Geodesic Lightlike Submanifolds of Indefinite Sasakian Manifolds*

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Abstract

In this paper, we study geodesic contact CR-lightlike submanifolds and geodesic screen CR-lightlike (SCR) submanifolds of indefinite Sasakian manifolds. Some necessary and sufficient conditions for totally geodesic, mixed geodesic, D̅-geodesic and D'-geodesic contact CR-lightlike submanifolds and SCR submanifolds are obtained.

Keywords: CR-Lightlike Submanifolds, Sasakian Manifolds, Totally Geodesic Submanifolds

1. Introduction

A submanifold $M$ of a semi-Riemannian manifold $\tilde{M}$ is called lightlike submanifold if the induced metric on $M$ is degenerate. The general theory of a lightlike submanifold has been developed by Kupeli [1] and Bejancu-Duggal [2].

The geometry of CR-lightlike submanifolds of indefinite Kaehler manifolds was studied by Guggal and Bejancu [2]. The geodesic CR-lightlike submanifolds in indefinite Kaehler manifolds were studied by Sahin and Günes [3,4].

Lightlike submanifold of indefinite Sasakian manifolds can be defined according to the behavior of the almost contact structure, and contact CR-lightlike submanifolds and screen CR-lightlike (SCR) submanifolds of indefinite Sasakian manifolds were studied by Duggal and Sahin in [5]. The study of the geometry of submanifolds of indefinite Sasakian manifolds has been developed by [6] and others.

In this paper, geodesic contact CR-lightlike submanifolds and geodesic screen CR-lightlike (SCR) submanifolds of indefinite Sasakian manifolds are considered. Some necessary and sufficient conditions for totally geodesic, mixed geodesic, $\tilde{D}$-geodesic and $D'$-geodesic contact CR-lightlike submanifolds and SCR submanifolds are obtained.

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Now, let $\nabla$ be the Levi-Civita connection on $\bar{M}$, we have
\[
X(\bar{g}(Y,Z)) = \bar{g}(\nabla_X Y, Z) + \bar{g}(Y, \nabla_X Z), \quad \forall X, Y, Z \in \Gamma(TM),
\]
(2.1)
\[
\nabla_X Y = \nabla^*_X Y + h'(X,Y), \quad \forall X, Y \in \Gamma(TM),
\]
(2.2)
\[
\nabla_X V = -A_X X + \nabla_X^* V, \quad \forall X \in \Gamma(TM),
\]
(2.3)
\[
V \in \Gamma(tr(TM)),
\]
where $\{\nabla^*_X A_X X\}$ and $\{h(X,Y), \nabla^*_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Using the projectors $l: tr(TM) \rightarrow S(TM)$ and $s: tr(TM) \rightarrow \text{ltr}(TM)$, from [1], we have
\[
\nabla_X Y = \nabla^*_X Y + h'(X,Y), \forall X, Y \in \Gamma(TM),
\]
(2.4)
\[
\nabla_X N = -A_X X + \nabla^*_X N + D'(X,N), \forall N \in \Gamma(ltr(TM)),
\]
(2.5)
\[
\nabla_X W = -A_X X + \nabla^*_X W + D'(X, W), \forall W \in \Gamma(TM),
\]
(2.6)
Denote the projection of $TM$ to $S(TM)$ by $P$, we have the decomposition
\[
\nabla_X P Y = \nabla^*_X P Y + h'(X, PY),
\]
(2.7)
\[
\nabla^*_X \xi = -A^*_X X + \nabla^*_X \xi,
\]
(2.8)
for any $X, Y \in \Gamma(TM), \xi \in \Gamma(Rad(TM)), N \in \Gamma(ltr(TM))$. From the above equations we have
\[
\bar{g}(h'(X,Y), \xi) = g(A^*_X X, Y),
\]
(2.9)
\[
\bar{g}(h'(X, PY), N) = g(A_X X, PY),
\]
(2.10)
\[
\bar{g}(h'(X, \xi), \bar{g}(V, V) = 0, A^*_X \xi = 0.
\]
(2.11)
Definition 2.1 A $(2n + 1)$-dimensional Semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a contact metric manifold if there is a $(1,1)$ tensor field $\phi$, a vector field $V$, called the characteristic vector field, and its dual 1-form $\eta$ such that
\[
\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \epsilon \eta(X) \eta(Y), \bar{g}(V, V) = \epsilon,
\]
(2.12)
\[
\phi^2(X) = -X + \eta(X)V, \bar{g}(X, V) = \epsilon \eta(X),
\]
(2.13)
\[
d\eta(X, Y) = \bar{g}(X, \phi Y), \forall X, Y \in \Gamma(TM),
\]
(2.14)
where $\epsilon = \pm 1$.
From the above definiton, it follows that
\[
\phi^2(X) = -X + \eta(X)V, \bar{g}(X, V) = \epsilon \eta(X),
\]
(2.13)
\[
d\eta(X, Y) = \bar{g}(X, \phi Y), \forall X, Y \in \Gamma(TM),
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\]
(2.13)
\[
d\eta(X, Y) = \bar{g}(X, \phi Y), \forall X, Y \in \Gamma(TM),
\]
(2.14)
so $h'(Y, \nabla_Y X) = 0$. That is to say $h'(Y, PX) = 0$.
In a similar way, we can get $h'(X, Y) = 0$. Thus, $M$ is totally geodesic.

