Non-integrability of Painlevé V Equations in the Liouville Sense and Stokes Phenomenon

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Abstract

In this paper we are concerned with the integrability of the fifth Painlevé equation \( P_V \) from the point of view of the Hamiltonian dynamics. We prove that the Painlevé \( V \) equation (2) with parameters \( \kappa_\alpha = 0, \kappa_\theta = -\theta \) for arbitrary complex \( \theta \) (and more generally with parameters related by Bäclund transformations) is non integrable by means of meromorphic first integrals. We explicitly compute formal and analytic invariants of the second variational equations which generate topologically the differential Galois group. In this way our calculations and Ziglin-Ramis-Morales-Ruiz-Simó method yield to the non-integrable results.

Keywords: Differential Galois theory, Painlevé V equation, Hamiltonian Systems, Stokes Phenomenon

1. Introduction

The six Painlevé equations \( (P_I - P_{VI}) \) were introduced and first studied by Paul Painlevé [1] and his student B. Gambier [2] who classified all the rational differential equations of the second order free of movable critical points. The solutions of these equations define some new functions, the so-called Painlevé transcendents or Painlevé functions. Although the Painlevé equations were discovered from strictly mathematical considerations they have recently appeared in several physical applications. Among field-theoretical problems which can be solved in terms of the considered below Painlevé \( V \) transcendent we mention the two-point correlation functions at zero temperature for the one-dimensional impenetrable Bose gas (Jimbo, Miwa, Mori, Sato [3]). In the present article we deal with the fifth Painlevé equation, \( P_V \),

\[
\begin{align*}
\ddot{y} &= \left( \frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 - \frac{1}{t} \dot{y} + \frac{(y-1)^2}{2t^2} \left( \kappa_\alpha^2 y - \kappa_\theta^2 \right) \\
- (\theta + 1) \frac{\dot{y}}{t} &+ \delta \frac{y}{(y+1)} \frac{(y+1)}{y-1}
\end{align*}
\]

where \( t \in \mathbb{C} \) and \( \kappa_\alpha, \kappa_\theta, \theta, \delta \) are arbitrary complex parameters. It is well known that when \( \delta = 0, \theta = -1 \) equation (1) can be solved by quadratures, [4]. If \( \theta \neq -1 \) and \( \delta = 0 \) the fifth Painlevé equation (1) is equivalent to the third Painlevé equation \( P_{III} \). In this paper we investigate the generic case of \( P_V \) when \( \delta \neq 0 \). Hence, by replacing \( t \) by \( \sqrt{-2\delta t} \) one can normalize the constant as \( \delta = -1/2 \) [5], and so we consider

\[
\begin{align*}
\ddot{y} &= \left( \frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 - \frac{1}{t} \dot{y} + \frac{(y-1)^2}{2t^2} \left( \kappa_\alpha^2 y - \kappa_\theta^2 \right) \\
- (\theta + 1) \frac{\dot{y}}{t} &+ \delta \frac{y}{(y+1)} \frac{(y+1)}{y-1}
\end{align*}
\]

As the Painlevé equations can be written as time-dependent Hamiltonian systems of \( 1+1/2 \) degrees of freedom (see Malquist [6] and Okamoto [5]) their integrability should be considered in the context of Hamiltonian systems. We recall that by this we mean the existence of enough meromorphic first integrals (in our case two). In [7] Morales-Ruiz raises the question about the integrability of the Painlevé transcendents as Hamiltonian systems. Later Morales-Ruiz in [8], and Stoyanova and Christov in [9] obtain a non-integrable result for Painlevé II family. Non-integrability of the Painlevé
VI equation for some particular values of the parameters is proved by Horozov and Stoyanova in [10] and by Stoyanova in [11]. In the present note we continue the study of Painlevé transcendents with the fifth Painlevé equation and obtain an analogous result for one family of the parameters. Our method uses the differential Galois approach to non-integrability of Hamiltonian systems [12] which is an extension of the Ziglin theory [13, 14]. In particular, studying the differential Galois group of the first and second variational equations along a particular rational non-equilibrium solution we can find non-integrable results. It appears that the corresponding variational equations have an irregular singularity and new difficulty have to be overcome.

Our main result is the following theorem:

Assume that \( \kappa_\mu = 0, \kappa_\nu = -\theta \) where \( \theta \) is an arbitrary complex parameter. Then the fifth Painlevé equation (2) is not integrable. We chose to investigate \( P_\nu \) (2) for these values of the parameters because then the Hamiltonian system (11) possesses a simple rational solution. The point is our method requires one single-valued solution.

By Bäcklund transformations of \( P_\nu \) it is possible to extend the result of main for an infinite subfamily of \( P_\nu \) (2):

Assume that \( \kappa_\mu + \kappa_\nu = \pm \theta + m \) where \( m \) is even and at least one \( \kappa_\mu, \kappa_\nu \) is integer. Then the fifth Painlevé equation (2) is not integrable.

The paper is organized as follows. In section 2 we recall the main results of the Ziglin-Ramis-Morales-Ruiz-Simó theory of non-integrability of the Hamiltonian systems, of the differential Galois theory and asymptotic theory for ordinary differential equations needed in the proofs. In section 3 we prove the non-integrability of the fifth Painlevé equation (2) for \( \kappa_\mu = 0, \kappa_\nu = -\theta \) and \( \theta \in \mathbb{N} \) (witt). In n1 we prove the non-integrability of \( P_\nu \) (2) for \( \kappa_\mu = 0, \kappa_\nu = -\theta = 1 \). In section 4 using Bäcklund transformations of Painlevé \( V \) equation we extend the results of section 3 to the entire orbit of the parameters and establish the main results of this article.

2. Preliminaries

2.1. Non-Integrability and Differential Galois Theory

In this section we briefly recall Ziglin-Ramis-Morales-Ruiz-Simó theory of non-integrability of Hamiltonian systems following [12] and [15].

Consider a Hamiltonian system

\[
\dot{x} = X_H(x)
\]

with a Hamiltonian \( H \) on a complex analytic \( 2n \)-dimensional manifold \( M \). Let \( x = x(t) \) be a particular solution of (3), which is not an equilibrium point of the vector field \( X_H \). Denote by \( \Gamma \) the phase curve corresponding to \( x = x(t) \). We can write the variational equations \( VE \) along \( \Gamma \)

\[
\dot{\xi} = \frac{\partial X_H}{\partial x}(x(t)) \xi
\]

We can always reduce the order of this system by one restricting \( VE \) to the normal bundle of \( \Gamma \) on the level variety \( M_0 = \{ x | H(x) = \hbar \} \). The new-obtained system is called normal variational equation (NVE). In his papers [13], [14], Ziglin showed that if system (3) has a meromorphic first integral, then the monodromy group \( \mathcal{M} \) of the normal variational equations has a rational integral.

Morales-Ruiz and Ramis generalized the Ziglin approach replacing the monodromy group \( \mathcal{M} \) by the differential Galois group of NVE, [12]. Solutions of NVE define the an extension \( L \subset K \) of the coefficient field \( K \) of NVE. The group of all the differential automorphisms of \( L \) which leave fixed the elements of \( K \) defines the differential Galois group \( \text{Gal}(L/K) \) of \( L \) over \( K \) (or of equations NVE). Then the main result of Ziglin-Morales-Ramis theory is 0.1 (Morales-Ramis) Suppose that the Hamiltonian system (3) possesses \( n \) independent first integrals in involution.

Then the connected component \( \mathcal{G}^0 \) of the unit element of the Galois group \( \text{Gal}(L/K) \) is Abelian.

