Uniform Convergence of Extremal Polynomials When Domains Have Corners and Special Cusps on the Boundary

Cem Koşar¹, Mehmet Kucukaslan², Fahreddin G. Abdullayev²
¹Department of Mathematics, Faculty of Science and Literature, Bitlis Eren University, Bitlis, Turkey
²Department of Mathematics, Faculty of Science and Literature, Mersin University, Mersin, Turkey
E-mail: ckosar77@gmail.com, {mkucukaslan, fabdul}@mersin.edu.tr
Received May 26, 2011; revised September 13, 2011; accepted September 22, 2011

Abstract
We study the approximation properties of the extremal polynomials in $A_p$-norm and $C$-norm. We prove estimates for the rate of such convergence of the sequence of the extremal polynomials on domains with corners and special cusps.

Keywords: Uniform Approximation, Extremal Polynomials, Conformal Mapping, $p$-Bieberbach Polynomials

1. Introduction and Main Results
1.1. Statement of Problem and Some Definitions
Let $G$ be a finite domain bounded by Jordan curve $L := \partial G$ and $\zeta_0 \in G$ be an arbitrary fixed point. Let $w = \phi(z)$ be conformal mapping of $G$ onto the disk $D(0, r_0) = \{ w : |w| < r_0 \}$ with normalization $\phi(\zeta_0) = 0$, $\phi'(\zeta_0) = 1$, where $r_0$ is a conformal radius of $G$ with respect to $\zeta_0$ and let $\psi := \phi^{-1}$ be an inverse mapping.

Let $p > 0$. We denote by $A_p(G)$ the set of functions $f(z)$ analytic in $G$ and normalized by $f(\zeta_0) = 0$, $f'(\zeta_0) = 1$ such that

$$\|f\|_{A_p(G)} := \left( \iint_G |f'(z)|^p \, d\sigma_z \right)^{\frac{1}{p}} < \infty,$$

where $d\sigma_z$ is a two-dimensional Lebesque measure on $G$.

We denote by $\phi_n$ of all algebraic polynomials $P_n(z)$, deg $P_n \leq n$ satisfying $P_n(\zeta_0) = 0$, $P_n'(\zeta_0) = 1$.

Let us consider following extremal problem:

$$\left\| \phi - P_n \right\|_{A_p(G)} : P_n \in \phi_n \rightarrow \min$$  \hspace{1cm} (1.1)

Using a method similar to the one given in ([1], p. 137), it is seen that there exists an extremal polynomial $P^*_n(z)$ furnishing to the problem (1.1), these polynomials $P^*_n(z)$ are determined uniquely in case of $p > 1$ ([1], p. 142). This extremal problem was first considered by Kucukaslan M. and Abdullayev F.G. and they were called the $p$-Bieberbach polynomial of degree $n$ for the pair $(G, \zeta_0)$ in [2], and denoted by $B_{n,p}(z)$.

The main goal of this paper is to investigate the approximation rate of convergence $B_{n,p}(z)$ to the function $\phi$ in uniform norm for some domains when has some certain singularity, i.e.

$$\|\phi - B_{n,p}\|_{C(\bar{G})} := \max_{z \in \partial G} \{|\phi(z) - B_{n,p}(z)|\} \leq \frac{C}{n^\gamma}, \hspace{1cm} (1.2)$$

where $\gamma := \gamma(p, G) > 0$, the constant $C$ is independent of $n$.

In case of $p = 2$, the solution of (1.1) coincides with the well known $n$-th Bieberbach polynomial $B_n(z) = B_{n,2}(z)$ (see, for example, [3,4]). The approximation properties in the uniform norm of $B_n(z)$ on $G$ first was observed by Keldysh in 1939 [3] for the domains with sufficiently smooth boundary. A considerable progress in this area has been achieved by Mergelyan [5], Suetin [4], Simonenko [6], Andrievskii [7,8], Gaier [9,10], Abdullayev [11-13], Israfilov [14,15] and the others.

In this paper, we are going to consider the case $p > 1$ the problem in (1.2). First, we will investigate the approximation rate of $B_{n,p}(z)$ to the function $\phi$ in...
The \( A^p \)- norm and using well known Simonenko and Andrievskii method (see, for example, [6,7]), the approximation rate of \( B_{\alpha,p}(z) \) to the function \( \varphi \) in uniform norm will be obtained.

Now we need some definitions:

**Definition 1.1.** ([16], p. 97) The Jordan arc or a curve \( L \) is called a \( K \)-quasiconformal (\( K \geq 1 \)) arc or curve if there is a \( K \)-quasiconformal mapping \( f \) of a region \( D \) containing \( L \) such that \( f(L) \) is line segment or circle.

Let \( F(L) \) denote the set of all sense-preserving plane homeomorphisms \( f \) of regions \( D \supset L \) such that \( f(L) \) is a line segment or circle and let

\[
K(L) = \inf \{ K(f) : f \in F(L) \},
\]

where \( K(f) \) is the maximal dilatation of such a mapping \( f \). Then \( L \) is \( K \)-quasiconformal if and only if \( K(L) < \infty \). If \( L \) is a \( K \)-quasiconformal, then \( K(L) \leq K \).

\( D = \overline{C} \) gives the global definition of a \( K \)-quasiconformal arc or curve consequently. This definition is common in the literature. At the same time, we can consider the domain \( D \supset L \) as the neighborhood of the curve \( L \). In this case, Definition 1.1 will be called local definition of quasiconformal arc or curve. Through this work we consider the local definition. The local definition has an advantage in determining the coefficients of quasiconformality for some simple arcs or curves.

