive arbitrary.

Here we firstly study, in \( X \), the local existence of the solution of the problem (1)-(2) and the continuous dependence of the parameter \( x \), will be proved.

Secondly, we study, in \( Y \), the global existence and Lyapunov uniform stability of the solution of the problem (1)-(2).

2. Integral Equation Representation
Consider the nonlocal Cauchy problem (1)-(2). Let \( f: [0, T] \times R^+ \rightarrow R^+ \) is continuous and satisfies the Lipschitz condition

\[ | f(t, x) - f(t, y) | \leq k | x - y |, \quad k > 0, \]

for all \( x, y \in R^+ \).

Lemma 2.1. The solution of the nonlocal Cauchy problem (1)-(2) can be expressed by the integral equation

\[ x(t) = B \left( x_0 - \sum_{j=1}^{m} b_j \int_{\tau_0}^{t} f(s, x(s)) ds \right) + \int_{\tau_0}^{t} f(s, x(s)) ds, \]

where \( B = \left( \sum_{j=1}^{m} b_j \right)^{-1} \).

Proof. Integrating the Equation (1), we obtain

\[ x(t) = x(0) + \int_{0}^{t} f(s, x(s)) ds. \]

Let \( t = \eta_j \) in (5), we obtain

\[ x(\eta_j) = x(0) + \int_{\tau_0}^{\eta_j} f(s, x(s)) ds, \]

and

\[ \sum_{j=1}^{m} b_j x(\eta_j) = \sum_{j=1}^{m} b_j x(0) + \int_{\tau_0}^{\eta_j} f(s, x(s)) ds. \]

Substitute from (2) into (7), we obtain

\[ x(0) = B \left( x_0 - \sum_{j=1}^{m} b_j \int_{\tau_0}^{\eta_j} f(s, x(s)) ds \right). \]
Substitute from (8) into (5), we obtain
\[ x(t) = B \left( x_1 - \sum_{j=1}^m b_j \int_0^t f(s, x(s)) \, ds \right) + \int_0^t f(s, x(s)) \, ds. \]

**Corollary 2.1.** The solution of the integral Equation (4) is nondecreasing.

**Proof.** Let \( x \) be a solution of the integral Equation (4), then for \( t_1 < t_2 \), we have
\[
x(t_1) = B \left( x_1 - \sum_{j=1}^m b_j \int_0^{t_1} f(s, x(s)) \, ds \right) + \int_0^{t_1} f(s, x(s)) \, ds < B \left( x_1 - \sum_{j=1}^m b_j \int_0^{t_2} f(s, x(s)) \, ds \right) + \int_0^{t_2} f(s, x(s)) \, ds = x(t_2),
\]
which proves that the solution \( x \) of the integral Equation (4) is nondecreasing.

**Corollary 2.2.** Let \( f \) be satisfies (3). The solution of the integral Equation (4) is positive for \( t \in [a, T] \).

**Proof.** Define the operator \( T : C[0,T] \to C[0,T] \) by
\[
T(x)(t) = B \left( x_1 - \sum_{j=1}^m b_j \int_0^t f(s, x(s)) \, ds \right) + \int_0^t f(s, x(s)) \, ds.
\]
Differentiating (4), we get
\[
x'(t) = f(t, x(t)).
\]
Multiplying by \( B = \left( \sum_{j=1}^m b_j \right)^{-1} \), we obtain
\[
B \sum_{j=1}^m b_j \int_0^t f(s, x(s)) \, ds \leq B \sum_{j=1}^m b_j \int_0^t f(s, x(s)) \, ds = \int_0^t f(s, x(s)) \, ds
\]
and the solution \( x \) of the integral Equation (4) is positive for \( t \in [a, T] \). This complete the proof. \( \square \)

3. Local Existence of Solution

**Theorem 3.1.** Let \( f \) be satisfies the Lipschitz condition. If \( T < k \left( 1 + |B| \sum_{j=1}^m b_j \right)^{-1} \) then the nonlocal Cauchy problem (1)-(2) has a unique nondecreasing positive solution.

