L-Topological Spaces Based on Residuated Lattices

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ABSTRACT

In this paper, we introduce the notion of L-topological spaces based on a complete bounded integral residuated lattice and discuss some properties of interior and left (right) closure operators.

Keywords: Residuated Lattice; L-Topological Space; Interior Operator; Left (Right) Closure Operator

1. Introduction

Residuation is a fundamental concept of ordered structures and the residuated lattices, obtained by adding a residuated monoid operation to lattices, have been applied in several branches of mathematics, including \( L \)-groups, ideal lattices of rings and multivalued logic. Commutative residuated lattices have been studied by Krull, Dilworth and Ward. These structures were generalized to the non-commutative situation by Blount and Tsinakis [1].

Definition 1.1. [1-4]. A residuated lattice is an algebra \( L = (L, \land, \lor, \cdot, \multimap, 0, 1) \) of type \((2, 2, 2, 2, 2, 0, 0)\) satisfying the following conditions:

(L1) \((L, \land, \lor)\) is a lattice,

(L2) \((L, \cdot, 1)\) is a monoid, i.e., \( \cdot \) is associative and \( x \cdot 1 = 1 \cdot x = x \) for any \( x \in L \),

(L3) \( x \cdot y \leq z \) if and only if \( x \leq y \multimap z \) if and only if \( y \leq x \multimap z \) for any \( x, y, z \in L \).

Generally speaking, 1 is not the top element of \( L \). A residuated lattice with a constant 0 is called a residuated lattice or full Lambek algebra (\( FL \)-algebra, for short). If \( x \leq 1 \) for all \( x \in L \), then \( L \) is called integral residuated lattice. An \( FL \)-algebra \( L \) satisfies the condition \( 0 \leq x \leq 1 \) for all \( x \in L \) is called \( FLw \)-algebra or bounded integral residuated lattice (see [2]). Clearly, if \( L \) is an \( FLw \)-algebra, then \((L, \land, \lor, 0, 1)\) is a bounded lattice.

A bounded integral residuated lattice is called commutative (see [5]) if the operation \( \cdot \) is commutative. We adopt the usual convention of representing the monoid operation by juxtaposition, writing \( ab \) for \( a \cdot b \).

The following theorem collects some properties of bounded integral residuated lattices (see [1-4,6]).

Theorem 1.1. Let \( L \) be a bounded integral residuated lattice. Then the following properties hold.

1) \( x \rightarrow x = x \leftrightarrow x = 1 \), \( 1 \rightarrow x = 1 \leftrightarrow x = x \).

2) \( x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) \).

3) \( x \rightarrow (y \land y) \leq x \land y, (x \rightarrow y) x \leq x \land y, x \leq y \rightarrow xy, y \leq x \rightarrow xy \).

4) \((x \rightarrow y) (y \rightarrow z) \leq x \rightarrow z, (y \rightarrow z) (x \rightarrow y) \leq x \rightarrow z \).

5) \( \multimap \) if \( x \leq y \), then \( xz \leq yz, zx \leq zy \), \( x \rightarrow z \geq y \rightarrow z, x \rightarrow z \geq x \rightarrow z \rightarrow y \rightarrow z, x \rightarrow x \leq z \rightarrow y \rightarrow z, \) and \( z \rightarrow x \leq z \rightarrow y \rightarrow z \).

6) \( x \leq y \) if and only if \( x \rightarrow y = 1 \) if and only if \( x \rightarrow y = 1 \).

7) \( \multimap z = y \rightarrow (x \rightarrow z), xy \rightarrow z = x \rightarrow (y \rightarrow z) \).

8) \((x \lor y) \multimap z = (x \rightarrow z) \land (y \rightarrow z), (x \lor y) \multimap z = (x \rightarrow z) \land (y \rightarrow z) \).

9) \( x \rightarrow (y \land z) = (x \rightarrow y) (x \rightarrow z), x \rightarrow (y \land z) = (x \rightarrow y) (x \rightarrow z) \).