Let us denote \( z = z(s), \ s \in [0, \text{mes} L] \) natural representation of \( L = \partial G \).

**Definition 1.2.** ([4]) We say that \( G \subset C_\theta \) if \( \partial G \) has a continuous tangent \( \theta (s) := \theta (z(s)) \) for every points \( z(s) \).

**Corollary 1.1.** ([17]) If \( G \subset C_\theta \), then \( K = 1 + \varepsilon \) for all \( \varepsilon > 0 \).

**Definition 1.3.** ([12]) We say that \( G \subset C_\theta (\lambda, \alpha), 0 < \lambda < 2, \alpha > 0 \), if \( L = \partial G \) is expressed as a union of a finite number of \( C_\theta \)-arcs, connecting at the points \( z_0, z_1, \ldots, z_m \) such that \( L \) is locally smooth at \( z_0 \) and in the local coordinate system \((x, y) \) origin at \( z_j, 1 \leq j \leq m \), the following conditions are satisfied:

1. For every \( z_j, 1 \leq j \leq m \), the domain \( G \) has \( \lambda_j, \pi_j, 0 < \lambda_j < 2 \), exterior angle at the corner \( z_j \), \( \min \{ \lambda_j \} = \lambda \).
2. For every \( z_j, p + 1 \leq j \leq m \), in \((x, y) \) coordinate system origin at \( z_j \) we have

\[
\begin{align*}
\{ z = x + iy : c_1 x^{\alpha_1} \leq y \leq c_2 x^{\alpha_2}, 0 \leq x \leq e_1 \} & \subset \overline{C} G, \\
\{ z = x + iy : |y| \geq e_2 x, 0 \leq x \leq e_1 \} & \subset CG,
\end{align*}
\]

for some constants \( -\infty < c_1 < c_2 < \infty, e_i > 0 \), \( i = 1, 2 \).

It is clear from the definition that each domain \( G \subset C_\theta (\lambda, \alpha) \) may have exterior \( \lambda, \pi \) angles, \( 0 < \lambda_j < 2 \), at the points \( z_j, 1 \leq j \leq p \), and interior zero angles at which the boundary arcs are touching with \( x^{\alpha} \)-speed at the points \( z_j, p + 1 \leq j \leq m \). If \( \alpha = 0 \) then the domain \( G \) does not have interior zero angles and \( G \subset C_\theta (\lambda, 0 < \lambda < 2) \), i.e., \( C_\theta (\lambda, 0) = C_\theta (\lambda) \).

If \( \lambda = 1 \) then the domain \( G \) has piecewise smooth boundary with only interior zero angles. We denote the class of domains by \( C_\theta (1; \alpha) \).

1.2. New Results

We introduce the following notation. For any \( 1 < p < 2 \) and \( 0 < \lambda < 2 \) we set \( \alpha_j (p, \lambda), \beta_j (p, \lambda) \) and \( \eta_j (p, \lambda) \) as follows:

\[
\begin{align*}
\alpha_j (p, \lambda) & := \begin{cases} 
1 - \lambda & \text{if } 0 < \lambda \leq 1, \ j = 1, \\
\frac{4 - (p + 2) \lambda}{2(2 - p) + (3 - 2) \lambda} & \text{if } \frac{1}{2} < \lambda < \frac{2}{3}, \ j = 2, \\
1 + \frac{\lambda - 1}{2 - \lambda} & \text{if } \frac{2}{3} \leq \lambda < 2, \ j = 3,
\end{cases} \\
\beta_j (p, \lambda) & := \begin{cases} 
\frac{\lambda - p}{4(2 - 2 \lambda)} & \text{if } 0 < \lambda \leq 1, \ j = 1, \\
\frac{\lambda - p}{4(2 - 2 \lambda)} + \frac{3}{8} - \frac{3}{4} & \text{if } \frac{1}{2} < \lambda < \frac{2}{3}, \ j = 2, \\
\frac{2p - 3}{4} & \text{if } \frac{2}{3} \leq \lambda < 2, \ j = 3,
\end{cases} \\
\eta_j (p, \lambda) & := \begin{cases} 
\frac{\lambda}{2 - \lambda} & \text{if } 0 < \lambda \leq 1, \ j = 1, \\
\frac{\lambda}{2 - \lambda} + \frac{1}{p} & \text{if } \frac{1}{2} < \lambda < \frac{2}{3}, \ j = 3,
\end{cases}
\end{align*}
\]

Throughout this paper, \( c, c_1, c_2, \ldots \) are positive and \( e, e_1, e_2, \ldots \) sufficiently small positive constants which in general depend on \( G \).