The opposite is not true in general, that is, if the connected component \( \mathcal{G}^0 \) of the unit element of the Galois group is Abelian it is not sure that the Hamiltonian system is integrable. This means that we need other obstruction to integrability. Already in [12] Morales-Ruiz suggested the quite natural conjecture that the higher Galois groups are also responsible for non-integrability. In [7] he announced this result and recently in a joint paper of Morales, Ramis and Simó [15] it was proved. Let us recall the corresponding notions and results.

Again we take a solution \( x(t) \) of the Hamiltonian system (3). We write the general solution as \( x(t,z) \), where \( z \) parametrizes the solutions near \( x(t) \), with \( z_0 \) corresponding to it. Then we write (3) as

\[
\dot{x}(t,z) = X_H(x(t,z))
\]

Denote the derivatives of \( x(t,z) \) with respect to \( z \) by \( x^{(1)}(t,z), x^{(2)}(t,z), \cdots \), etc. Let us differentiate the last equation with respect to \( z \) and evaluate it at \( z_0 \). The functions \( x^{(k)}(t,z) \) satisfy equations of the type

\[
\dot{x}^{(k)}(t,z) = X_H^{(k)}(x(t,z)) x^{(k)}(t,z) + P(x^{(1)}(t,z), \cdots, x^{(k-1)}(t,z))
\]
where \( P \) denotes polynomial term in the components of its arguments. The coefficients depend on \( t \) through \( X^{(j)}(x(t,z)), j < k \). One can show that the system of non-homogeneous equations for \( x^{(k)}(t,z) \), (4), can be arranged to a homogeneous linear system of higher dimension. These recently built systems define successive extensions of the main field \( K \) of the coefficients the NVE, i.e. we have \( K \subset L_1 \subset L_2 \cdots \subset L_k \) where \( L_1 \) is as above, \( L_2 \) is the extension obtained by adjoining the solutions of (4) for \( k = 2 \), etc. We can define the Galois groups \( \text{Gal}(L_n/K), \text{Gal}(L_j/K), \cdots \). Then the theorem from [15] asserts that 0.2 (Mora\( \)les-Ramis-Sim\( \)ó) If the Hamiltonian system (3) is integrable then for each of them being possibly zero. Then the matrix \( Q = \text{diag}\left(q_1/x, \ldots, q_n/x\right) \) with \( q_j \in \mathbb{C} \), not necessary distinct.

We now turn to the Galois group of equation (5) over the field of formal Laurent series \( \mathbb{C}((x)) \). FGG ([18, 20]) The formal differential Galois group of equation (5) over \( \mathbb{C}((x)) \) is the Zariski closure of the group generated by the formal monodromy and the exponential torus. FM ([17])

The formal monodromy matrix \( \gamma \in \text{GL}_n(\mathbb{C}) \) relative to \( Y(x) \) in (6) is defined by

\[
Y\left(xe^{2\pi i}\right)(x) = \gamma
\]

and the formal monodromy group is the closed subgroup of the corresponding Galois group topologically generated by \( \gamma \). The formal monodromy group is independent on the choice of the fundamental solution \( Y(x) \) and it is a formal invariant of the differential equation (5).

ET ([20]) The exponential torus \( T \) relative to the solution \( Y(x) \) (6) is the group of differential \( \mathbb{C}((x)) \)-automorphism \( T = \text{Gal}(E/\mathbb{C}((x))) \) where

\[
E = \mathbb{C}((x))(e^{\varphi_{1/x}}, \ldots, e^{\varphi_{n/x}})
\]

We may identify \( T \) with the subgroup of \( \left( \mathbb{C}^{*} \right)^{n} \). Next lemma gives the relationship between the formal monodromy \( \gamma \) and the exponential torus \( T \).

Now we turn to the Galois group over the field of convergent Laurent series \( \mathbb{C}((x)) \). The general theory of summability ensures that the matrix \( H(x) \) in (6) is multisummable along any non-singular ray \( d \). In the case when all non-zero polynomials \( q_j(1/x) - q_j(1/x) \) have the same degree 1 this means that \( H(x) \) is either convergent or 1-summable.

We need to recall some definitions and theoretical results. All angular directions and sectors are to be considered on the Riemann surface of the (natural) logarithm.

Section ([23]) 1. A sector is defined to be a set of the form

\[
S = S(d, \alpha, \rho) = \left\{ x = re^{\varphi} \mid 0 < r < \rho, d - \frac{\alpha}{2} < \varphi < d + \frac{\alpha}{2} \right\}
\]

where \( d \) is an arbitrary real number (bisetcing direction of \( S \)), \( \alpha \) is a positive real (the opening of \( S \)), and \( \rho \) is either a positive real number or \( +\infty \) (the radius of \( S \)).

2. A closed sector is a set of the form

\[
\overline{S} = \overline{S}(d, \alpha, \rho) = \left\{ x = re^{\varphi} \mid 0 < r < \rho, d - \frac{\alpha}{2} < \varphi < d + \frac{\alpha}{2} \right\}
\]
with $d$ and $\alpha$ as before, but $\rho$ is a positive real number (i.e. never equal to $+\infty$). A singular direction for equation (5) relative to $Y(x)$ in (6) is a bisecting ray of any maximal angular sector where

$$\text{Re} \left( \frac{d_i - d_j}{x} \right) < 0 \text{ for some } i, j = 1, \cdots, n.$$

Following Balser [23] we define a Gevrey function and a Gevrey series. Let $f = f(x)$ be analytic in some sector $S(d, \alpha, \rho)$ at $x = 0$. We say that $f$ is asymptotic to $f(x) = \sum f_n x^n \in \mathbb{C}[x]$ in Gevrey order $\alpha$ if for every closed subsector $\overline{S_1}$ of $S$ there exist positive constants $C_{\alpha,}, A_{\alpha} > 0$ such that for every non-negative integer $N$ and every $x \in \overline{S_1}$ one has

$$\left| f(x) - \sum_{n=0}^{N-1} f_n x^n \right| \leq C_{\alpha,} A^N x^N \mathbb{N}[x]^N$$

We, according to standard, denote by $A_{\alpha}(S)$ the ring of all Gevrey functions of order $\alpha$ in $S$.

Corresponding to the notion of a Gevrey function is the notion of a Gevrey series. Let $1$ A formal power series $f = \sum_{n \geq 0} f_n x^n$ is said to be of Gevrey order $\alpha$ if there exist two positive constants $C, A > 0$ such that

$$\left| f_n \right| \leq CA^nn! \text{ for every } n \in \mathbb{N}$$

We denote by $\mathbb{C}(\{x\})$ the set of all power series of Gevrey order $1$.

(Ramis) (22) 1. Let $f \in \mathbb{C}\{x\}_1$ such that there exists an open sector $V$ whose opening $> \pi$ and a holomorphic function $f \in A_{\alpha}(V)$ such that $f$ is asymptotic to $f$ on $V$ in Gevrey 1 sense. We will say that $f$ is 1-summable in the direction $d$ ( $d$ being the bisecting line of $V$) and $f$ is the 1-summ of $f$ in the direction $d$ . 2. If $f \in \mathbb{C}\{x\}_1$ is 1-summable in all but a finite number of directions, we will say that it is 1-summable. We will denote $f \in \mathbb{C}\{x\}_1$. This summability definition is very useful but it does not say how to compute the sum. Another definition of 1-summability is given in terms of Borel and Laplace transforms. In the next two definitions we follow Balser [23], van der Put [18] and Singer [20].

Borel The formal Borel transform $B$ of order 1 to a formal power series $f(t) = \sum_{n \geq 0} f_n t^n$ is called the formal series

$$B(\sum f_n x^n) = \sum_{n \geq 0} \frac{f_n}{n!} \xi^n$$

Then $f \in \mathbb{C}(\{x\})_1 \in \mathbb{C}(\{\xi\})_1 \Rightarrow B(f) \in \mathbb{C}(\{\xi\})$ (i.e. convergent) [20].