If bounded integral residuated lattice \( L \) is complete, then

\( x \rightarrow z = \lor \{ y \in L | xy \leq z \}, x \rightarrow z = \lor \{ y \in L | xy \leq z \} \).

Thus, it follows from some results in [7] that

Theorem 1.2. Let \( L \) be a complete bounded integral residuated lattice and \( a, b, a_j, b_j \in L (j \in J) \). Then the following properties hold.

1) \( a \lor_{j \in J} b_j = \lor_{j \in J} ab_j \) and \( \lor_{j \in J} a_j b = \lor_{j \in J} a_j b \), i.e., the operation \( \lor \) is infinitely \( \lor \)-distributive.

2) \((\lor_{j \in J} a_j) \rightarrow b = \lor_{j \in J} (a_j \rightarrow b) \) and \((\lor_{j \in J} a_j) \rightarrow b = \lor_{j \in J} (a_j \rightarrow b) \).

3) \( a \multimap (\lor_{j \in J} b_j) = \lor_{j \in J} (a \multimap b_j) \) and \( a \multimap (\lor_{j \in J} b_j) = \lor_{j \in J} (a \multimap b_j) \), i.e., the two residuation operations \( \rightarrow \) and \( \multimap \) are all right infinitely \( \land \)-distributive (see [8]).

4) \((\land_{j \in J} a_j) \rightarrow b \geq \land_{j \in J} (a_j \rightarrow b) \) and \((\land_{j \in J} a_j) \rightarrow b \geq \land_{j \in J} (a_j \rightarrow b) \)
5) \( a \rightarrow (\vee_{j \in J} b_j) \geq \vee_{j \in J} (a \rightarrow b_j) \) and
\( a \rightarrow (\vee_{j \in J} b_j) \geq \vee_{j \in J} (a \rightarrow b_j) \).

Let us define on \( L \) two negations, \(-^L \) and \(-^R \):
\(-^L x = x \rightarrow 0 \) and \(-^R x = x \rightarrow 0 \).

For any \( x, x_j (j \in J), b \in L \), it follows from Theorems 1.1 and 1.2 that
\(-^L \neg^R x \geq x, \ -^R \neg^L x \geq x, \ x \rightarrow \neg^L y = \neg^L (xy),\)
\( x \rightarrow \neg^R y \geq \neg^R y = \neg^L (xy), \ x \rightarrow \neg^R y \rightarrow \neg^L x, \)
\( \neg^L \neg^R \neg^L x = \neg^L x, \ -^R \neg^L \neg^R x \geq \neg^R x, \)
\( x \rightarrow y \leq \neg^R y \rightarrow \neg^L x, \ x \rightarrow y \leq \neg^L y \rightarrow \neg^L x, \)
\(-^L (\vee_{j \in J} x_j) = \wedge_{j \in J} \neg^L x_j, \ -^L (\wedge_{j \in J} x_j) = \wedge_{j \in J} \neg^L x_j, \)
\(-^L (\wedge_{j \in J} x_j) \geq \wedge_{j \in J} \neg^R x_j, \ -^L (\wedge_{j \in J} x_j) \geq \wedge_{j \in J} \neg^R x_j.\)

A bounded residuated lattice \( L \) is called an involutive residuated lattice (see [3]) if \(-^L \neg^R x = \neg^R \neg^L x = x \) for any \( x \in L \). In a complete involutive residuated lattice \( L \),
\( x \rightarrow y \geq \neg^R y \rightarrow \neg^L x, \ x \rightarrow y \leq \neg^L y \rightarrow \neg^L x, \)
\(-^L (\wedge_{j \in J} x_j) = \wedge_{j \in J} \neg^L x_j, \ -^L (\vee_{j \in J} x_j) = \vee_{j \in J} \neg^L x_j.\)

In the sequel, unless otherwise stated, \( L \) always represents any given complete bounded integral residuated lattice with maximal element 1 and minimal element 0.