**Theorem 1.2.** Let \( 1 < p < 2 \) and assume that

\[
G \subset C_\theta (\lambda, \alpha) \quad \text{for some } 0 < \lambda \leq \frac{1}{2}
\]

\[
0 < \alpha < \min \{ \alpha_j (p, \lambda), \beta_j (p, \lambda) \}.
\]

Then the \( p \)-Bieberbach polynomial \( B_{\alpha,p}(z) \) satisfies

\[
\| p - B_{\alpha,p} \|_{\infty} \leq c_n p
\]

for each \( \gamma \) with \( 0 < \gamma < \frac{2}{p} \).
Theorem 1.3. Let \( \frac{6(2-\lambda)}{6-\lambda} < p < 2 \) and assume that 
\( G \in C_0(\lambda;\alpha) \) for some \( \frac{1}{2} < \lambda < \frac{2}{3} \) and
\[ 0 \leq \alpha < \min \{ \alpha_j(p,\lambda), \alpha_z(p,\lambda) \}. \]
The \( p \)-Bieberbach polynomial \( B_{n,p}(z) \) satisfies
\[ \| \varphi - B_{n,p} \|_{L(T)} \leq c_n n^{-\gamma} \]
for each \( \gamma \) with \( 0 < \gamma < \eta_2 \frac{2}{p} (2 + 2\alpha - p) \).

Theorem 1.4. Let \( \frac{3}{2} < p < 2 \) and assume that 
\( G \in C_0(\lambda;\alpha) \) for some \( \frac{2}{3} < \lambda < 2 \) and
\[ 0 \leq \alpha < \min \{ \alpha_j(p,\lambda), \frac{2p-3}{4} \}. \]
The \( p \)-Bieberbach polynomial \( B_{n,p}(z) \) satisfies
\[ \| \varphi - B_{n,p} \|_{L(T)} \leq c_n n^{-\gamma} \]
for each \( \gamma \) with \( 0 < \gamma < \eta_3 \frac{2}{p} (2 + 2\alpha - p) \).

For \( 0 < \lambda < 1 \) we obtain
\[ \max \{ \alpha_j(p,\lambda), j = 1,2,3 \} = \alpha_1(p,\lambda) = \frac{1-\lambda}{\lambda} \]
and so, we have the following theorem.

Theorem 1.5. Let \( \max \{ 1, \frac{(1-4\lambda^2)(2-\lambda)}{2\lambda(\lambda^2-2\lambda+2)} \} < p < 2 \)
and assume that \( G \in C_0(\lambda;\alpha) \) for some \( \frac{2}{9} < \lambda < 1 \) and
\[ \frac{1-\lambda}{\lambda} \leq \alpha < \beta_1(p,\lambda). \]
Then the \( p \)-Bieberbach polynomial \( B_{n,p}(z) \) satisfies
\[ \| \varphi - B_{n,p} \|_{L(T)} \leq c_n \sqrt{\ln n} n^{-\gamma} \]
for each \( \gamma \) with \( 0 < \gamma < \eta_4 \frac{2}{p} (2 + 2\alpha - p) \).

Analogously result can be written for case \( 1 \leq \lambda < 2 \).

Theorem 1.6. Let \( 2 < p < 2 + \frac{2}{|\lambda-1|} \) and assume that 
\( G \in C_0(\lambda;\alpha) \) for some \( 0 < \lambda < 2 \), and \( \alpha \geq 0 \). Then, for any \( n \geq 3 \) and arbitrary small \( \varepsilon > 0 \).
\[ \| \varphi - B_{n,p} \|_{L(T)} \leq c_n n^{-\gamma} \]
for each \( \gamma \) with \( 0 < \gamma < \frac{1}{p} \), if \( 0 < \lambda < 1 \) and
\[ 0 < \gamma < \frac{1}{p} \left( \frac{2}{p} - 1 \right) (\lambda - 1), \]
if \( 1 \leq \lambda < 2 \).

Corollary 1.7. Let \( p = 2 \) and assume that 
\( G \in C_0(\lambda;\alpha) \) for some \( 0 < \lambda < 2 \) and
\[ 0 \leq \alpha < \frac{1}{2} \min \{ \frac{1}{2}, \frac{\lambda}{2-2\lambda} \}. \]
Then the Bieberbach polynomial \( B_{n,p}(z) \) satisfies
\[ \| \varphi - B_{n,p} \|_{L(T)} \leq c_n \sqrt{\ln n} n^{-\gamma} \]
for each \( \gamma \) with \( 0 < \gamma < \min \{ \frac{1}{2}, \frac{\lambda}{2-2\lambda} \} - 2\alpha \).

Remark 1.1. 1) Theorems 1.2-1.2, extend the corresponding results in [3-5, 10,12,13] to case \( p \neq 2 \).
2) Corollary 1.2 is extending the one result [10] to domains bounded by a piecewise smooth curve with interior zero angles and in \( \alpha = 0 \) coincides with the corresponding result of this work.

2. Some Auxiliary Results

The notation “\( a \leq b \)”, we mean that \( a \leq c_b \) for a constant \( c_b \), which doesn’t depend on \( a \) and \( b \). The relation “\( a \leq b \)” indicates that \( c_b \leq a \leq c_b \), where \( c_2, c_3 \) are independent of \( a \) and \( b \).

Let \( G \subset \mathbb{C} \) be finite domain bounded by Jordan curve \( L \) and let \( w = \Phi(z) \) be the conformal mapping of \( \Omega := \mbox{ext} \overline{G} \) onto
\[ \hat{\Lambda} = \left\{ w : |w| > 1 \right\} \setminus \left\{ w : |w| < 1 \right\}, \]
normalized by \( \Phi(\infty) = \infty, \Phi'(\infty) > 0 \) \( \left( \hat{\phi}(\zeta_0) = 0, \hat{\phi}'(\zeta_0) > 0 \right) \).

The level curve (exterior or interior) can be defined for \( t > 0 \) as
\[ L_t := \left\{ z : |\hat{\phi}(z)| = t, \mbox{if} \ t < 1 ; |\Phi(z)| = t, \mbox{if} \ t > 1 \right\}, \ L_t \equiv L_t.