**Proof.** Define the operator \( T : C[0,T] \to C[0,T] \) by
\[
T(x)(t) = B \left( x_1 - \sum_{j=1}^m b_j \int_0^t f(s, x(s)) \, ds \right) + \int_0^t f(s, x(s)) \, ds.
\]

Let \( x, y \in C[0,T] \), then
\[
T(x) - T(y) = -B \sum_{j=1}^m b_j \int_0^t f(s, x(s)) \, ds + \int_0^t f(s, x(s)) \, ds + B \sum_{j=1}^m b_j \int_0^t f(s, y(s)) \, ds - \int_0^t f(s, y(s)) \, ds
\]
\[
= -B \sum_{j=1}^m b_j \left[ f(s, x(s)) - f(s, y(s)) \right] \, ds + \int_0^t \left[ f(s, x(s)) - f(s, y(s)) \right] \, ds,
\]
\[
\|T(x) - T(y)\| \leq k \left| B \right| \sum_{j=1}^m b_j \left\| x(s) - y(s) \right\| \, ds + k \int_0^t \left\| x(s) - y(s) \right\| \, ds
\]
\[
\leq kT \left| B \right| \sum_{j=1}^m b_j \left\| x - y \right\| + kT \left\| x - y \right\| \leq kT \left( 1 + \left| B \right| \sum_{j=1}^m b_j \right) \left\| x - y \right\| \leq K \left\| x - y \right\|
\]
but if
\[
K = kT \left( 1 + \left| B \right| \sum_{j=1}^m b_j \right) < 1,
\]
then we get
\[
\|T(x) - T(y)\| \leq K \left\| x - y \right\|,
\]
which proves that the map \( T : C[0,T] \to C[0,T] \) is contraction.

Applying the Banach contraction fixed point theorem we deduce that the integral Equation (4) has a unique solution \( x \in C[0,T] \).

To complete the proof, we prove that the integral Equation (4) satisfies nonlocal problem (1)-(2).

Differentiating (4), we get
\[
x'(t) = f(t, x(t)).
\]

Let \( t = \eta_j \) in (4), we obtain
\[
x(\eta_j) = B \left( x_1 - \sum_{j=1}^m b_j \int_0^{\eta_j} f(s, x(s)) \, ds \right) + \int_0^{\eta_j} f(s, x(s)) \, ds,
\]
then
\[
\sum_{j=1}^m b_j x(\eta_j) = x_1.
\]
This implies that there exist a unique nondecreasing positive solution \( x \in C[0,T] \) of the nonlocal Cauchy problem (1)-(2). This complete the proof. \( \blacksquare \)

4. Continuous Dependence of the Solution

Consider the nonlocal Cauchy problem

\[
\left\{ \begin{array}{l}
x'(t) = f(t, x(t)), t \in [0, T], \\
\sum_{j=1}^{m} b_{j} x(\eta_{j}) = \bar{x}_{i}, \quad \text{and} \quad \eta_{j} \in (0, a) \subset [0, T].
\end{array} \right.
\]

Definition 4.1. The solution of the nonlocal Cauchy problem (1)-(2) continuously dependence on \( x_{i} \) if

\[
\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0, \text{ such that } |x_{i} - \bar{x}_{i}| < \delta, \quad \text{then } |x(t) - \bar{x}(t)| < \varepsilon
\]

where \( \bar{x}(t) \) is the solution of the nonlocal Cauchy problem \( \bar{P} \).

Now we have the following theorem

**Theorem 4.1.** The solution of the nonlocal Cauchy problem (1)-(2) continuously dependence on \( x_{i} \).

**Proof.** Let \( x(t), \bar{x}(t) \) are the solutions of (1)-(2) and \( \bar{P} \) respectively.

Then we can get

\[
x(t) - \bar{x}(t) = B(x_{i} - \bar{x}_{i}) - B \sum_{j=1}^{m} \int_{0}^{\eta_{j}} f(s, x(s)) \, ds + B \sum_{j=1}^{m} \int_{0}^{\eta_{j}} f(s, \bar{x}(s)) \, ds + \int_{0}^{\eta} [f(s, x(s)) - f(s, \bar{x}(s))] \, ds
\]

\[
\|x(t) - \bar{x}(t)\| \leq |B|x_{i} - \bar{x}_{i}| + |B| \sum_{j=1}^{m} b_{j} \|f(s, x(s)) - f(s, \bar{x}(s))\| ds + \int_{0}^{\eta} |f(s, x(s)) - f(s, \bar{x}(s))| ds
\]

\[
\leq |B|x_{i} - \bar{x}_{i}| + k |B| \sum_{j=1}^{m} b_{j} \sup_{t \in [0, T]} |x(s) - \bar{x}(s)| ds + k \sup_{t \in [0, T]} |x(s) - \bar{x}(s)| ds
\]

\[
\leq |B|x_{i} - \bar{x}_{i}| + k |B| \sum_{j=1}^{m} b_{j} \sup_{t \in [0, T]} |x(t) - \bar{x}(t)| \int_{0}^{\eta_{j}} ds + k \sup_{t \in [0, T]} |x(t) - \bar{x}(t)| \int_{0}^{\eta_{j}} ds
\]