The family of all \( L \)-fuzzy set in \( X \) will be denoted by \( L^X \). For any family \( \mu, \mu_1 \in L^X \) of \( L \)-fuzzy sets,
we will write \(-^L \mu, \neg^R \mu, \vee_{j \in J} \mu_j \) and \( \wedge_{j \in J} \mu_j \) to denote the \( L \)-fuzzy sets in \( X \) given by
\((-^L \mu)(x) = \neg^L \{ \mu((x)) \}, \neg^R \mu(x) = \neg^R \{ \mu((x)) \},\)
\( \vee_{j \in J} \mu_j(x) = \vee_{j \in J} \mu_j(x), \wedge_{j \in J} \mu_j(x) = \wedge_{j \in J} \mu_j(x).\)

Besides this, we define \( 1_X, 0_X \in L^X \) as follows:
\( 1_X(x) = 1 \forall x \in X \) and \( 0_X(x) = 0 \forall x \in X \).

2. \( L \)-Topological Spaces

A completely distributive lattice \( L \) is called a \( F \)-lattice, if \( L \) has an order-reversing involution \( \cdot : L \rightarrow L \). When \( L \) is a \( F \)-lattice, Liu and Luo [9] studied the concept of \( L \)-topology. Below, we consider the notion of \( L \)-topological space based on a complete bounded integral residuated lattice.

**Definition 2.1.** Let \( \tau \subseteq L^X \). If \( \tau \) satisfies the following three conditions:
(LFT1) \( 0_X, 1_X \in \tau, \)
(LFT2) \( \mu, \nu \in \tau \Rightarrow \mu \wedge \nu \in \tau, \)
(LFT3) \( \mu \in \tau \Rightarrow \vee_{j \in J} \mu_j \in \tau, \)
then \( \tau \) is called an \( L \)-topology on \( X \) and \((L^X, \tau)\) an \( L \)-topological space.

When \( L = [0,1] \), called an \( L \)-topological space \((L^X, \tau)\) an \( F \)-topological space.

Every element in \( \tau \) is called an open subset in \( L^X \).
Let \( \tau^L = \{ \neg^L \mu \in \tau \} \) and \( \tau^R = \{ \neg^R \mu \in \tau \} \). The elements of \( \tau^L \) and \( \tau^R \) are, respectively, left closed subsets and right closed subsets in \( L^X \).

**Definition 2.2.** Let \( \tau \) be an \( L \)-topology on \( X \) and \( \mu \) \( L \)-fuzzy subset of \( X \). The interior, left closure and right closure of \( \mu \) w.r.t. \( \tau \) are, respectively, defined by
\( \text{int}(\mu) = \{ \eta \in \tau | \eta \leq \mu \}, \)
\( \text{cl}_L(\mu) = \{ \xi \in \tau | \mu \leq \xi \}, \)
\( \text{cl}_R(\mu) = \{ \xi \in \tau | \mu \leq \xi \}. \)

int, \( \text{cl}_L \) and \( \text{cl}_R \) are, respectively, called interior, left closure and right closure operators.

For the sake of convenience, we denote \( \text{int}(\mu), \text{cl}_L(\mu) \) and \( \text{cl}_R(\mu) \) by \( \mu^o, \mu_L \) and \( \mu_R \), respectively.

In view of Definitions 2.1 and 2.2, for any \( \mu \in L^X \),
\( \mu^o = \text{int}(\mu) = \{ \eta \in \tau | \eta \leq \mu \} \in \tau, \)
\( \mu_L = \text{cl}_L(\mu) = \{ \xi \in \tau | \mu \leq \xi \} \in \tau, \)
\( \mu_R = \text{cl}_R(\mu) = \{ \xi \in \tau | \mu \leq \xi \} \in \tau, \)
where
\( \mu_1 = \text{int}(\mu) = \{ \xi \in \tau | \mu \leq \xi \} \in \tau, \)
\( \mu_2 = \text{cl}_L(\mu) = \{ \xi \in \tau | \mu \leq \xi \} \in \tau, \)
i.e., \( \mu^o \) is just the largest open subset contained in \( \mu \), \( \mu_L \) and \( \mu_R \) are, respectively, the smallest left closed and right closed subsets containing \( \mu \).