The inverse of the Borel transform is the Laplace transform: laplac Let $f$ be analytic and of exponential size 1, i.e. $\left| f(\xi) \right| \leq A \exp(B|\xi|)$ along the ray $r_d$ from 0 to 0 in direction $d$. Then the integral

$$\mathcal{L}_d f(x) = \int_{r_d} f(\zeta) \exp \left( -\frac{x}{\zeta} \right) d\left( \frac{\zeta}{x} \right)$$

is said to be the Laplace complex transform $\mathcal{L}_d$ of order 1 in the direction $d$ of $f$. The following proposition gives useful criteria for a Gevrey series of order 1 to be 1-summable crit(18]) Let $f \in \mathbb{C}(\{x\}_1)$ and let $d$ be a direction. Then the following are equivalent:

1. $f$ is 1-summable in the direction $d$.
2. The convergent power series $B(f)$ has an analytic continuation $h$ in a full sector

$$\{\xi \in \mathbb{C} | 0 < |\xi| < \infty, \arg(\xi) - d \}$$

In addition, this analytic continuation has exponential growth of order $\leq 1$ at $\infty$ on this sector, i.e. $|h(\xi)| \leq A \exp(B|\xi|)$. In this case, $f = \mathcal{L}_d(h)$ is its 1-sum. The good is that the two definitions $R$ and $\text{cri}$ are in fact equivalent [22].

To define the Stokes multipliers relative to the solution (6) we need the following fundamental result of Ramis sum(Ramis, [22]) Let $Ly = g$ be a linear non-homogeneous ordinary differential equation of order $n$ with polynomial coefficients and $g(x) \in \mathbb{C}\{x\}_1$. We suppose the Newton polygon on the origin admits only one strictly positive slope 1. If the series $f \in \mathbb{C}(\{x\}_1)$ is a formal power series solution of $Ly = g$ then $f$ is 1-summable or convergent. In this paper we will not apply the theory of the Newton polygon. We note that when $q_j ((1/x) - q_j/x, j = 1, \cdots, n \in \mathbb{C}$ the Newton polygon of equation (5) admits only one strictly positive slope 1. sum says that there exists a unique holomorphic function $f(x)$ in all but a finite number of directions, solution of the differential equation $Ly = g$ such that asymptotic to $f(x)$ in Gevrey-1 sense. Moreover $f(x)$ can be obtained from $f(x)$ by a Borel-Laplace transform.

Let us assume that the matrix $L$ in (6) is in a diagonal form, i.e. there are no logarithms in the solutions of equation (5). Due to our assumption the fundamental set of solutions of equation (5) is spanned by the functions

$$y_j(x) = x^{l_j} e^{\gamma_j x} y_j(x), \quad j = 1, \cdots, n$$

where $l_j, q_j \in \mathbb{C}$ and $y_j(x) \in \mathbb{C}\{x\}_1$. The formal series $\hat{y}_j(x)$ are “in general” divergent. It is well known fact that each of these divergent series $\hat{y}_j(x)$ is a solution of a linear homogeneous differential equation with polynomial coefficients

$$b_n(x) y_j^{(n+1)} + \cdots + b_1(x) y_j + b_0(x) y = 0$$

whose other solutions are $x^{l_j-j} \frac{d^{l_j-j}}{dx^{l_j-j}} y_j(x)$. Next, the series $\hat{y}_j(x), j = 1+n$ lie at the first row of the matrix $H(x)$. We remark that the behaviour of the elements of
the first row of the matrix \( H(x) \) (and from here of their derivations) is enough for our purpose. This is based on the important result that the set \( C^1(x)_{1,\delta} \) of all \( f(x) \) such that are 1-summable in a direction \( d \) is a differential algebra over \( C \) (see Balser [23], Chapter 3.3, Theorem 2). We will consider the equations (10) (not equation (5)) and we will apply sum to equation (10). In this way for any open sector \( V \) with vertex 0, with opening \( > \pi \) and bisected by non-singular direction \( d \) to any series \( \widehat{y}_j(x) \) we associate a unique holomorphic function \( f_j(x) \) solutions of the equation (10) and 1-sum of the corresponding series \( \widehat{y}_j(x) \) along \( d \) such that \( f_j(x) \) is asymptotic to \( \widehat{y}_j(x) \) in Gevrey-1 sense (R).

Replacing the series \( \widehat{y}_j(x) \) (and their derivations) in matrix \( H(x) \) by their 1-sums we obtain a holomorphic matrix \( H(x) \) on a sector with opening \( > \pi \), bisected by a non-singular direction \( d \), asymptotic to \( H(x) \) in Gevrey-1 sense on this sector and denotes the 1-sum of \( H(x) \). The new-obtained matrix \( Y(x) = H(x) x^s e^{Q(Y(x))} \) is an actual fundamental matrix of equation (5) and denotes the 1-sum of \( Y(x) \) along any nonsingular ray \( d \).

Further, following [18] and [20], relative to equations (5), (10) and to the solution (6) (the matrix \( Q(1/x) \)) we define:

1. Eigenvalues \( q_{ij} = q_{ij}/x \) of equation (10), where \( q_{ij}/x \) are the eigenvalues of equation (5);

2. A Stokes direction \( d \) for \( q_{ij} \) is a direction such that \( Re(q_{ij}) = 0 \);

3. Let \( d_1, d_2 \) be consecutive Stokes directions. We say that the pair \( (d_1, d_2) \) is a negative Stokes pair if \( Re(q_{ij}) < 0 \) for \( arg(x) \in (d_1, d_2) \);

4. A singular direction is the bisector of a negative Stokes pair.

sd1 Equations (10) have at least one zero eigenvalue, this corresponding to the series \( \widehat{y}_j(x) \). In this way the singular directions corresponding to the series \( \widehat{y}_j(x) \), from here to equations (5) and (10) are exact these defined by (4) (this defined more exactly the expression “for some \( i, j = 1, \ldots, n \)” in sd).

Let \( d \) be a singular direction for equations (10) at \( x = 0 \) and let \( d^+ = d + \varepsilon \) and \( d^- = d - \varepsilon \) where \( \varepsilon > 0 \) is small be two non-singular neighboring directions of \( d \). Let \( Y_{d^+} \) and \( Y_{d^-} \) denote the 1-sums of \( Y \) along \( d^+ \) and \( d^- \) respectively. st Wit0h respect to a given formal fundamental solution \( Y \) as in (6) the Stokes matrix (or multiplier) \( St_d \in GL_n(C) \) corresponding to the singular line \( d \) at \( x = 0 \) is defined by \( Y_{d^-} = Y_{d^+} : St_d \cdot R \) (Ramis) ([16]) With respect to a formal solution \( Y \) given as in (6) the analytic differential Galois group of equation (5) at 0 is the Zariski closure in \( GL_n(C) \) of the subgroup generated by the formal monodromy \( \gamma \), the exponential torus \( T \) and the Stokes matrices \( St_d \) for all singular rays \( d \). It is possible to generalize the above theorem to a global linear differential equation: Mi ([19]) The global Galois group \( G \) of equation (5) is topologically generated in \( GL_n(C) \) by the local Galois subgroup \( G_d \) where \( a \) runs over the set of singular points of (5).