Therefore, for \( \delta > 0 \) such that

\[
|x_{i} - \bar{x}_{i}| < \delta(\varepsilon),
\]

we can find

\[
\varepsilon = \left(1 - kT \left(1 + |B| \sum_{j=1}^{m} b_{j}\right)\right)^{-1} |B| \delta
\]

such that \( \|x - \bar{x}\| \leq \varepsilon \), which complete the proof theorem.

5. Global Existence of Solution

**Theorem 5.1.** Let \( f \) be satisfies the Lipschitz condition, then the nonlocal Cauchy problem (1)-(2) has a unique nondecreasing positive solution.

**Proof.** Define the operator \( T : C[\tau_{0}, T] \rightarrow C[\tau_{0}, T] \) by the Equation (9).

Let \( x, y \in C[\tau_{0}, T] \), then

\[
Tx(t) - Ty(t) = -B \sum_{j=1}^{m} \int_{0}^{\eta_{j}} f(s, x(s)) \, ds + \int_{0}^{\eta} f(s, y(s)) \, ds + B \sum_{j=1}^{m} \int_{0}^{\eta_{j}} f(s, y(s)) \, ds - \int_{0}^{\eta} f(s, y(s)) \, ds
\]

\[
\|Tx(t) - Ty(t)\| \leq k |B| \sum_{j=1}^{m} b_{j} \left|\int_{0}^{\eta_{j}} x(s) - y(s) \, ds\right| + k \sup_{t \in [0, T]} |x(s) - y(s)| ds
\]

\[
e^{-N(\tau_{0})} \|Tx(t) - Ty(t)\| \leq k |B| \sum_{j=1}^{m} b_{j} \left|e^{-N(\tau_{0})} \int_{0}^{\eta_{j}} x(s) - y(s) \, ds\right| + k e^{-N(\tau_{0})} \sup_{t \in [0, T]} |x(s) - y(s)| ds
\]
\[ e^{-\lambda(t-t_0)} \left| T_x(t) - T_y(t) \right| \leq k \sum_{j=1}^{m} b_j \int_{0}^{\rho} e^{-\lambda(s-t_0)} e^{-\lambda(s-t_0)} \left| x(s) - y(s) \right| ds \]
\[
\quad + k \int_{0}^{\rho} e^{-\lambda(s-t_0)} e^{-\lambda(s-t_0)} \left| x(s) - y(s) \right| ds
\]
\[
\leq k \sum_{j=1}^{m} b_j \left\| x - y \right\| \int_{0}^{\rho} e^{-\lambda(s-t_0)} ds + k \left\| x - y \right\| \int_{0}^{\rho} e^{-\lambda(s-t_0)} ds
\]
\[
\leq k \sum_{j=1}^{m} b_j \left\| x - y \right\| \frac{e^{-\lambda(t)} - e^{-\lambda(t_0)}}{\lambda} + k \left\| x - y \right\| \frac{1 - e^{-\lambda t}}{\lambda}
\]
\[
\leq \frac{k}{\lambda} \left\{ \sum_{j=1}^{m} b_j \left( e^{-\lambda(t)} - e^{-\lambda(t_0)} \right) + \left( 1 - e^{-\lambda t_0} \right) \right\} \left\| x - y \right\|
\]

where

\[ K = \frac{k}{\lambda} \left( \sum_{j=1}^{m} b_j + 1 \right). \]

Choose \( N \) large enough such that \( K < 1 \), then

\[ \left\| T_x - T_y \right\| \leq K \left\| x - y \right\|, \]

therefore the map \( T : C[t_0, T] \rightarrow C[t_0, T] \) is contraction.

Applying the Banach contraction fixed point theorem we deduce that the integral Equation (4) has a unique solution \( x \in C[t_0, T] \).

To complete the proof, we prove that the integral Equation (4) satisfies nonlocal problem (1)-(2).