For any \( \mu \in L^X \),
\( \neg^L \mu^o = \neg^L \{ \text{int}(\mu) \} = \{ \neg^L \eta | \eta \leq \mu, \eta \in \tau \} = \{ \neg^L \eta | \eta \leq \mu, \eta \in \tau \} \in \tau, \)
\( \geq \neg^L \{ \neg^L \eta | \neg^L \eta \leq \mu, \eta \in \tau \} = \{ \neg^L \mu \}. \)

Similarly, \( \neg^R \mu^o = \{ \neg^R \mu \} \).

**Theorem 2.1.** If \( L \) is an involutive residuated lattice and \( \mu \in L^X \), then
1) \( \neg^L \mu^o = \neg^L \mu_L \) and \( \neg^R \mu^o = \neg^R \mu_R \);  
2) \( \mu^o \leq \neg^R \mu_L \) and \( \mu^o \leq \neg^L \mu_R \);  
3) \( \neg^L \mu^o \leq \neg^R \mu_R \) and \( \neg^R \mu^o \leq \neg^L \mu_L \).
\[ \mu_\ell = -^\ell \left( -^R \mu \right)^\ell \quad \text{and} \quad \mu_\ell = -^R \left( -^\ell \mu \right)^R. \]

**Proof.** When \( L \) is an involutive residuated lattice, \( -^R \left( -^\ell \mu \right) = -^\ell \left( -^R \mu \right) = \mu^\ell \mu \in L^R \).

1) If \( \eta \in L^X \) and \( -^\ell \mu \leq -^\ell \eta \), then
\[ \mu = -^R \left( -^\ell \mu \right) \geq -^R \left( -^\ell \eta \right) = \eta. \]

Thus, \( -^\ell \left( \mu^\ell \right) = \left( -^\ell \mu \right)^\ell \). Similarly,
\[ -^R \left( \mu^R \right) = \left( -^R \mu \right)^R. \]

2) It follows from 1) that
\[ \mu^\ell = -^R \left( -^R \mu \right) = -^R \left( -^\ell \mu \right)^L, \]
\[ \mu^R = -^\ell \left( -^\ell \mu \right) = -^\ell \left( -^R \mu \right)^R. \]

3) By 2), we see that
\[ \left( -^\ell \mu \right)^\ell = -^R \left( -^R \mu \right)^R = -^R \left( -^\ell \mu \right)^L, \]
\[ \left( -^R \mu \right)^\ell = -^\ell \left( -^R \mu \right)^R = -^\ell \left( -^R \mu \right)^L, \]
\[ \left( -^\ell \mu \right) \left( -^R \mu \right)^\ell = \mu^\ell \left( -^\ell \mu \right)^R, \]
\[ -^R \left( -^\ell \mu \right)^\ell = -^\ell \left( -^R \mu \right)^R = -^\ell \left( -^R \mu \right)^L, \]
\[ -^R \left( -^R \mu \right)^\ell = -^\ell \left( -^R \mu \right)^R = -^\ell \left( -^R \mu \right)^L. \]