Let us denote \( G_t := \mbox{int} L_t, \quad \Omega_t := \mbox{ext} L_t \) and
\[ d(z,L) := \inf \{ |z - \zeta| : \zeta \in L_t \}.

Let \( L \) be a \( K \)-quasiconformal curve and \( D \subset \overline{C} \). Then the region \( D \) can be chosen to be the region \( G_{R_0} \setminus G_{R_0}' \), for a certain number \( 1 < R_0 \leq 2 \) depending on \( \hat{\phi}, \Phi, f \) and \( r_0 = R_0^{-1} \). In this case, it is known that the function \( \alpha^*(\cdot) = f^{-1}\left( f(\cdot) \right)^{-1} \) is a \( K^2 \)-quasiconformal reflection across \( L \) as shown in ([18], p. 28), by analogously in ([19], p. 75), that is, \( \alpha^*(\cdot) \) is a \( K^2 \)-quasiconformal mapping leaving points on \( L \) fixed and satisfying the conditions \( \alpha \left( \frac{G}{\hat{\tau}} \right) \subset G \setminus \overline{G_{R_0}} \)
\( \alpha \left( G \setminus \overline{G_{R_0}} \right) \subset G_{R_0} \setminus \overline{G} \) for some \( 1 \leq \hat{\tau} < R_0, \ r_0 < \hat{\tau} < 1. \)

By using the facts in ([16], p. 97) and ([19], p. 76, [20], p. 26) we can find a \( C(K) \)-quasiconformal reflection \( \alpha(\cdot) \) across \( L \) such that it satisfies the following.
\[ |z_i - \alpha(z)| \leq |z_i - z|, \forall z_i \in L, \epsilon \leq |z| < \frac{1}{\epsilon}, \]
\[ |\alpha_z| \leq |\alpha_z|, \epsilon \leq |z| < \frac{1}{\epsilon}, \]
\[ |\alpha_z| \leq |\alpha_z|', \epsilon \leq |z| < \frac{1}{\epsilon}, \]  
(2.1)
and Jacobian \( J_a = |\alpha_z|^2 \) of \( \alpha(z) \) satisfied \( J_a > 1 \).

Therefore, by means of the extension theorem of a quasiconformal mapping, without loss of generality we may assume that
\[ \alpha^*(z) = \alpha(z), z \in D. \]

**Lemma 2.1.** ([18]) Let \( L \) be a \( K \)-quasiconformal curve;
\[ z_1 \in L; z_2, z_3 \in G \cap \{ z : |z - z_1| \leq c_2 d(z_1, L_{R_0}) \}, \]
\[ w_j = \phi(z_j), \]
\( \{ \text{or } z_2, z_3 \in G \cap \{ z : |z - z_1| \leq c_2 d(z_1, L_{R_0}) \}, w_j = \Phi(z_j) \} \]
j = 1, 2, 3.
Then, 1) The statements \( |z_i - z_j| \leq |z_i - z_j| \) and \( \epsilon \leq |z_j - z_j| \leq |z_j - z_j| \) are equivalent. So are \( |z_i - z_j| \leq |z_i - z_j| \) and \( |w_i - w_j| \leq |w_i - w_j| \),
2) If \( |z_i - z_j| \leq |z_i - z_j| \), then
\[ \left( |w_i - w_j| \leq |z_i - z_j| \right)^{\frac{1}{k+1}} \leq \left( |w_i - w_j| \leq |z_i - z_j| \right)^{\frac{1}{k+1}} \]
and, consequently, for any \( z_1 \in L_{R_0}, z_2 \in L_{R_0} \)
\[ |w_i - w_j| \leq |z_i - z_j| \leq |w_i - w_j| \],
where \( 1 \leq R_0 < 2 \) and \( R_0 = R_0^{-1} \) are fixed constants.

**Lemma 2.2.** ([21]) Let \( L = \partial G \) be a quasiconformal curve. Then, For every \( z \in L \) there exists an arc \( \beta(\xi_0, z) \) in \( G \) joining \( \xi_0 \) to \( z \) with following properties:
1) \( d(\xi, L) \leq |\xi - z| \) for every \( \xi \in \beta(\xi_0, z) \).
2) If \( \beta(\xi_1, \xi_2) \) is the subarc of \( \beta(\xi_0, z) \) joining \( \xi_1 \) to \( \xi_2 \), then \( mes \beta(\xi_1, \xi_2) \leq |\xi_1 - \xi_2| \) for every pair \( \xi_1, \xi_2 \in \beta(\xi_0, z) \).

**Lemma 2.3.** ([7]) Let \( L = \partial G \) be a \( K \)-quasiconformal curve. Then \( mes \ell \approx mes \alpha(\ell) \), for every rectifiable arc \( \ell \in G \).

Let \( G \) be arbitrary Jordan domain and \( \Gamma := \Gamma(\xi_0, \xi) \in \Omega, z_0 \in L, \xi \in \Omega \) an rectifiable arc except for one of its endpoints of its \( z_0 \in L \) which satisfies the following conditions:
1) \( mes \Gamma(\xi, \xi) \leq |\xi| - |\xi| \), for all \( \xi, \xi \in \Gamma \).
2) There exists a monotone increasing function \( g(t) \) such that \( d(\xi, L) \leq g(\xi - \xi) \) for all \( \xi \in \Gamma \).