Differentiating (4), we get

\[ x'(t) = f(t, x(t)). \quad (11) \]

Let \( t = \eta_j \) in (4), we obtain

\[ x(\eta_j) = B \left( x_0 - \sum_{j=1}^{m} b_j \int_{0}^{s_j} f(s, x(s)) ds \right) + \int_{0}^{s_j} f(s, x(s)) ds, \]

then

\[ \sum_{j=1}^{m} b_j x(\eta_j) = x_0. \]

This implies that there exist a unique nondecreasing positive solution \( x \in C[t_0, T] \) of the nonlocal Cauchy problem (1)-(2), This complete the proof. \( \blacksquare \)

6. Lyapunov Uniform Stability of the Solution

Consider here the nonlocal Cauchy problem

\[ \left\{ \begin{array}{l}
 x'(t) = f(t, x(t)), t \in [t_0, T], \\
 \sum_{j=1}^{m} b_j x(\eta_j) = \bar{x}_0, \quad \text{and} \quad \eta_j \in (0, a) \subset [t_0, T].
\end{array} \right. \]

Definition 6.1. The solution of the nonlocal Cauchy problem (1)-(2) is uniform stable, if for some \( \epsilon > 0, \delta(\epsilon) > 0 \), such that

\[ |x(t) - \bar{x}(t)| < \epsilon. \]

where \( \bar{x}(t) \) is the solution of the nonlocal Cauchy problem \( \bar{P} \).

Now we have the following theorem

Theorem 6.1. The solution of the nonlocal Cauchy problem (1)-(2) is uniformly stable.

Proof. Let \( x(t), \bar{x}(t) \) are the solutions of (1)-(2) and \( \bar{P} \) respectively.

Then we can get

\[ x(t) - \bar{x}(t) = B(x_0 - \bar{x}_0) - B \sum_{j=1}^{m} b_j \int_{0}^{s_j} f(s, x(s)) ds + B \sum_{j=1}^{m} b_j \int_{0}^{s_j} f(s, \bar{x}(s)) ds + \sum_{j=1}^{m} \{ f(s, x(s)) - f(s, \bar{x}(s)) \} ds \]

\[ \left\| x(t) - \bar{x}(t) \right\| \leq \left\| B \right\| \left\| x_0 - \bar{x}_0 \right\| + \left\| B \right\| \sum_{j=1}^{m} b_j \int_{0}^{s_j} \left\| f(s, x(s)) - f(s, \bar{x}(s)) \right\| ds + \sum_{j=1}^{m} \left\| f(s, x(s)) - f(s, \bar{x}(s)) \right\| ds \]

\[ \quad \leq \left\| B \right\| \left\| x_0 - \bar{x}_0 \right\| + k \left\| B \right\| \sum_{j=1}^{m} b_j \int_{0}^{s_j} \left\| x(s) - \bar{x}(s) \right\| ds + k \int_{0}^{s_j} \left\| x(s) - \bar{x}(s) \right\| ds \]

\[ e^{-\lambda(t-t_0)} \left\| x(t) - \bar{x}(t) \right\| \leq e^{-\lambda(t-t_0)} \left\| B \right\| \left\| x_0 - \bar{x}_0 \right\| + k \left\| B \right\| \sum_{j=1}^{m} b_j \int_{0}^{s_j} e^{-\lambda(t-t_0)} \left\| x(t) - \bar{x}(t) \right\| ds \]

\[ + k \int_{0}^{s_j} e^{-\lambda(t-t_0)} \left\| x(t) - \bar{x}(t) \right\| ds \]
\[ \|x - \bar{x}\| \leq |\beta| |x_1 - \bar{x}_1| + k |\beta| \sum_{j=1}^{m} |p_j| \|x - \bar{x}\| \int_0^t e^{-N(t-s)} ds + k \|x - \bar{x}\| \int_0^t e^{-N(t-s)} ds \]
\[ \leq |\beta| |x_1 - \bar{x}_1| + k |\beta| \sum_{j=1}^{m} |p_j| \left( \frac{e^{-N(t-b_0)} - e^{-N(t)}}{N} \right) + \frac{1 - e^{-N(t)}}{N} \|x - \bar{x}\| \]
\[ \leq |\beta| |x_1 - \bar{x}_1| + \frac{k}{N} \left( \|x - \bar{x}\| \right) \]
\[ \|x - \bar{x}\| \leq |\beta| \left[ 1 - \frac{k}{N} \left( \|x - \bar{x}\| \right) \right] \]

Therefore, \( |x_1 - \bar{x}_1| < \delta(\varepsilon) \Rightarrow \|x - \bar{x}\| < \varepsilon \), which complete the proof of theorem.

REFERENCES