**Theorem 2.2.** Let \( \mu, \nu \in L^X \). Then the following properties hold:

1) \( (1_X)^\ell = 1_X, (0_X)^L = 0_X \).
2) \( \mu^\ell \leq \mu, \mu \leq \mu^\ell, \mu \leq \mu^R, \mu \leq \mu^R \).
3) If \( \mu \leq \nu, \) then \( \mu^\ell \leq \nu^\ell, \mu^R \leq \nu^R, \mu_\ell \leq \nu_\ell, \mu_R \leq \nu_R. \)
4) \( \left( \mu^\ell \right)^\ell = \mu^\ell, \left( \mu^L \right) = \mu^L \text{ and } \left( \mu^R \right) = \mu^R. \)
5) Clearly, \( \left( \mu \lor \nu \right)^\ell \leq \mu^\ell \lor \nu^\ell \). Noting that \( \mu^\ell \lor \nu^\ell \in \tau \), we see that
\[ \left( \mu \lor \nu \right)^\ell = \left( \mu \lor \nu \right)^\ell \leq \mu^\ell \lor \nu^\ell. \]

Thus, \( \mu \lor \nu \leq \mu^\ell \lor \nu^\ell \).
6) There exist \( \mu \in \tau \) such that \( \mu^\ell = -^\ell \mu, \mu^R = -^R \mu \).

If \( -^\ell (x \lor y) = -^\ell x \lor -^\ell y \forall x, y \in L \), then
\[ -^R \left( \mu \lor \nu \right) = -^R \left( \mu \lor \nu \right) \lor -^\ell \mu^\ell \lor -^R \mu^R = \left( \mu \lor \nu \right)^R \in \tau. \]
Thus, \( \left( \mu \lor \nu \right)^R \leq \mu^\ell \lor \nu^R \).

Therefore, \( \left( \mu \lor \nu \right) = \mu^\ell \lor \nu^R \).
7) Similar to (6).

**Theorem 2.3.** Let \( f : L^X \rightarrow L^X \) be a mapping. Then the following two statements hold.

1) If the operator \( f \) on \( L^X \) satisfying the following conditions:

(C1) \( f(1_X) = 1_X, \)
(C2) \( f(\mu) \leq \mu \lor \mu, \mu \leq f(\mu), \mu \leq \mu \lor \mu \).
(C3) \( f(\mu \lor \nu) = f(\mu) \lor f(\nu) \forall \mu, \nu \in L^X, \)
(C4) \( f(\mu) \leq f(\mu) \lor f(\nu) \forall \mu, \nu \in L^X, \)

Then \( f \) is an L-operator on \( X \). Moreover, if the operator \( f \) also fulfills

(C4) \( f(\mu) \leq f(\mu) \lor f(\nu) \forall \mu, \nu \in L^X, \)
then with the L-topology \( \tau \), \( f(\mu) = \mu^\ell \) for every \( \mu \in L^X \), i.e., \( f \) is the interior operator w.r.t \( \tau \). 2) If the operator \( f \) on \( L^X \) satisfying the following conditions:

(C1') \( f(0_X) = 0_X, \)
(C2') \( f(\mu) \leq f(\mu) \lor f(\nu) \forall \mu, \nu \in L^X, \)
(C3') \( f(\mu \lor \nu) = f(\mu) \lor f(\nu) \forall \mu, \nu \in L^X, \)

Then \( f \) is an L-operator on \( X \), moreover, if the operator \( f \) also fulfills

(C4) \( f(\mu) \leq f(\mu) \lor f(\nu) \forall \mu, \nu \in L^X, \)
then with the L-topology \( \tau \), \( f(\mu) = \mu^\ell \) for every \( \mu \in L^X \), i.e., \( f \) is the left closure operator w.r.t \( \tau \). 2) If the operator \( f \) on \( L^X \) satisfying the following conditions:

(C1') \( f(1_X) = 1_X, \)
(C2') \( f(\mu) \leq f(\mu) \lor f(\nu) \forall \mu, \nu \in L^X, \)
(C3') \( f(\mu \lor \nu) = f(\mu) \lor f(\nu) \forall \mu, \nu \in L^X, \)

Then \( f \) is an L-operator on \( X \), moreover, if the operator \( f \) also fulfills