2.3. \( P_v \) as a Hamiltonian System

The fifth Painlevé equation, \( P_v \), is equivalent to the Hamiltonian system, [25],

\[
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}
\]

with the Hamiltonian \( H \),

\[
H = \frac{1}{t}\left[ p(p+t)q(q-1) + \alpha_s q t - \alpha_s pq - \alpha_p (q-1) \right]
\]

In fact, putting

\[
\kappa_s = \alpha_s, \kappa_0 = \alpha_s, \theta = \alpha_s - \alpha_0 - 1
\]

we see that equation for \( y = 1 - \frac{1}{q} \) is nothing but \( P_v \) (2). The non-autonomous system (11) can be turned into autonomous one with two degrees of freedom by introducing new dynamic variables—\( t \) and the conjugate to \( t \)—\( F \). The new Hamiltonian becomes

\[
H = H + F
\]

The symplectic structure \( \omega \) is canonical in the variables \( (q,p,t,F) \), i.e. \( \omega = dp \wedge dq + dF \wedge dt \). In what follows we denote by \( s \) the time variable.

3. Non-Integrability of \( P_v \) for \( \kappa_s = 0, \kappa_0 = -\theta \)

In this section we begin studying \( P_v \) with the following values of the constants: \( \kappa_s = 0, \kappa_0 = -\theta \) where \( \theta \) is an arbitrary parameter. The results of this part are the base of the main theorems.

For these values of the constants the corresponding autonomous Hamiltonian system becomes

\[
\dot{q} = -\frac{1}{t}\left[ (2p+t)q(q-1) + \theta q \right]
\]

\[
\dot{p} = -\frac{1}{t}\left[ (2q-1)p(p+t) + (\theta+1)t + \theta p \right]
\]
i = 1
\[
\dot{F} = -\frac{1}{t} \left[ pq(q-1) + (\theta + 1)q \right] \\
+ \frac{1}{t} \left[ p(p + t)q(q-1) + (\theta + 1)qt + \theta pq \right]
\]

We choose as our non-equilibrium particular solutions
\[ q = 0, \quad p = -s, \quad t = s, \quad F = 0. \]

As we plan to compute the second variational equations it would be convenient to put
\[ q = 0 + \varepsilon q_1 + \varepsilon^2 q_2 + \ldots \]
\[ p = -s + \varepsilon p_1 + \varepsilon^2 p_2 + \ldots \]
\[ t = s + \varepsilon t_1 + \varepsilon^2 t_2 + \ldots \]
\[ F = 0 + \varepsilon F_1 + \varepsilon^2 F_2 + \ldots \]

It is easy to see that as a normal to the phase curves of these solutions we can pick the plane in the homogeneous equation corresponding to this equation.

Because of the equation \( i = 1 \) we can use \( t \) as an independent variable instead of \( s \).

For the first variational equations we obtain the system
\[
\dot{q}_1 = \left[ 1 + \frac{\theta}{t} \right] q_1, \quad \dot{p}_1 = -\left[ 1 + \frac{\theta}{t} \right] p_1
\]

This system can be solved by quadratures and
\[
\Phi(t) = \left[ t^\theta e^t, 0, t^{-\theta} e^{-t} \right]
\]

is a fundamental matrix solution. Thus the differential Galois group \( G \) of the first variational equation is Abelian, it is conjugated to the diagonal matrices
\[
G = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, \quad c \in \mathbb{C}^* \right\}
\]

and there is no obstruction to integrability. Next, for the second variational equations we obtain
\[
\dot{q} = \left[ 1 + \frac{\theta}{t} \right] q - q_1^2 - \frac{2}{t} p q_1 \\
\dot{p} = -\left[ 1 + \frac{\theta}{t} \right] p + \frac{1}{t} p_1 + 2 p_2 q_1
\]

where we have replaced \( q(t) = q_2(t), \quad p(t) = p_2(t) \).

As our approach uses the Stokes multipliers we find that it is more suitable to study the second variational equations as a scalar equation than a system. Further, as the equation for \( q(t) \) does not depend on \( p(t) \) it is enough for our purpose to consider only the scalar homogeneous equation corresponding to this equation.

Furthermore, we find our problem will get more simple if we write the equation for \( q(t) \) as a fourth order linear homogeneous equation, not as a third order (the natural method). Then the equations of \( q(t) \) will have a Galois group contained in \( G_{L_2}(\mathbb{C}) \) (not in \( G_{L_3}(\mathbb{C}) \)). This apparent complication preserves the non-commutativity of the unit element of the corresponding differential Galois group. The equation of \( q(t) \) can be written as a fourth order linear homogeneous equation
\[
L(q) = \frac{d^4 q}{dt^4} + \frac{5}{t^3} \frac{d^3 q}{dt^3} + \frac{8}{t^2} \frac{d^2 q}{dt^2} - \frac{2}{t} \left[ \frac{6}{t^2} + \frac{\theta(\theta - 1)}{t^3} + \frac{\theta(\theta - 1)(\theta + 1)}{t^4} \right] dq - \frac{2}{t^2} \left[ \frac{4 \theta}{t^2} + \frac{\theta(\theta + 1)}{t^3} \right] q = 0
\]

As it said from here on we will study the differential Galois group only of Equation (15).

Equation (15) has over two singular points - the point \( t = 0 \) is a regular singularity and the point \( t = \infty \) is an irregular singularity of rank one. Furthermore, this equation is reducible \( L = L_1 L_2 \) and it is solvable in terms of second order linear differential equation
\[
L_1(q) = f(t), \quad L_2 = \dot{\theta}^2 - \left[ 1 + \frac{\theta - 1}{t} \right] \dot{\theta} - \frac{1}{t}
\]

The second equation \( L_1(q) = 0 \) is
\[
L_1(q) = \dot{q} - \frac{2(\theta - 1)}{t} \dot{q} + \left[ \frac{4}{t^2} + \frac{2 \theta(\theta - 1)}{t^3} \right] q = 0
\]

and the system \( \{ t^{2\theta} e^{2t}, t^{2\theta-1} e^{2t} \} \) is a fundamental system of solution of (17).

Let us consider the homogeneous equation \( L_2(q) = 0 \). Changing the dependent variable
\[
q = y \exp \left[ \frac{1}{2} \int \left( 1 + \frac{\theta - 1}{t} \right) dt \right]
\]

we transform equation \( L_2(q) = 0 \) to the reduced form
\[
y - \left[ \frac{1}{4} + \frac{\theta + 1}{2t} + \frac{\theta^2 - 1}{4t^2} \right] y = 0
\]

Equation (18) is known as the Whittaker equation [26]
\[
y = \left( \frac{1}{4} - \frac{\kappa}{t} + \frac{4\mu^2 - 1}{4t^2} \right) y = 0
\]

with parameters \( \kappa = -\frac{\theta + 1}{2} \) and \( \mu = \frac{\theta}{2} \). Then we have witt The identity component of the differential Galois group of equation (18) is Abelian if and only if \( \theta \in \mathbb{N}^- \).

It is well known from the work of Martinet and Ramis [16] (see [12] too) that the identity component \( G^0 \) of the differential Galois group of the Whittaker equation is Abelian if, and only if, \( \left( \kappa + \mu - \frac{1}{2}, \kappa - \mu - \frac{1}{2} \right) \) belong to \( \mathbb{N} \times \mathbb{N}^- \) or \( \mathbb{N} \times \mathbb{N} \). In our case \( \kappa + \mu - \frac{1}{2} = -1, \kappa - \mu - \frac{1}{2} = -1 - \theta. \)

Hence the identity component \( G^0 \) of the differential Galois group of equation (18) is Abelian if and only if, \( -1 - \theta \in \mathbb{N} \). This proves the lemma.

We finish with some notes. The fundamental set of solutions of equation (18) is spanned by the set

\[
\left\{ y_1 = t^{\frac{\theta + 1}{2}} e^{\frac{\theta}{2}}, y_2 = t^{\frac{\theta - 1}{2}} e^{-\frac{\theta}{2}} \right\},
\]

i.e. only one of the Stokes multipliers of the Whittaker equation (18) is different from zero when \( \theta \in \mathbb{N}^- \).