**Lemma 2.4.** ([22]) Let \( G \subset C_p(\lambda, \alpha) \) for some \( 0 < \lambda < 2, \alpha \geq 0 \) and on the arc \( \Gamma \) a measurable function \( f(\xi) \) be given such that \( |f(\xi)| \leq |\xi - z_0|^{-\delta} \) for all \( \xi \in \Gamma \). Then for all \( z \in \Gamma \),
\[ \| f(\xi) \|_{L^2(G)} \approx \| \ln(\frac{1}{\epsilon}) \|_{L^2(\Gamma)} \leq 1 < p < 2, \]
\[ \frac{2^\frac{1}{2}}{2^\frac{1}{2} - \frac{1}{2}} \leq p < 2 + \frac{2}{2 - 1}. \]

3. Approximation in the \( A_p(\alpha) \) - Norm

Suppose that \( G \subset C_p(\lambda, \alpha) \) for some \( 0 < \lambda < 2 \) and \( \alpha \geq 0 \) is given. For the sake of simplicity, but without loss of generality, we assume that \( m = 2, z_1 = 1, z_2 = -1 \) \((-1,1) \subset G \) and let the local coordinate axes be parallel to \( OX \) and \( OY \) in the coordinate system; \( L^1 := \{ z : z \in L, Imz \geq 0 \}, L^2 := \{ z : z \in L, Imz \leq 0 \} \). Then \( z_0 \) is taken as an arbitrary point on \( L^1 \) (or on \( L^2 \) subject to the chosen direction).

We recall that the domain \( G \subset C_p(\lambda, \alpha) \) has exterior \( 2\pi \) \((0 < \lambda < 2)\) non zero angle and \( 1^{1/\alpha} \) type interior zero angle in the neighborhood of the points \( z_1 = -1 \) and \( z_2 = 1 \), respectively.

We can say that the function \( w = \phi(z) \) for the domain \( G \subset C_p(\lambda, \alpha) \) satisfies the conditions described in Lemma 2 in the neighborhood of point \( z_1 = -1 \). So, we can easily get from Lemma 2
\[ d(L, z) \approx |\phi(z) - \phi(1)|^{1/1+\alpha} \quad |z - 1| \leq |\phi(z) - \phi(1)|^{1/1+\alpha}, \]
(3.1)
for all \( z \in M := \{ z \in G : |z| \geq |z_1| \} \).

On the other hand, using properties of the function \( w = \phi(z) \) in the neighbourhood of the point \( z_1 = -1 \) (see, [7,23]) we obtain
\[ |z + 1| \leq \left[ -\ln(\phi(z) - \phi(-1)) \right]^{\alpha^{-1}} \]
(3.2)
Because each \( L^j, j = 1,2 \) is a \( (1+\epsilon_1) \)- quasiconformal arc, \( \alpha(\ell) \) must be the quasiconformal reflection across \( L^j \). Let us consider the curve
\[ \gamma_1 := \alpha_1 \left\{ z = x + iy : y = \frac{2\overline{c}_1 + \overline{c}_2}{3}(x + 1)^{1/\alpha} \right\}; \]
\[ \gamma_2 := \alpha_2 \left\{ z = x + iy : y = \frac{2\overline{c}_2 + \overline{c}_2}{3}(x + 1)^{1/\alpha} \right\}; \]
\[ y_1 := z = x + iy : y = \overline{c}_1(x - 1); \]
\[ y_2 := z = x + iy : y = \overline{c}_4(x - 1); \]
for some \(-\infty < \tilde{c}_i < \tilde{c}_j < \infty\), where a constants \(\tilde{c}_i\) and \(\tilde{c}_j\) from Definition 1.3.

It is easy to check from Lemma 2.2 that

\[ \text{mes } \gamma_i' (\xi, \xi_j) \leq |\xi - \xi_j|, \]

for all \(\xi, \xi_j \in \gamma_i', i, j = 1, 2\).

Let \(N > N(R_0)\) sufficiently large natural number. For \(n > N\) and arbitrary \(0 < \varepsilon < 1\), let us choose \(R = r_0 + cn^{-1}\) such that \(r_0 < R < R_0\) and points \(\gamma_i', i, j = 1, 2\) such that they are in the intersection of \(L_i\) and \(\gamma_i'\).

These points divide \(L_i\) into four parts:

\[
L_i' := R \left( z_1', z_1' \right), L_i^1 := R \left( z_1', z_1^1 \right), \\
L_i^2 := R \left( z_1^2, z_1^2 \right), L_i^4 := R \left( z_1^1, z_1^1 \right)
\]

are traversed in the positive direction (counterclockwise). \(L_i' := L_i \cup L_i^1 \cup L_i^2 \cup L_i^4\) and \(\gamma_i' (R)\) are subarcs of \(\gamma_i'\) joining points \(-1, 1\) with \(\gamma_i'\). Let us denote

\[
\Gamma_i := \gamma_i' (R) \cup \gamma_i^1 (R) \cup \gamma_i^2 (R) \cup \gamma_i^4 (R), \\
U := \text{int} (\Gamma_i) \text{ and } \Gamma_i := U \setminus \Gamma_i.
\]

We can extend the function \(\varphi (z)\) to \(U\) in the following way

\[
\varphi (z) := \begin{cases} 
\varphi (z), & z \in \overline{G}, \\
\frac{\varphi (\alpha_i (z))}{n}, & z \in U_i, i = 1, 2.
\end{cases} \tag{3.3}
\]