(C4) \( f(\mu) \leq f(\mu) \lor f(\nu) \forall \mu, \nu \in L^X, \)
then with the L-topology \( \tau \), \( f(\mu) = \mu^\ell \) for every \( \mu \in L^X \), i.e., \( f \) is the right closure operator w.r.t \( \tau \). 2) If the operator \( f \) on \( L^X \) satisfying the following conditions:

(C1') \( f(1_X) = 1_X, \)
(C2') \( f(\mu) \leq f(\mu) \lor f(\nu) \forall \mu, \nu \in L^X, \)
(C3') \( f(\mu \lor \nu) = f(\mu) \lor f(\nu) \forall \mu, \nu \in L^X, \)

Then \( f \) is an L-operator on \( X \), moreover, if the operator \( f \) also fulfills

(C4) \( f(\mu) \leq f(\mu) \lor f(\nu) \forall \mu, \nu \in L^X, \)
then with the L-topology \( \tau \), \( f(\mu) = \mu^\ell \) for every \( \mu \in L^X \), i.e., \( f \) is the left closure operator w.r.t \( \tau \). 2) If the operator \( f \) on \( L^X \) satisfying the following conditions:

(C1') \( f(0_X) = 0_X, \)
(C2') \( f(\mu) \leq f(\mu) \lor f(\nu) \forall \mu, \nu \in L^X, \)
(C3') \( f(\mu \lor \nu) = f(\mu) \lor f(\nu) \forall \mu, \nu \in L^X, \)

Then \( f \) is an L-operator on \( X \), moreover, if the operator \( f \) also fulfills

(C4) \( f(\mu) \leq f(\mu) \lor f(\nu) \forall \mu, \nu \in L^X, \)
then with the L-topology \( \tau \), \( f(\mu) = \mu^\ell \) for every \( \mu \in L^X \), i.e., \( f \) is the right closure operator w.r.t \( \tau \).
\[ f(\neg^l (\vee_{j \in J} \eta_j)) = f(\bigwedge_{j \in J} \neg^l \eta_j) \leq \bigwedge_{j \in J} f(\neg^l \eta_j) = \bigwedge_{j \in J} \neg^l \eta_j = \neg^l (\vee_{j \in J} \eta_j). \]

Combing with (C2), we have that
\[ f(\neg^l (\vee_{j \in J} \eta_j)) = \neg^l (\vee_{j \in J} \eta_j). \]

Thus, \( \vee_{j \in J} \eta_j \in \tau_i \) and so \( \tau_i \) is an \( L \)-topology on \( X \).

For any \( \mu \in L^X \),
\[
\begin{align*}
    f(\mu^c) &= f\left(\bigwedge \{ \neg^l \xi \mid \mu \leq \neg^l \xi, \xi \in \tau_i \right}\) \\
   &\leq \bigwedge\{ f(\neg^l \xi) \mid \mu \leq \neg^l \xi, \xi \in \tau_i \} \\
   &= \bigwedge\{ \neg^l \xi \mid \mu \leq \neg^l \xi, \xi \in \tau_i \} = \mu^c,
\end{align*}
\]
i.e., \( f(\mu) \leq f(\mu^c) \leq \mu^c \). Moreover, if (C4) holds and \( \neg^l : L^X \to L^X \) is a bijection, then
\[
\begin{align*}
    f(\mu) &\geq \bigwedge \{ \eta \in L^X \mid f(\eta) = \eta \geq \mu \} \\
   &= \bigwedge\{ \neg^l \xi \mid \mu \leq \neg^l \xi, \xi \in \tau_i \} = \mu^c.
\end{align*}
\]

Therefore, \( f(\mu) = \mu^c \), i.e., \( f \) is the left closure operator w.r.t \( \tau_i \).

We can prove in an analogous way that \( \tau_2 \) is an \( L \)-topology on \( X \) and the corresponding \( f \) is the right closure operator w.r.t \( \tau_2 \).

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