Conversely, if $h'(X, Y) = h'(X, Y) = 0$, since
\[
\left( \nabla_X h' \right)(Y, Z) = \nabla_X h'(Y, Z) - h'(\nabla_X Y, Z)
- h'(Y, \nabla_X Z) = 0,
\]
\[
\left( \nabla_X h' \right)(Y, Z) = \nabla_X h'(Y, Z) - h'(\nabla_X Y, Z)
- h'(Y, \nabla_X Z) = 0,
\]
so $h'$ and $h'$ are parallel, which completes the proof.

4. Geodesic Contact CR-Lightlike Submanifolds

**Definition 4.1** Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold, tangent to the structure vector field $V$, immersed in an indefinite Sasakian manifold $(\overline{M}, \overline{g})$. We say that $M$ is a contact CR-lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

1. [A] $\text{Rad} TM$ is a distribution on $M$ such that $\text{Rad} TM \cap \phi(\text{Rad} TM) = \{0\}$.
2. [B] There exist vector bundles $D_0$ and $D'$ over $M$ such that $S(TM) = \{\phi(\text{Rad} TM) \oplus D' \} \perp D_0 \perp V$,
\[
\phi D_0 = D_0, \phi D' = L_1 \perp L_2,
\]
where $D_0$ is non-degenerate and $L_1 = \text{tr}(TM)$, $L_2$ is a vector subbundle of $S(TM^\perp)$. So we have the decomposition
\[
TM = \{D \perp \phi D' \} \perp V, \quad D = \text{Rad} TM \perp \phi(\text{Rad} TM) \perp D_0.
\]

If we denote $\hat{D} = D \perp V$, then we have
\[
TM = \hat{D} \oplus D', \phi \hat{D} = \hat{D}.
\]

**Definition 4.2** A contact CR-lightlike submanifold of an indefinite Sasakian manifold is called $D$-geodesic contact CR-lightlike submanifold if its second fundamental form $h$ satisfies $h(X, Y) = 0$, for any $X, Y \in \Gamma(\hat{D})$.

**Definition 4.3** A contact CR-lightlike submanifold of an indefinite Sasakian manifold is called mixed geodesic contact CR-lightlike submanifold if its second fundamental form $h$ satisfies $h(X, Z) = 0$, for any $X \in \Gamma(\hat{D})$ and $Z \in \Gamma(D')$.

**Definition 4.4** A contact CR-lightlike submanifold of an indefinite Sasakian manifold is called $D'$-geodesic contact CR-lightlike submanifold if its second fundamental form $h$ satisfies $h(Z, U) = 0$, for any $Z, U \in \Gamma(D')$.