Then,

\[
\varphi (z) = \begin{cases} 
0, & z \in \overline{G}, \\
\varphi (\alpha_i (z)) \alpha_i (z), & z \in U_i, i = 1, 2. \tag{3.4}
\end{cases}
\]

From the Cauchy-Pompeiu formula ([16], p. 148), we get

\[
\varphi (z) = \frac{1}{2 \pi i} \int_{\Gamma_i} \frac{\varphi (\xi)}{\xi - z} d\xi - \frac{1}{2 \pi} \int_{\Gamma_i} \frac{\varphi (\xi)}{\xi - z} d\sigma_i, z \in \overline{G}.
\]

Then, using the above notations we obtain

\[
\varphi (z) = \frac{1}{2 \pi} \int_{\Gamma_i} \frac{f (\xi)}{\xi - z} d\xi + \frac{1}{2 \pi i} \sum_{j, j' (\xi)} \frac{\varphi (\xi) - \varphi (-1)}{\xi - z} d\xi + \frac{1}{\pi} \int_{\Gamma_i} \varphi (\xi) d\sigma_i, 
\]

where

\[
f (\xi) = \begin{cases} 
\tilde{\varphi} (\xi), & \xi \in L_i \alpha (\xi) \in L_i^2, \\
\varphi (1), & \xi \in L_i^4.
\end{cases}
\]

Since the first part of right hand side in (3.5) is analytic function in \(G\), there exists a polynomial \(P_{n+1} (z)\), where \(\text{deg } P_{n+1} (z) \leq n - 1\) ([24], p. 142), such that

\[
\left| \frac{1}{2 \pi i} \int_{L_i} f (\xi) (\xi - z)^{-2} d\xi - P_{n+1} (z) \right| < \frac{c}{n}, \tag{3.6}
\]

Let \(Q_\alpha (z) := \frac{1}{2 \pi i} \int_{L_i} P_{n+1} (z) d\xi\). Then, \(Q_\alpha (z) = 0\) and from (3.5) and (3.6) we have

\[
\left| \varphi' (z) - Q_\alpha (z) \right| \leq \frac{c}{n} + \frac{1}{2 \pi} \sum_{j, j' (\xi)} \int_{\Gamma_i} \frac{\varphi (\xi) - \varphi (-1)}{\xi - z} d\xi
\]

and let us take integrals over \(G\) of the \(p\)-th power of each side we obtain

\[
\left| \int_{G} \left| \varphi' (z) - Q_\alpha (z) \right|^p d\sigma_i \right| \leq \frac{c}{n} + \frac{1}{n^p} \int_{G} \left| \varphi (\xi) - \varphi (-1) \right|^p d\xi
\]

(3.7) and (3.8) give us

\[
\| \varphi' - Q_\alpha \|_{L^p (G)}^p \leq \frac{1}{n^p} + \sum_{j, j' (\xi)} \int_{\Gamma_i} \frac{\varphi (\xi) - \varphi (-1)}{\xi - z} d\xi + \frac{1}{n^p} \int_{\Gamma_i} \left| \varphi' (\xi) \right|^p d\sigma_i. \tag{3.9}
\]
Let us consider two cases in the last double integral in (3.9) as: if \( 1 < p < 2 \) and \( p \geq 2 \).

If \( 1 < p < 2 \), then, using Hölder Inequality ([25], p. 105) we obtain

\[
\int_{\mathbb{R}} \left| \varphi' (\alpha, (\xi)) \right|^p d\sigma_\xi \leq \left( \int_{\mathbb{R}} \left| \varphi' (\alpha, (\xi)) \right|^p d\sigma_\xi \right)^{\frac{p}{2}} \left( \int_{\mathbb{R}} \left| \varphi' (\alpha, (\xi)) \right|^p d\sigma_\xi \right)^{1 - \frac{p}{2}}.
\]

(3.10)

Thus, (3.9) and (3.10) give us

\[
\| \varphi' - Q'_n \|_{L^p (\mathbb{R})} \leq \frac{1}{n^\theta} + \sum_{\xi = 1}^{\sigma_n} \int_{\mathbb{R}} \frac{\phi'(\xi) - \varphi'(-1)^j}{(\xi - z)^2} d\sigma_\xi.
\]

(3.11)

Now, we need some technical estimates to attack the problem in (3.11).

**Lemma 3.1.** Let \( G \in C_0 (\lambda, \alpha) \) for some \( 0 < \lambda < 2 \), \( \alpha > 0 \). Then for all \( \epsilon > 0 \)

\[
\left\| \int_{\mathbb{R}} \frac{\phi'(\xi) - \varphi'(-1)^j}{(\xi - z)^2} d\sigma_\xi \right\|_{L^p (\mathbb{R})} \leq \left\{ \begin{array}{ll}
\epsilon \left( \frac{1}{p - 1} \right)^{\frac{1}{2}} \left( \frac{1}{p + 1} \right)^{\frac{1}{2}}, & 1 < p \leq 2, \quad j = 1, \\
\epsilon \left( \frac{1}{p - 1} \right)^{\frac{1}{2}} \left( \frac{1}{p + 1} \right)^{\frac{1}{2}}, & 2 < p \leq 2 + \frac{2}{\lambda - 1}, \quad j = 1, \\
\epsilon \left( \frac{1}{p - 1} \right)^{\frac{1}{2}} \left( \frac{1}{p + 1} \right)^{\frac{1}{2}}, & 1 < p \leq 2, \quad j = 2, \\
\epsilon \left( \frac{1}{p - 1} \right)^{\frac{1}{2}} \left( \frac{1}{p + 1} \right)^{\frac{1}{2}}, & 2 < p \leq 2 + \frac{2}{\lambda - 1}, \quad j = 2.
\end{array} \right.
\]