Observe that the identity component \( G^0 \), as well the Galois group \( G \) of equation (18) is a subgroup of \( \text{SL}_2(\mathbb{C}) \). In particular, when \( \theta \in \mathbb{N}^- \)

\[
G = G^0 = \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \in \mathbb{C}^*, \mu \in \mathbb{C}
\]

and when \( \theta \in \mathbb{N}^- \)

\[
G = G^0 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \in \mathbb{C}^*
\]

Equation \( L_2(q) = 0 \) has a solution space that is spanned by functions \( q_1 = t^\theta e^t, q_2 = \int_0^t x^{-\theta - 1} e^{-xt} dx \).

Therefore the identity component of its Galois group is not a subgroup of \( \text{SL}_2(\mathbb{C}) \), i.e. equations \( L_2(q) = 0 \) and (18) have different identity components of the corresponding Galois groups. As in equation (18) only one of the Stokes multipliers of equation \( L_2(q) = 0 \) is different from zero when \( \theta \in \mathbb{N}^- \). Furthermore, the identity components of the Galois group of both of the equations \( L_2(q) = 0 \) and (18) are Abelian (resp. not Abelian) in the same way, as both of them depend on the same integral \( \int_0^t x^{-\theta - 1} e^{-xt} dx \). In particular, when \( \theta \in \mathbb{N}^- \) for the Galois group of equation \( L_2(q) = 0 \) we have

\[
G = G^0 = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \lambda \in \mathbb{C}^*
\]

When \( \theta \in \mathbb{N}^- \) to define the identity component \( G^0 \) of the Galois group is a complicated task. But even in the worst case when

\[
G^0 = \begin{pmatrix} \lambda & \mu \\ 0 & 1 \end{pmatrix}, \lambda \in \mathbb{C}^*, \mu \in \mathbb{C}
\]

\( G^0 \) is not Abelian group. However, as witt gives us necessary and sufficient condition for Abelian differential Galois group of equation (18) (and from above remark of equation \( L_2(q) = 0 \)), thus as a corollary we have the following theorem. witt Assume that \( \kappa = 0, \kappa_0 = -\theta \in \mathbb{N}^- \). Then the Painlevé \( V \) equation (2) is not integrable in the Liouville sense.

Due to the reducibility of \( L, L = L_1L_2 \), with equation (15) one can associate a matrix equation

\[
\begin{pmatrix} \hat{Q} + A_1 & C \\
0 & A_1 \end{pmatrix} = 0
\]

where \( Q = (Q_1, Q_2) \), the matrix equation \( \hat{Q}_1 + A_1Q_1 = 0 \) is completely reducible and \( C \) may be taken to be the matrix

\[
C = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}
\]

In this way the matrix equation \( Q'_1 + A_1Q_1 = 0 \) and the matrix \( A_1 \) may be taken to be the corresponding matrix equation and the corresponding matrix to the equation (17) in the standard sense. Namely, if we put \( q' = q, q'' = \hat{q} \) then a function \( q \) is a solution of the scalar equation (17) if and only if the column vector \( Q = (q', q'') \) is a solution of the following matrix equation

\[
Q'_1 + A_1Q_1 = 0
\]

\[
A_1 = \begin{pmatrix} 0 & -1 \\ 4 + \frac{4(2\theta - 1)}{t} + 2\theta(2\theta - 1) - 4 & -1 \\
-1 & -1 - \frac{2(2\theta - 1)}{t} - 4 \end{pmatrix}
\]

We will not fix on the equation \( \hat{Q}_1 + A_1Q_1 = 0 \) but will note that its differential Galois group is Abelian. Hence, there is no obstruction to integrability. Note that the equation \( \hat{Q}_2 + A_1Q_2 = 0 \) and the matrix \( A_2 \) may be taken to be the corresponding equation and the corresponding matrix to equation \( L_2(q) = 0 \) in the standard sense, i.e.

\[
A_2 = \begin{pmatrix} 0 & -1 \\ -1 & -1 - \frac{2(2\theta - 1)}{t} \end{pmatrix}
\]

Next, the matrix
\[
Q = \begin{pmatrix}
Q_2 & UQ_1 \\
0 & Q_1
\end{pmatrix}
\]

(20)
is a fundamental matrix solution of equation (19) if and only if \( U \) satisfies \( \dot{U} + (A_U - UA_U) = -C \) [27]. Furthermore, the matrices \( Q_2 \) and \( Q_1 \) in (20) are the fundamental matrix solutions of equations \( Q_2 + A_U Q_1 = 0 \) and \( Q_1 + A_U Q_2 = 0 \) respectively.

Now, it is easy to see that the identity component of the differential Galois group of the equation (15) is not Abelian for \( \theta \notin \mathbb{N}^* \). Indeed, let us denote \( G \) to be the differential Galois group of equation (19). Then we have that \( \sigma(Q), \sigma \in G \) is again a fundamental matrix solution of equation (19), and a calculation shows that

\[
(Q^{-1} \sigma(Q))' = 0.
\]

Therefore \( \sigma(Q) = QR(\sigma) \) for some \( R(\sigma) \in GL_4(\mathbb{C}) \). Expressing any such \( R(\sigma) \) in block notation,

\[
R(\sigma) = \begin{pmatrix} G_2 & G_3 \\ G_4 & G_1 \end{pmatrix}
\]

let us write the equation \( \sigma(Q) = QR(\sigma) \) explicitly

\[
\sigma(Q) = \begin{pmatrix} Q_2 & UQ_1 \\ 0 & Q_1 \end{pmatrix} \begin{pmatrix} G_2 & G_3 \\ G_4 & G_1 \end{pmatrix}
\]

\[
= \begin{pmatrix} Q_2G_2 + UQ_1G_4 & Q_2G_3 + UQ_1G_1 \\ Q_4G_2 & Q_4G_1 \end{pmatrix}
\]

\[
= \begin{pmatrix} \sigma(Q_2) & \sigma(UQ_1) \\ \sigma(0) & \sigma(Q_1) \end{pmatrix}
\]

The last two equations imply \( G_4 = 0 \) and \( \sigma(Q) = QG_1 \). Hence one can identify the matrix \( G_1 \) with the representation of the differential Galois group of equation (17) in \( GL_2(\mathbb{C}) \). Next, from \( G_4 = 0 \) follows that \( \sigma(Q_2) = QG_2 \) and one can identify the matrix \( G_2 \in GL_2(\mathbb{C}) \) with the differential Galois group of equation \( L_2(q) = 0 \). Now from Witt and Re it results that for \( \theta \notin \mathbb{N}^* \) the identity component \( G_2^0 \) of the Galois group \( G_2 \) is not commutative and as a corollary the identity component of the differential Galois group of equation (21) is not Abelian. Thus from the Morales-Ramis-Simó theorem [15] and why the corresponding Hamiltonian system is not integrable. This proves witt.

3.1. Non-Integrability for \( \theta = -1 \)

To prove non-integrability for \( \theta \notin \mathbb{N}^* \) we will study the matrix \( UQ_1 \) in the matrix solution (20). Let for simplicity \( \theta = -1 \). In the last section from this particular case, \( \theta = -1 \), we will extend the present results over \( \theta \notin \mathbb{N}^* \) by Bäcklund transformations.