**Theorem 4.1** Let $M$ be a contact CR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$.

Then $M$ is totally geodesic if and only if
\[
\overline{g}(h(X, A_0 X) = \overline{g}(Y, D'(X, W)), \quad \nabla X \phi Y \text{ has no components in } \phi L_1, \quad Y \in \Gamma(TM) - \text{span}\{V\} \} \text{ or } X \text{ has no components in } \phi L_1.
\]

**Proof.** We know that $M$ is totally geodesic if and only if $h(X, Y) = 0$, for any $X, Y \in \Gamma(TM)$. By the definition of the second fundamental form, $h(X, Y) = 0$ is equivalent to $\overline{g}(h(X, Y), \xi) = 0, \overline{g}(h(X, Y), W) = 0$, for any $\xi \in \Gamma(\text{Rad} TM), W \in \Gamma(S(TM^\perp))$.

From (2.4) and (2.7) we have
\[
\overline{g}(h(X, Y), \xi) = \overline{g}(\nabla_X Y, \xi)
= \overline{g}(\phi \nabla_X Y, \phi \xi) + \eta(\nabla_X Y)\eta(\xi)
= \overline{g}(\nabla_X \phi Y, \phi \xi) + \overline{g}(\overline{g}(X, Y)\phi \xi)
= \overline{g}(\nabla_X \phi Y, \phi \xi) + \eta(Y)\overline{g}(X, \phi \xi)
\]
and
\[
\overline{g}(h'(X, Y), W) = \overline{g}(\nabla_X Y', W)
= x(\overline{g}(X, W') - \overline{g}(Y, \nabla_X W')
= -\overline{g}(Y, \nabla_X W')
= -\overline{g}(Y, -A_0 X + \nabla_X W + D'(X, W))
= \overline{g}(Y, A_0 X) - \overline{g}(Y, D'(X, W))
\]

Thus, from (4.1) and (4.2), the proof is completed.

**Theorem 4.2** Let $M$ be a contact CR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then $M$ is mixed geodesic if and only if $A_0 X \phi Y \text{ has no components in } \phi \text{Rad} TM \perp L_2$.

**Proof.** By the definition, $M$ is mixed geodesic if and only if
\[
\overline{g}(h(X, Y), \xi) = 0, \overline{g}(h(X, Y), W) = 0.
\]

Then we have
\[
\overline{g}(h(X, Y), \xi) = \overline{g}(\nabla_X Y, \xi)
= \overline{g}(\phi \nabla_X Y, \phi \xi) + \eta(\nabla_X Y)\eta(\xi)
= \overline{g}(\nabla_X \phi Y, \phi \xi)
= \overline{g}(\nabla_X \phi Y, \phi \xi) + \overline{g}(\overline{g}(X, Y)\phi \xi)
= \overline{g}(\nabla_X \phi Y, \phi \xi) + \eta(Y)\overline{g}(X, \phi \xi)
= -\overline{g}(A_0 X, \phi \xi) + \eta(Y)\overline{g}(X, \phi \xi)
= -\overline{g}(A_0 X, \phi \xi)
\]
and
\[ \bar{g}(h(X,Y),W) = \bar{g}(\nabla_Y Y, W) \]
\[ = \bar{g}(\phi \nabla_Y Y, \phi W) + \eta(\nabla_Y Y) \eta(W) \]
\[ = \bar{g}(\phi \nabla_Y Y, \phi W) \]
\[ = \bar{g}(\nabla_Y Y, \phi W) \]
\[ = \bar{g}(\nabla_Y Y, \phi W) + \bar{g}(\bar{g}(X,Y)V + \eta(Y)X, \phi W) \]
\[ - \bar{g}(A_{\phi} Y, \phi W). \]

Thus, the proof of the theorem is complete.