Proof. Let us choose \( f (\xi) := \phi'(\xi) - \varphi'(-1)^j \), \( j = 1, 2 \), in Lemma 2.4, we obtain

\[
\left\| \int_{\mathbb{R}} \frac{\phi'(\xi) - \varphi'(-1)^j}{(\xi - z)^2} d\sigma_\xi \right\|_{L^p (\mathbb{R})} \leq \left\{ \begin{array}{ll}
\epsilon \left( \frac{1}{p - 1} \right)^{\frac{1}{2}} \left( \frac{1}{p + 1} \right)^{\frac{1}{2}}, & 1 < p \leq 2, \\
\epsilon \left( \frac{1}{p - 1} \right)^{\frac{1}{2}} \left( \frac{1}{p + 1} \right)^{\frac{1}{2}}, & 2 < p \leq 2 + \frac{2}{\lambda - 1},
\end{array} \right.
\]

(3.12)

where \( \ell_{i,j} := \text{mes} \gamma'_j (R) \). On the other hand, according to Lemma 2.1 and Corollary 1.0, we have

\[
d (z_j, L') \approx n^{-1}. \quad \text{Then, from (2.1), (3.1) and (3.2), for all } \epsilon > 0, \text{ we get}
\]
Then, the following inequalities are obtained from (3.12) and (3.13)

\[ \int \hat{\phi}(\xi) - \phi(-1) \left( \xi - z \right) \frac{d\xi}{d\zeta} \leq \ln \left( \frac{1}{n} \right) \left( \frac{1}{1 + |z - z'|} \right) - \varepsilon, \quad i = 1,2 \]

and

\[ \int \hat{\phi}(\xi) - \phi(1) \left( \xi - z \right) \frac{d\xi}{d\zeta} \leq \ln \left( \frac{1}{n} \right) \left( \frac{1}{1 + |z - z'|} \right) - \varepsilon, \quad i = 1,2 \]

Combining (3.14) and (3.15) the proof is completed. □

Now, let us estimate \( \text{mes} \alpha_i (V_i') \) by choosing \( K = 1 + \varepsilon, \varepsilon > 0 \) and from Lemma 2.1 we have

\[ \text{mes} \alpha_i (V_i') \leq \left( \frac{1}{n} \right)^{1 - \varepsilon}. \]

On the other hand if we use the method of Gaier in [9]

Lemma 3.4. Let \( G \in C_{\theta}(\lambda; \alpha) \) for some \( 0 < \lambda < 2, \alpha \geq 0 \). Then for all \( \varepsilon > 0 \)

\[ \text{mes} \alpha_i (V_i') \leq \left( \frac{1}{n} \right)^{1 - \varepsilon}. \]

After then using the second and third inequality in the first one we obtain the desired proof. □

Lemma 3.4. Let \( G \in C_{\theta}(\lambda; \alpha) \) for some \( 0 < \lambda < 2, \alpha \geq 0 \). Then for all \( \varepsilon > 0 \)

\[ \int_{U_k} |\varphi'(\alpha_i(z))|^p \, d\sigma_z \leq \left( \frac{1}{n} \right)^{1 - \varepsilon}, \quad i = 1,2 \]

where \( 2 \leq p < 2 + \frac{1}{1 - \theta}, \theta = \min \{ \lambda, 2 - \lambda \}. \)

Proof. Let \( z \in \overline{G}, \ z' \) – the point of \( L := \partial G \) nearest to \( z \). Then we have

\[ |z - z'|^{1 - \varepsilon} \leq |\varphi(z) - \varphi(z')| \leq |z - z'|^{1 - \varepsilon}, \]

and from ([18], Lemma 3)
After this estimation, we get
\[
\left\lbrack \int |\varphi'(\alpha_i(z))|^p \, d\sigma_z \right\rbrack \leq \left\lbrack \int |\varphi'(z)|^p \, d\sigma_z \right\rbrack \leq \left\lbrack \int \left( \frac{1}{|\psi'(w)|} \right)^{\beta - \varepsilon} \right\rbrack \leq (r_0 - |w|)^{1-\beta + \varepsilon},
\]
where \( r := \min \{ |\varphi(\alpha(z))| : z \in L \} \) and
\[ 2 \leq p < 2 + \frac{1}{1 - \theta}, \]

By using Lemma 3.1, 3.2, 3.3, 3.4 and (3.11) we get the following results. We need this notation:
\[
\phi_n(\lambda, \alpha) := \begin{cases} 
\ln \left( \frac{1}{n} \right)^{\frac{1}{\alpha}} \left( \frac{1}{1 + \frac{2(1-p\alpha)}{2n}} \right)^{\varepsilon} &, \quad \alpha \geq \max \{ \alpha_i(p, \lambda) \}, \quad i = 1, 2, 3, \\
\left( \frac{1}{n} \right)^{\frac{\alpha}{2-\lambda}} &, \quad \alpha < \alpha_i(p, \lambda), \quad 0 < \lambda < \frac{1}{2}, \\
\left( \frac{1}{n} \right)^{\frac{\alpha}{2-\lambda}} \left( \frac{1}{1 + \frac{2(1-p\alpha)}{2n}} \right)^{\varepsilon} &, \quad \alpha < \alpha_i(p, \lambda), \quad \frac{1}{2} < \lambda \leq \frac{2}{3}, \\
\left( \frac{1}{n} \right)^{\frac{\alpha}{2-\lambda}} &, \quad \alpha > \alpha_i(p, \lambda), \quad \frac{2}{3} < \lambda < 2.
\end{cases}
\]

