A Note on the Paper “Generalized $\phi$-Contraction for a Pair of Mappings on Cone Metric Spaces”

Mohamed Abd El-Rahman Ahmed  
Department of Mathematics, Faculty of Science, Assiut University, Assiut, Egypt  
Email: mahmed68@yahoo.com

Received August 10, 2011; revised September 20, 2011; accepted September 28, 2011

ABSTRACT

We note that Theorem 2.3 [1] is a consequence of the same theorem for one map.

Keywords: Generalized $\phi$-Contraction; Weakly Compatible; Solid Cone

1. Introduction

Huang and Zhang [2] initiated fixed point theory in cone metric spaces. On the other hand, the authors [3] gave a lemma and showed that some fixed point generalizations are not real generalizations. In this note, we show that Theorem 2.3 [1] is so.

Following [2], let $E$ be a real Banach space and $\theta$ be the zero vector in $E$, and $P \subseteq E$. $P$ is called cone iff

1) $P$ is closed, nonempty and $P \neq \{\theta\}$,

2) $ax+by \in P$ for all $a, b, x, y \in P$ and nonnegative real numbers $a, b$,

3) $P \cap (-P) = \{\theta\}$.

For a given cone $P$, we define a partial ordering $\preceq$ with respect to $P$ by $x \preceq y$ iff $y-x \in P$. $x \prec y$ (resp. $x \ll y$) stands for $x \preceq y$ and $x \neq y$ (resp. $y-x \in \text{int}(P)$), where $\text{int}(P)$ denotes the interior of $P$. In the paper we always assume that $P$ is solid, i.e., $\text{int}(P) \neq \emptyset$. It is clear that $x \ll y$ leads to $x \preceq y$ but the reverse need not to be true.

The cone $P$ is called normal if there exists a number $K > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$.

The least positive number satisfying above is called the normal constant of $P$.

Definition 1.1 [2]. Let $X$ be a nonempty set. A function $d : X \times X \to E$ is called cone metric iff

\[ (M_1) \quad \theta \preceq d(x,y), \]

\[ (M_2) \quad d(x,y) = d(y,x) = \theta \iff x = y, \]

\[ (M_3) \quad d(x,y) = d(y,x), \]

\[ (M_4) \quad d(x,y) \preceq d(x,z) + d(z,y), \]

for all $x, y, z \in X$, $(X,d)$ is said to be a cone metric space.

In [3], the authors gave the following important lemma.

Lemma 1.1 Let $X$ be a nonempty and $f : X \to X$. Then there exists a subset $Y \subseteq X$ such that $f(Y) = f(X)$ and $f : Y \to X$ is one-to-one.

Definition 1.2 [4]. Let $(X,d)$ be a cone metric space and $f, g : X \to X$ be mappings. Then, $z \in X$ is called a coincidence point of $f$ and $g$ iff $f(z) = g(z)$.

Definition 1.3 [1]. Let $(X,d)$ be a cone metric space. The mappings $f, g : X \to X$ are weakly compatible iff for every coincidence point $z \in X$ of $f$ and $g$,

$f(g(x)) = g(f(x))$.

Definition 1.4 (see [1]). Let $P$ be a solid cone in a real Banach space $E$. A nondecreasing function $\phi : P \to P$ is called a comparison function iff

1) $\phi(\theta) = \theta$ and $\theta \ll \phi(x) < x$ for $x \in P - \{\theta\}$;

2) $x \in \text{int}(P)$ implies $x - \phi(x) \in \text{int}(P)$;

3) $\lim_{n \to \infty} \phi^n(x) = 0$ for all $x \in P - \{\theta\}$.

In [1], the authors established the following fixed point theorem.

Theorem 1.1 Let $(X,d)$ be a cone metric space, $P$ a solid cone and $f, g : X \to X$. Assume that $(f, g)$ is a generalized $\phi$-contraction; i.e.,

$d(f(x), f(y)) \preceq \phi(u)$

for all $x, y \in X$ and some $u$ where

\[ u \in \left\{ d(g(x), g(y)), d(f(x), g(x)) - d(f(y), g(y)) + d(g(x), f(y)) \right\}. \]

Suppose that $f(X) \subseteq g(X)$, $f(X)$ or $g(X)$ is a complete subspace of $X$, and $f$ and $g$ are weakly compati-
ble. Then the mappings \( f \) and \( g \) have a unique common fixed point in \( X \).

2. Main Result

In Theorem 1.1, if we choose \( g = I_x \) (where \( I_x \) is the identity map on \( X \)), then we have the following theorem.

**Theorem 2.1** Let \( (X, d) \) be a cone metric space, \( P \) a solid cone and \( f : X \to X \). Assume that \( f \) is a generalized \( \phi \)-contraction; i.e., \( d(f(x), f(y)) \leq \phi(u) \) for all \( x, y \in X \) and some \( u \) where

\[
\begin{align*}
    & u \in \left\{ d(x, y), d(f(x), y)+d(f(y), y) \right\}, \\
    & d(x, f(x)) + d(y, f(y)) \leq 2
\end{align*}
\]

Suppose that \( f(X) \) or \( X \) is a complete subspace of \( X \). Then the mapping \( f \) has a unique fixed point in \( X \).

Now, we state and prove our main result in the following way.

**Theorem 2.2** Theorem 1.1 is a consequence of Theorem 2.1.

**Proof.** By Lemma 1.1, there exists \( Y \subseteq X \) such that \( g(Y) = g(X) \) and \( g : Y \to X \) is one-to-one. Define a map \( h : g(Y) \to g(Y) \) by \( h(g(x)) = f(x) \) for each \( x \in g(Y) \). Since \( g \) is one-to-one on \( Y \), then \( h \) is well-defined. \( d(f(x), f(y)) \leq \phi(u) \) for all \( x, y \in X \) and some \( u \) where

\[
\begin{align*}
    & u \in \left\{ d(g(x), g(y)), d(f(x), g(x)), \\
    & d(f(y), g(y)), d(g(x), f(y)) + d(g(y), f(x)) \right\}, \\
    & d(f(x), f(y)) + d(g(y), f(x)) \leq 2
\end{align*}
\]

Since \( f(X) \) or \( g(Y) = g(X) \) is complete, by using Theorem 2.1, there exists \( x_0 \in X \) such that \( h(g(x_0)) = g(x_0) = f(x_0) \). Hence, \( f \) and \( g \) have a point of coincidence which is also unique. Since \( f \) and \( g \) are weakly compatible, then \( f \) and \( g \) have a unique common fixed point.

**Remark 2.1** Since Theorem 1 [4] is a special case of Theorem 1.1, then it is a consequence of Theorem 2.1, too.

**REFERENCES**