**Theorem 4.3** Let \( M \) be a contact CR-lightlike submanifold of an indefinite Sasakian manifold \( \bar{M} \). Then \( M \) is \( \bar{D} \)-geodesic if and only if
\[ \nabla_\eta \phi \xi \in \Gamma(\phi RadTM \perp \phi L_2), \nabla_\eta Y \] has no components in \( \phi L_2, \nabla_\eta Y \in \Gamma(\bar{D}). \]

**Proof.** \( M \) is \( \bar{D} \)-geodesic if and only if
\[ \bar{g}(h(X,Y),\xi) = 0, \bar{g}(h'(X,Y),W) = 0, \] for any
\( X,Y \in \Gamma(D'), \xi \in \Gamma(RadTM) \) and \( W \in \Gamma(S(TM^\perp)). \)

Thus the assertions of the theorem follows.

**5. Geodesic Contact SCR-Lightlike Submanifolds**

**Definition 5.1** Let \( (M,g,S(TM),S(TM^\perp)) \) be a lightlike submanifold, tangent to the structure vector field \( V \), immersed in an indefinite Sasakian manifold \( \bar{M}, g \). We say that \( M \) is a contact SCR-lightlike submanifold of \( \bar{M} \) if the following conditions are satisfied
\[ \text{[(A)] There exist real non-null distributions } D \text{ and } D^\perp, \text{ such that } \]
\[ S(TM) = D \perp D^\perp \perp V, \phi(D^\perp) \subset S(TM^\perp), \]
\[ D \cap D^\perp = \{0\}, \]
where \( D^\perp \) is the orthogonal complementary to \( D \perp V \) in \( S(TM) \). \[ \text{[(B)] } \phi D = D, \phi RadTM = RadTM, \phi ltr(TM) = ltr(TM). \]

Hence we have the decomposition
\[ TM = \bar{D} \perp D^\perp \perp V, \bar{D} = D \perp RadTM. \]

Let us denote \( \bar{D} = D \perp V. \)

**Definition 5.2** A contact SCR-lightlike submanifold of an indefinite Sasakian manifold is called mixed geodesic contact SCR-lightlike submanifold if its second fundamental form \( h \) satisfies \( h(X,Y) = 0 \), for any \( X \in \Gamma(\bar{D}) \) and \( Y \in \Gamma(D^\perp). \)

**Theorem 5.1** Let \( M \) be a contact SCR-lightlike submanifold of an indefinite Sasakian manifold \( \bar{M} \). Then \( M \) is totally geodesic if and only if
\[ h(X,Y), \xi) = 0, \bar{g}(h'(X,Y),W) = 0, \] for any
\( X,Y \in \Gamma(D'), \xi \in \Gamma(RadTM) \) and \( W \in \Gamma(S(TM^\perp)). \)

**Proof.** We know \( M \) is totally geodesic if and only if
\[ \bar{g}(h(X,Y),\xi) = 0, \bar{g}(h'(X,Y),W) = 0, \] for any
\( X,Y \in \Gamma(\bar{D}), Y \in \Gamma(D^\perp). \)
From (2.1) and Lie derivative we obtain
\[
\bar{g}(h(X,Y),\xi) = \bar{g}(\nabla_X Y,\xi)
\]
\[
= X(\bar{g}(Y,\xi)) - \bar{g}(\nabla_Y X,\xi)
\]
\[
= \bar{g}(Y,[\xi,X]) - \bar{g}(Y,\nabla_Y X)
\]
\[
= \bar{g}(Y,[\xi,X]) - \xi(\bar{g}(X,Y)) + \bar{g}(X,\nabla_Y \xi)
\]
\[
= \bar{g}(Y,[\xi,X]) - \xi(\bar{g}(X,Y)) + \bar{g}(X,[\xi,Y]) + \bar{g}(\nabla_Y \xi, X)
\]
\[
= -(L_\xi \bar{g})(X,Y) - \bar{g}(h(X,Y),\xi).
\]
Hence we have
\[
2 \bar{g}(h(X,Y),\xi) = -(L_\xi \bar{g})(X,Y).
\]
In a similar way, we can get
\[
2 \bar{g}(h(X,Y),W) = -(L_\xi \bar{g})(X,Y),
\]
thus the proof is completed.