Lemma 3.5. Let \( 1 < p \leq 2 \) and assume that \( G \in C_\alpha(\lambda; \alpha) \) for some \( 0 < \lambda < 2, \quad \alpha \geq 0 \). Then, for any \( n \geq 3 \) and arbitrary small \( \varepsilon > 0 \)
\[
\| \phi - B_{n,p} \|_{\phi_n(\lambda)} \leq \theta_n(\lambda, \alpha).
\] (3.16)

Lemma 3.6. Let \( 2 < p < 2 + \frac{2}{1-\lambda} \) and assume that \( G \in C_\alpha(\lambda; \alpha) \) for some \( 0 < \lambda < 2, \quad \alpha \geq 0 \). Then, for any \( n \geq 3 \) and arbitrary small \( \varepsilon > 0 \)
\[
\| \phi - Q' \|_{\phi_n(\lambda)} \leq \begin{cases} 
\ln \left( \frac{1}{n} \right)^{\frac{1}{\alpha}} \left( \frac{1}{1 + \frac{2(1-p\alpha)}{2n}} \right)^{\varepsilon} &, \quad 1 < p < 2, \quad j = 1, \\
\left( \frac{1}{n} \right)^{\frac{\alpha}{2-\lambda}} \left( \frac{1}{1 + \frac{2(1-p\alpha)}{2n}} \right)^{\varepsilon} &, \quad 2 \leq p < 2 + \frac{2}{1-\lambda}, \quad j = 1, \\
\ln \left( \frac{1}{n} \right)^{\frac{1}{\alpha}} \left( \frac{1}{1 + \frac{2(1-p\alpha)}{2n}} \right)^{\varepsilon} &, \quad 1 < p < 2, \quad j = 2, \\
\left( \frac{1}{n} \right)^{\frac{\alpha}{2-\lambda}} \left( \frac{1}{1 + \frac{2(1-p\alpha)}{2n}} \right)^{\varepsilon} &, \quad 1 < p < 2, \\
\left( \frac{1}{n} \right)^{\frac{\alpha}{2-\lambda}} \left( \frac{1}{1 + \frac{2(1-p\alpha)}{2n}} \right)^{\varepsilon} &, \quad 1 < p < 2, \\
\left( \frac{1}{n} \right)^{\frac{\alpha}{2-\lambda}} &, \quad 2 \leq p < 2 + \frac{1}{1 - \theta}.
\end{cases}
\] (3.18)

Proof. The proof of Lemma 3.5-3.6 is similar. So, we give them together. From (3.11) and Lemma 3.1-3.4 we obtain
where \( \mu := \min \left\{ \frac{\lambda}{2 - \lambda}, \frac{1}{2} \right\} \),
\( \lambda_c := \min \{2\lambda; 1\} \), \( \theta_c := \min \{1, 2 - \lambda\} \)

Case 1. Let \( 1 < p < 2 \). In this case from (3.18) we get

\[
\|\varphi' - \varphi'_{\beta_{c}(G)}\|_{\beta_{c}(G)} \leq \|n^{-1}\left(\frac{1}{n}\right)^{\frac{1}{1-\theta_c}} + \left(\frac{1}{n}\right)^{\frac{1}{1-\theta_c} - \frac{1}{2p}}\|_{\beta_{c}(G)}
\]

Case 2. Let \( 2 \leq p < \min \left\{ 2 + \frac{2}{(p-1)^2}, \frac{1}{1 - \theta_c} \right\} \), \( \theta_c := \min \{1, 2 - \lambda\} \).

In this case from (3.18) we have

\[
\|\varphi - Q_{n}\|_{\beta_{c}(G)} \leq \left\{ \begin{array}{ll}
\left(\frac{1}{n}\right)^{1-\theta_c}, & 0 < \lambda < 1,
\left(\frac{1}{n}\right)^{\frac{1}{p} - \frac{1}{1-\theta_c}}, & 1 \leq \lambda < 2,
\left(\frac{1}{n}\right)^{1-\theta_c}, & 1 < p < 2,
\end{array} \right.
\]

Now, let us consider the polynomial
\( \tilde{Q}_n(z) := Q_n(z) + [1 - Q'_{\beta_{c}}(z_0)](z - z_0) \). It is clear
\( \tilde{Q}_n \in \varphi_n \) that \( \tilde{Q}_n(z_0) = 0, \tilde{Q}_n'(z_0) = 1 \). So, we have

1. \( 1 < p < 2 \),
2. \( 0 < \lambda < 1, 2 \leq p < 2 + \frac{2}{|\lambda - 1|} \),
3. \( 1 \leq \lambda < 2, 2 \leq p < 2 + \frac{2}{|\lambda - 1|} \)

4.1) in (3.19) to prove Theorems 1.2-1.6.

4. References

[8] V. V. Andrievskii, “Convergence of Bieberbach Polynomials in Domains with Quasiconformal Boundary,”


