On BCL⁺-Algebras

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ABSTRACT
This paper presents the BCL⁺-algebras, which is derived the fundamental properties. Results are generalized with version of BCL-algebras [5], using some unusual for a binary relation * and a constant 1 (one) in a non-empty set X, one may take different axiom systems for BCL⁺-algebras.

Keywords: BCL-Algebra; BCL⁺-Algebra; Logic Algebra

1. Introduction
The BCK/BCI/BCH-algebra (see [1-4]) has been a major issue, but BCL-algebra (see [5]) is a new algebra structure—and we started to grasp the properties. This paper presents the BCL⁺-algebras, we show that under our formulation, the BCL⁺-algebra is a variant of a BCL-algebra. We can define by taking some axioms and important properties in this way for the BCL⁺-algebras.

A BCL-algebra may be defined as a non-empty set X with a binary relation * and a constant 0 (zero) satisfying the following axioms:

Definition 1.1. [5] An algebra \((X;*,0)\) of type \((2,0)\) is said to be a BCL-algebra if and only if for any \(x,y,z \in X\), the following conditions:
1) BCL-1: \(x*x = 0\);
2) BCL-2: \(x*y = 0\) and \(y*x = 0\) imply \(x = y\);
3) BCL-3: \((((x*y)*z)*((x*z)*y))*((z*y)*x) = 0\).

Such set X in Definition 1.1 is called the underlying set of a BCL-algebra \((X;*,0)\), which needs the following theorem:

Theorem 1.1. [5] Algebra \((X;*,0)\) of type \((2,0)\) is a BCL-algebra if and only if it satisfies the following conditions for all \(x,y,z \in X\),
1) BCL-1: \(x*x = 0\);
2) BCL-2: \(x*y = 0\) and \(y*x = 0\) imply \(x = y\);
3) BCL-3: \((((x*y)*z)*((x*z)*y))*((z*y)*x) = 0\).

2. Main Result
The BCL⁺ product, denoted by *#. We call the binary operation * on X the *# product on X, and the constant 1(one) of X the unit element of X. For brevity we often write X instead of \((X;*,1)\). We begin with the following definition:

Definition 2.1. An algebra \((X;*,1)\) is called a BCL⁺-algebra if it satisfies the following laws hold: for any \(x,y,z \in X\),
1) BCL⁺-1: \(x*x = 1\);
2) BCL⁺-2: \(x*y = 1\) and \(y*x = 1\) imply \(x = y\);
3) BCL⁺-3: \(((x*y)*z)*((x*z)*y) = (z*y)*x\).

Such definition, clearly, the BCL⁺-algebra is a generalization of the BCL-algebra, imply a BCL-algebra is a BCL⁺-algebra, however, the converse is not true. We illustrate with the next theorem.

Theorem 2.1. A BCL⁺-algebra is existent.

Proof. The proof of this Theorem 2.1 is not difficult and uses only example. Let \(X = \{0,1,2,3\}\). Define an operation * on X, which are given in Table 1.

Then \((X;*,1)\) is a proper BCL⁺-algebra. It is easy to verify that there are

\[
\begin{align*}
\text{BCI-1:} & \quad (2*3)*(2*1)*(1*3) \\
& = (1*1)*3 \\
& = 1*3 \\
& = 3 \neq 0; \\
\text{BCI-2:} & \quad (2*(2*3))*3 \\
& = (2*1)*3 \\
& = 1*3 \\
& = 3 \neq 0; \\
\text{BCH-3:} & \quad \text{1) The left side of the equation is} \\
& = (2*3)*1 = 1*1 = 1;
\end{align*}
\]
2) The right side of the equation is 
\[(2 \ast 1) \ast 3 = 1 \ast 3 = 3.\]
In the expression we see that \(1 \neq 3\).

**BCL-3:**
\[
( ((2 \ast 3) \ast 1) \ast ((2 \ast 1) \ast 3) ) = ((1 \ast 1) \ast (1 \ast 3)) \ast (3 \ast 2)
\]
\[
= (1 \ast 3) \ast 3
\]
\[
= 3 \ast 3
\]
\[
= 1 \neq 0.
\]

**BCL-3:** 1) The left side of the equation is
\[
( (2 \ast 3) \ast 1 ) \ast ( (2 \ast 1) \ast 3 )
\]
\[
= (1 \ast 1) \ast (1 \ast 3)
\]
\[
= 1 \ast 3
\]
\[
= 3;
\]
2) The right side of the equation is 
\[(1 \ast 3) \ast 2 = 3 \ast 2 = 3 .\]
In the expression we see that BCL-3 is valid. In fact, it is not difficult to verify that BCL-1 and BCL-2 are valid.

A BCL-\(^-\) algebra \((X; \ast, 1)\) is a partially ordered relation \(\leq\) on \(X\), now we obtain the following definition:

**Definition 2.2.** Suppose that \((X; \ast, 1)\) is a BCL\(^-\) algebra, the ordered relation if 
\[x \leq y \text{ if and only if } x \ast y = 1,\]
for all \(x, y \in X\),
then \((X; \leq)\) is partially ordered set and \((X; \ast, 1)\) is an algebra of partially ordered relation.

**Corollary 2.1.** Let every \(x \in X\). Then 1(one) is maximal element in a BCL\(^-\) algebra \((X; \ast, 1)\) such that 
\[1 \leq x \text{ imply } x = 1.\]

**Definition 2.3.** A BCL\(^-\) algebra \(X\) is called proper BCL\(^-\) algebra if \(X\) is not a BCL-algebra.

**Example 2.1.** Let \(X = \{0, a, b, c, 1\}\). We define an operation \(\ast\) on \(X\) by Table 2.

In fact, it is not difficult to verify that \((X; \ast, 1)\) is a BCL\(^-\) algebra.
3) \([(x*y)*z]*((x*z)*y)]*((z*y)*x) = 1;
4) \(x*(1*x) = x\).

Proof. The proof is routine. Necessity. To prove 1). By BCL^{-3}.

\[x*x = (x*x)*1 = ((1*x)*x)*((1*x)*x) = 1. \tag{2.11}\]

Then 1) holding.

Sufficiency. Substituting \(x*1\) for \(y\) and \(x\) for \(z\) in 3), by BCL^{-3} and 1), it follows

\[[((x*(x*1))*x)*(x*(x*1))*x)]*((x*(x*1))*1) = ((1*x)*(1*x))*(1*x). \tag{2.12}\]

Also, substituting 1\(\star\)\(x\) for \(x\) in (2.11), by BCL^{-3} and 1), we have

\[((1*x)*(1*x))*(1*x) = ((x*x)*x)*((x*x)*x) = (1*x)*(1*x) = x*x = 1. \tag{2.13}\]

Using Theorem 2.2 with 4), we obtain

\[x*1 = x*(1*x) = x. \tag{2.14}\]

Hence \((X;*,1)\) is a BCL^{-}-algebra.

REFERENCES