As we are going to apply the theory of irregular points at \( t = 0 \) we change the dependent variable \( t = -1/z \). Then for \( \theta = -1 \) equations (15), \( L_2(q) = 0 \) and (17) become

\[
L(q) = \frac{d^2q}{dz^2} + \left[ \frac{4}{z} - \frac{5}{z^2} \right] \frac{d^4q}{dz^4} + \left[ \frac{2}{z^2} + \frac{3}{z^3} + \frac{8}{z^4} \right] \frac{d^4q}{dz^4} = 0
\]

(21)

\[
L_2(q) = q^3 - \frac{1}{z^2} q' + \frac{1}{z^2} q = 0,
\]

(22)

\[
L_1(q) = q^3 + \frac{4}{z} q' + \frac{4}{z^2} q = 0
\]

(23)

Equations (22) and (23) have solution spaces spanned by the sets \( \{ z, z e^{-\psi} \} \) and \( \{ (2 z^{-2} + z^{-1}) e^{-\psi}, (z^{-2} + z^{-1}) e^{-\psi} \} \) respectively. As a formal fundamental set of solutions around the irregular singularity at 0 of the equation (21) we can take the formal fundamental set of solutions \( \{ q_1 = z e^{-\psi}, q_2 = z \} \) of equation \( L_2(q) = 0 \) and

\[
q_3 = z e^{-\frac{1}{2}} \varphi(z), \quad q_4 = z e^{-\psi}(z)
\]

where \( \varphi(z) \) and \( \psi(z) \) are two formal series

\[
\varphi(z) = z - z^2 + 2 z^3 - 3 z^4 + \ldots = \sum_{n=0}^{\infty} (-1)^n n! z^{n+1}
\]

\[
\psi(z) = \frac{1}{2} z - \frac{1}{2} z^2 - 2 z^3 + 2 z^4 - 3 z^5 + \ldots
\]

\[
= \sum_{n=0}^{\infty} (-1)^n n! \frac{z^{n+1}}{2^{n+1} z^{n+1}}
\]

Then a formal fundamental matrix of equation (21) is

\[
Q = H(z) z^t e^{R(\psi)z}
\]

(24)

where

\[
R = \text{diag}(-\frac{1}{z}, 0, -\frac{2}{z}, \frac{2}{z}) \quad J = \text{diag}(1,1,-2,-2)
\]

(25)

and

\[
H(z) = \begin{pmatrix}
1 & 1 & z \varphi(z) & z \psi(z) \\
\frac{1}{z} + \frac{1}{z^2} & \frac{1}{z} & h_{23} & h_{24} \\
0 & 0 & 2 + z & 1 + z \\
0 & 0 & -1 - 2 z + \frac{4}{z} + \frac{2}{z^2} & -1 + \frac{2}{z^2}
\end{pmatrix}
\]

(26)

The precise form of \( h_{23} \) and \( h_{24} \) are not important for our purpose but we do note that they are elements of
let \( \mathbb{C}(z) \). We remark that the formal series \( \varphi(z) \) and \( \psi(z) \) are the result of formal operations that have nothing to do with sectors.

We now turn to the Galois group of equation (5) over \( \mathbb{C}(z) \). The formal Galois group over \( \mathbb{C}(z) \) is the Zariski closure of the group generated by the formal monodromy \( \gamma \) and the exponential torus \( T \). The last ones on the other hand represent differential automorphisms of the extension \( F \) of the field \( \mathbb{C}(z) \) by the linear matrix \( \mathbb{C}(z)z^r e^{R(z)} \) over \( \mathbb{C}(z) \); \( F = \mathbb{C}(z) \left( e^{1/z}, e^{2/z} \right) \) in our case. Furthermore, the formal monodromy \( \gamma \) is trivial. Therefore \( \text{Gal}(F/\mathbb{C}(z)) \) is equal to the exponential torus \( T \)

\[
\text{Gal}(F/\mathbb{C}(z)) = T = \left\{ \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^2 & 0 \\ 0 & 0 & 0 & e^2 \end{pmatrix} \mid c \in \mathbb{C}^* \right\} \quad (27)
\]

Let now \( F \) be the Picard-Vessiot extension of \( \mathbb{C}(z) \) for the equation (21). To determine the convergent Galois group at \( z = 0 \) over \( \mathbb{C}(z) \) we will need to compute the Stokes matrices at \( z = 0 \) relative to the solution \( Q(z) \), (24).

Let us focus now on \( \varphi(z) = \sum_{n=0}^{\infty} a_n z^{n+1} \) and \( \psi(z) = \sum_{n=0}^{\infty} b_n z^{n+1} \). The first series \( \varphi(z) = \sum_{n=0}^{\infty} n! z^{n+1} \) is the so called Euler series, the second is a modified Euler series. The properties of the Euler series and the corresponding Stokes phenomenon are well studied, for example in a paper of Ramis [22] and in a paper of Singer [20]. In the next following two papers we will compute the Stokes matrix relative to the series \( \varphi(z) \) and \( \psi(z) \).

These series are divergent and they obviously satisfy a Gevrey-1 growth condition. Then we can consider the formal Borel transforms of \( \varphi(z) \) and \( \psi(z) \)

\[
\varphi(\zeta) = B\varphi(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n} \varphi^{n+1}_{n+1}}{n+1} = \log(1+\zeta)
\]

\[
\psi(\zeta) = B\psi(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n} \varphi^{n+1}_{n+1}}{2n+1} = \log(2+\zeta) - \ln 2
\]

The Gevrey-1 growth condition ensures that \( \varphi(\zeta) \) and \( \psi(\zeta) \) are analytic in the neighborhood of the origin of the \( \zeta \) plane. For any \( d \neq 1 \) the functions \( \log(1+\zeta) \) and \( \log(2+\zeta) \) have analytic continuations along \( d \). For such a ray their Laplace transforms (see Example 4.2.22 in [20])

\[
\varphi_d(z) = \mathcal{L}_d \left( \log(1+\zeta) \right)(z)
\]

\[
= \int_d \log(1+\zeta) e^{-\zeta} d\zeta = \int_d \frac{1}{1+\zeta} e^{-\zeta} d\zeta = \int_d \frac{1}{1+\zeta} e^{-\zeta} d\zeta
\]

\[
\psi_d(z) = \mathcal{L}_d \left( \log(2+\zeta) - \ln 2 \right)(z)
\]

\[
= \int_d \log(2+\zeta) e^{-\zeta} d\zeta = \int_d \frac{2}{2+\zeta} e^{-\zeta} d\zeta
\]

define the corresponding 1-sum of the series \( \varphi(z) \) and \( \psi(z) \) in the direction \( d \). We note that functions

\[
q_1(z) = ze^{\frac{z}{2}} \varphi(z) \quad \text{and} \quad q_2(z) = ze^{\frac{z}{2}} \psi_d(z)
\]

again satisfy equation (21). Indeed, one can complete the set \( \{ q_1, q_2 \} \) to the fundamental set of solutions of equation (21) by the particular solutions of equations (21) the following non homogeneous differential equations with polynomial coefficients

\[
z^4 \varphi' + (2z^3 + 3z^2) \varphi + 2\varphi = 2z + z^2
\]

\[
z^4 \psi' + (2z^3 + 3z^2) \psi' + 2\psi = z + z^2
\]

with unique formal solutions

\[
\varphi(z) = \sum_{n=0}^{\infty} (-1)^n n! z^{n+1} \quad \text{and} \quad \psi(z) = \sum_{n=0}^{\infty} (-1)^n n! z^{n+1}
\]

respectively. Further, solving equations (29) by the variation of constants we get a particular solution

\[
\varphi(z) = e^{z/2} \int_0^z \frac{e^{-t/2}}{x} dx
\]

of the first equation and a particular solution

\[
\psi(z) = e^{z/2} \int_0^z \frac{e^{-t/2}}{u} du
\]

of the second equation, where for convenience the integrals are taken in the direction \( \mathbb{R}^+ \). Next, let

\[
\varphi(z) = \int_0^z \frac{e^{-t/2}}{x} dx
\]

define a new variable \( \zeta \) by

\[
-\zeta/z = -1/1 + 1/z \quad \text{gives} \quad \varphi(z) = \int_0^z e^{-\zeta/2} \frac{1}{1+\zeta} d\zeta
\]