**Theorem 5.2** Let \( M \) be a contact SCR-lightlike submanifold of an indefinite Sasakian manifold \( \bar{M} \). Then \( M \) is mixed geodesic if and only if \( \nabla_x \phi Y \in \Gamma(D^+) \), \( A_{\phi Y} X \in \Gamma(D^+) \), for any \( X \in \Gamma(\bar{D}) \), \( Y \in \Gamma(D^+) \).

**Proof.** For any \( X \in \Gamma(\bar{D}) \), \( Y \in \Gamma(D^+) \),
\[
\xi \in \Gamma(RadTM), \ W \in \Gamma(S(TM^+))
\]
denote by
\[
\phi X = P'X + Q'X, \phi W = B'W + C'W,
\]
where \( P'X \in \Gamma(\bar{D}) \), \( Q'X \in \Gamma(\phi D^+) \), \( B'W \in \Gamma(D^+) \)
and \( C'W \in \Gamma(S(TM^+) - \phi D^+) \).

If \( M \) is mixed geodesic, then
\[
h(X,Y) = \nabla_x Y - \nabla_x Y = 0.
\]
From the definition, there exists \( W \in \Gamma(S(TM^+)) \) such that \( \phi W = Y \). Thus we have
\[
0 = \nabla_x Y = \nabla_x Y - \nabla_x Y
\]
\[
= \phi(-A_{\phi W} X + \nabla_x W) - \nabla_x Y
\]
\[
= -P' A_{\phi W} X - Q' A_{\phi W} X + B' \nabla_x W + C' \nabla_x W - \nabla_x Y.
\]
From the definition of the \( Q' \) and \( C' \), we know that \( Q' A_{\phi W} X = C' \nabla_x W, W = 0 \). So we have
\[
\nabla_x W \in \Gamma(\phi D^+), \ A_{\phi W} X \in \Gamma(\bar{D}).
\]
From \( \phi W = Y \) and (2.13), we have \( W = -\phi Y \), thus the proof is completed.

**Theorem 5.3** Let \( M \) be a contact SCR-lightlike submanifold of an indefinite Sasakian manifold \( \bar{M} \). Then \( D^+ \) defines a totally geodesic foliation if and only if \( h'(X,\phi Z) \) and \( h'(X,\phi N) \) has no components in \( \Gamma(\phi(D^+)) \), \( \forall X \in \Gamma(D^+) \), \( Z \in \Gamma(D^+) \).

**Proof.** From the definition, we have that \( D^+ \) is a totally geodesic foliation if and only if \( \nabla_x Y \in \Gamma(D^+) \), for any \( X, Y \in \Gamma(D^+) \), which is equivalent to
\[
g(\nabla_x Y, Z) = g(\nabla_x Y, N) = 0,
\]
\( \forall Z \in \Gamma(D^+), N \in \Gamma(ltr(TM)) \).

Then we have
\[
g(\nabla_x Y, Z) = \bar{g}(\nabla_x Y, Z) = -\bar{g}(\nabla_x Y, Z)
\]
\[
-\bar{g}(\phi Y, \phi \nabla_x Z) - \eta(Y) \eta(\nabla_x Z)
\]
\[
= \bar{g}(\phi Y, \phi \nabla_x Z)
\]
\[
-\bar{g}(\phi Y, \nabla_x \phi Z + g(X, Z) V + \eta(Z) X)
\]
\[
= \bar{g}(\phi Y, \nabla_x \phi Z)
\]
\[
= \bar{g}(\phi Y, h'(X, \phi N))
\]
and
\[
g(\nabla_x Y, N) = \bar{g}(\nabla_x Y, N)
\]
\[
= \bar{g}(\phi \nabla_x Y, \phi N) + \eta(\nabla_x Y) \eta(N)
\]
\[
= \bar{g}(\phi \nabla_x Y, \phi N)
\]
\[
= \bar{g}(\nabla_x \phi Y + g(X, Z) V + \eta(Y) X, \phi N)
\]
\[
= \bar{g}(\nabla_x \phi Y, \phi N)
\]
\[
= \bar{g}(\phi Y, \nabla_x \phi N)
\]
\[
= \bar{g}(\phi Y, h'(X, \phi N)).
\]
Thus the assertion is proved.

6. References