In the same manner let \( \psi(z) = \int_0^z e^{2z+1/2} du \) and setting
\[ -\zeta/z = -2u + 2/z \] gives \( \psi(z) = \int_0^z \frac{e^{-\zeta z}}{2 + \zeta} \, dz \). More general, these integrals exist if \( d \neq \mathbb{R}^+ \). Therefore, applying to equations (29) Ramis’ sum, for any ray \( d \) except the negative real axis, we are able to canonically associate unique functions \( \varphi_d(z) \) and \( \psi_d(z) \), analytic in a large sector around this ray, with the divergent series

\[ \sum (-1)^n n!z^{n+1} \quad \text{and} \quad \sum (-1)^n n! z^{n+1} \]

respectively, and so these series are 1-summable and \( \varphi_d(z) \) and \( \psi_d(z) \) are their 1-sum. Furthermore, the functions

\[ q_i(z) = \frac{z}{e^{i\varphi_d(z)}} \quad \text{and} \quad q_i(z) = \frac{z}{e^{i\psi_d(z)}} \]

will satisfy equations (28) respectively and from here the equation (21). In this way the matrix

\[ Q(z) = H(z)z^\prime e^{\theta(z)} \]

with \( z^\prime \) and \( e^{\theta(z)} \) as before but

\[ H(z) = \begin{pmatrix}
1 & 1 & \frac{1}{z} & \frac{1}{z^2} & \frac{1}{z^3} & h_{23} & h_{24} \\
0 & 0 & 2 + z & 2 + z & 2 + z & 4 + z & 4 + z \\
0 & 0 & -1 + 2 + z & -1 + 2 + z & -1 + 2 + z & -1 + 2 + z & -1 + 2 + z
\end{pmatrix} \]

where \( h_{23} = z^3 \left[ \frac{1}{z} + \frac{2}{z^2} \right] \varphi_d(z) + \frac{d\varphi_d(z)}{dz} \)

and

\[ h_{24} = z^3 \left[ \frac{1}{z} + \frac{2}{z^2} \right] \psi_d(z) + \frac{d\psi_d(z)}{dz} \]

is an actual fundamental matrix of equation (21). The matrix \( H(z) \) is holomorphic in an open angular sector \( \pi \) at \( z = 0 \) of opening angle \( 2\pi \) \((-\pi < \arg z < \pi)\) and \( H(z) \) is asymptotic to \( H(z) \) (26) on this sector in Gevrey-1 sense.

We are now in a position to describe the analytic elements which, together with the formal Galois group (27) determine the analytic Galois group of equation (21). As the Stokes multipliers depend on the functions \( \varphi(z) \) and \( \psi(z) \) (resp. on \( \varphi_d(z) \) and \( \psi_d(z) \)) we consider the following homogeneous ODE with polynomial coefficients obtained from equation (21) by changing \( q(z) \) to \( \varphi(z) = z^{-1}e^{3/z}q(z) \)

\[ z^6q^{(4)}(z)+(8z^3+3z^4)q''(z)+(14z^3+6z^3+2z^3)q''(z)+(4z^3-4z)q'(z)+4q(z) = 0 \]

(\#)

The functions \( \varphi(z) \) and \( \psi(z) \) are their formal solutions. The other entries of the fundamental set of solutions of equation (\#) are \( e^{z^3} \) and \( e^{2z^3} \). So the eigenvalues of this equation are \( \left\{ 0, 0, 1, 2 \right\} \).

Now we are almost ready to compute the Stokes constants corresponding to the singular direction \( \mathbb{R}^+ \) \((d = \pi)\). Relative to equation (\#) we define:

- eigenvalues of equation (\#): \( \left\{ 0, 0, 1, 2 \right\} \);
- the Stokes direction \( d = \frac{\pi}{2} \); such that

\[ \text{Re} \left( \frac{2}{z} \right) > 0 \]

- the singular direction \( d = \pi \) as the bisector of the negative Stokes pair \( \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \).

We select two directions \( d_+ \) and \( d_- \) which are respectively slightly to the left and to the right of the critical direction \( d = \pi = \mathbb{R}^+ \). Let

\[ \varphi_d = \int_{d_+ \frac{1}{1+\zeta}} e^{\zeta} d\zeta \quad \text{and} \quad \varphi_d = \int_{d_- \frac{1}{1+\zeta}} e^{\zeta} d\zeta \]

\[ \psi_d = \int_{d_+ \frac{1}{2+\zeta}} e^{\zeta} d\zeta \quad \text{and} \quad \psi_d = \int_{d_- \frac{1}{2+\zeta}} e^{\zeta} d\zeta \]

be the associated Laplace transforms of \( \varphi \) and \( \psi \) of the directions \( d_+ \) and \( d_- \). The difference between them amounts to integrating on a path coming form infinity along the critical line on the right till the origin and then doing to infinity by following the critical line on the left. As there is no singularity between 0 and \(-1\), and no other between \(-1\) and \(\infty\), Cauchy’s formula immediately implies that the difference \( \varphi_d - \varphi_d \) is given by \( 2\pi i \text{Res}_{\zeta = \pi} \left( e^{-\zeta^2}/(1 + \zeta) \right) = 2\pi i \). In the same manner, the difference between \( \psi_d \) and \( \psi_d \) is given by \( 2\pi i \text{Res}_{\zeta = \pi} \left( e^{-\zeta^2}/(2 + \zeta) \right) = 2\pi i \). In the same manner, the Stokes matrix \( S_{\pi} \) is the Stokes matrix in the direction \( d = \pi \). Furthermore, \( q_i - q_i = (\varphi_d - \varphi_d) e^{-2\zeta^2} = 2\pi i e^{-2\zeta^2} = 2\pi i q_i \). Then for the Stokes matrix \( S_{\pi} \) we have \( (S_{\pi})_{ij} = 2\pi i \). In the same manner,
\[ q_i^* - q_i = (\psi_j^* - \psi_i^*) z e^{-2i/3} = 2\pi i z = 2\pi i (z) \]. Then we have that \((s_{t_k})_{z_k} = 2\pi i\).

From the above reasoning for the Stokes matrix \(S_t\), we obtain

\[
S_t = \begin{pmatrix}
1 & 0 & 2\pi i & 0 \\
0 & 1 & 0 & 2\pi i \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(30)

Thus we have n1. Assume that \(\kappa_n = 0, \kappa_0 = 1, \theta = -1\). Then the Painlevé-V equation (2) is not integrable in the Liouville sense.

By a theorem of Schlesinger [28] the local differential Galois group of equation (21) at infinity is generated topologically by the monodromy group at \(\infty\). We note that the actual monodromy around \(\infty\) and around 0 of equation (21) are the same. Therefore the local differential Galois group of equation (21) at infinity can be interpreted as a subgroup of the local differential Galois group at the origin. Next, observe that the differential Galois group of equation (21) is a connected group. As the formal monodromy \(\gamma\) is trivial then by the Ramis’ theorem the Galois group is topologically generated by the exponential torus \(T\) and the Stokes matrix \(S_t\).

The Zariski closure of the subgroup \(T, n \in \mathbb{Z}\) is the same \(T\) and the elements \(f_c\) of \(T\) is

\[
f_c = \begin{pmatrix}
c & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & c & 0 \\
0 & 0 & 0 & c^2
\end{pmatrix}
\]

where \(c \in \mathbb{C}^*\).

The matrix \((S_{t_k})^n, n \in \mathbb{Z}\) is of the kind \(I + nX\) where \(I = Id\) and \(X\) is a unipotent matrix. Denote by \(S_\lambda\) the Zariski closure of the subgroup \((S_{t_k})^n, n \in \mathbb{Z}\) then the elements \(s_{n,\lambda}\) of \(S_\lambda\) are

\[
s_{n,\lambda} = \begin{pmatrix}
1 & 0 & 2\pi i \lambda & 0 \\
0 & 1 & 0 & 2\pi i \lambda \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

where \(\lambda \in \mathbb{C}\).

When \(\lambda\) is different from zero and \(c \neq \pm 1\) the commutator of \(f_c\) and \(s_{n,\lambda}\) is

\[
f_c s_{n,\lambda} f_c^{-1} s_{n,\lambda}^{-1} = \begin{pmatrix}
1 & 0 & 2\pi i \lambda (c^{-1} - 1) & 0 \\
0 & 1 & 0 & 2\pi i \lambda (c^{-2} - 1) \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(31)

which is not identically equal to \(Id\).

The case \(c = 1\) implies that \(\sigma(e^{-1/3}) = e^{-1/3}\) for any \(\sigma \in \text{Gal}(L(q) = 0)\), i.e. \(e^{-1/3} \in \mathbb{C}(z)\), which is an obviously contradiction. In the same manner \(c = -1\) implies that \(e^{1/3}\) is invariant under any morphism \(\sigma \in \text{Gal}(L(q) = 0)\), i.e. \(e^{-1/3} \in \mathbb{C}(z)\). From these remarks, formula (31) and the fact that \(\text{Gal}(L(q) = 0)\) is a connected group follow that the identity component of the differential Galois group of equation (21) is not Abelian. Thus from why and 0.2 we have that for \(\kappa_n = 0, \kappa_0 = 1, \theta = -1\) the corresponding Hamiltonian system does not possess two meromorphic first integrals. This proves n1. galois We will not determine precisely the differential Galois group of equation (21). But we can say that this group is not finite and it coincides with its connected component \(C^0\) of the unit element. Furthermore the differential Galois group of equation (21) is conjugate to the following matrices

\[
\text{Gal}(L(q) = 0) = \begin{pmatrix}
c & 0 & d & 0 \\
0 & 1 & 0 & a \\
0 & 0 & c & 0 \\
0 & 0 & 0 & c^2
\end{pmatrix}
\]

(32)

4. Generalization

In this paragraph we will extend the results of the previous section to the entire orbits of the parameters using the Bäcklund transformations of the Painlevé fifth equation, given by the following list of restriction of this group on the parameter space, [29]

\[
s_j(\alpha_i) = -\alpha_i, \quad s_i(\alpha_j) = \alpha_j + \alpha_i (j = i \pm 1),
\]

\[
s_j(\alpha_j) = \alpha_j, (j \neq i, i \pm 1)
\]

(33)

\[
\pi(\alpha_i) = \alpha_{i+1}, \quad \sigma(\alpha_0) = \alpha_0, \quad \sigma(\alpha_i) = \alpha_i, \\
\sigma(\alpha_2) = \alpha_2, \quad \sigma(\alpha_3) = \alpha_3
\]

We note that the Bäcklund transformations group of the fifth Painlevé equation is isomorphic to the extended affine Weyl group of \(A^{(1)}_n\) type, [5]. It is well known (see Okamoto [5]) that the group of Bäcklund transformations of the \(P_n\) equation is represented as the group of birational canonical transformations of the Painlevé system (that is the corresponding non-auto-

nomous Hamiltonian system) associated with the fifth Painlevé equation. In particular the Bäcklund transformations remain the property non-integrability.

We define (following Masuda et al. [29]) the translation operators \(T_i\) \((i = 0, 1, 2, 3)\) by

\[
T_i = \pi s_3 s_2 s_1, T_2 = s_1 \pi s_3 s_2, T_3 = s_2 s_1 \pi s_3, T_0 = s_1 s_2 s_1 \pi
\]

(34)
These operators acts on parameters $\boldsymbol{\alpha}_i$ as

$$T_i (\boldsymbol{\alpha}_{i-1}) = \alpha_{i-1} + 1, T_i (\alpha_j) = \alpha_j - 1. T_i (\alpha_j) = \alpha_j (j \neq i - 1, i)$$

4.1. Generalization of the Results of the Paragraph 3

In n1 we have proved that for $\kappa_e = 0, \kappa_0 = -\theta = 1$ the Painlevé $V$ equation (2) is not integrable. Let us recall that these values of the parameters are particular case of the family $\kappa_e = 0, \kappa_0 = -\theta$ taken at $\theta = -1$. In the following proposition using the operators $T_j$, (34), (35), we will show that the result of n1 can be extended for $\theta \in \mathbb{N}^*$. p1 Assume that $\kappa_e = 0, \kappa_0 = -\theta = m$ where $m \in \mathbb{N}^*$. Then the Painlevé $V$ equation (2) is not integrable.

We will prove the statement by building the appropriate transformations which extend the initial parameter family $\kappa_e = 0, \kappa_0 = -\theta = 1$ or $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (0, 0, 0, 1)$ to this of the proposition.

The Bäcklund transformation $T_2 T_1 T_0$ (starting from $T_0$) maps the parameter family $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ to $(\alpha_0, \alpha_1, \alpha_2 - 1, \alpha_3 + 1)$. Now, applying $m - 1$ times $T_2 T_1 T_0$ to the initial parameter family we obtain $(\alpha_0, \alpha_1, \alpha_2 - m + 1, \alpha_3 + m - 1) = (0, 0, 1 - m, m)$. If we recall that $\theta = \alpha_3 - \alpha_0 - 1$ then we obtain that $\theta = -m$.

The proof follows from the fact that the Bäcklund transformations are birational canonical transformations [5] and n1.

As a corollary from Witt and p1 we have the following generic non-integrable result: g1 Assume that $\kappa_e = 0, \kappa_0 = -\theta$ where $\theta$ is an arbitrary complex parameter. Then the Painlevé $V$ equation (2) is not integrable.

The next lemma describes the orbit of the vector $\alpha^0 = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ as $0, 0, 1, \theta, -\theta$ under the Bäcklund transformation group of the fifth Painlevé equation. orbit Let $(q, p) = (0, -i)$ be a rational solution of the Hamiltonian system (11), (12) with parameters $\alpha^0 = (0, 0, 1, \theta, -\theta)$. Then beginning with $\alpha^0$ by the Bäcklund transformations (33) we obtain a rational solution of (11), (12) with new $\alpha_j = 0, 1, 2, 3$ as at least two of them are integer and at least one of these integer is $\alpha_j$. Furthermore the parameters satisfy either $\alpha_i + \alpha_j + \alpha_k - 1, 2Z$ or $\alpha_0 + \alpha_1 + \alpha_2 - 1, 2Z$. relations.

Let $(\alpha') = (\alpha'_0, \alpha'_1, \alpha'_2, \alpha'_3)$ be the vector of parameters obtained by $i$ successive transformations $s_i, \pi, \sigma$ from $\alpha^0$. We will prove the statement inductively. At first for $\alpha^0$ the statement is true.

Let $i = 1, \ i.e. \ we \ have \ applied \ on \ \alpha^0$ some of the transformations (33). Under $s_0$ and $s_1$ the vector $\alpha^0$ does not change, i.e. $\alpha^i = \alpha^0$ and the statement is true. Let us denote $S^0 = \alpha'_0 + \alpha'_1 + \alpha'_2 - \alpha'_0 - 1$ and $S^1 = \alpha'_0 + \alpha'_1 + \alpha'_2 - \alpha'_0 - 1.2$. Under $s_2, \alpha^0$ becomes $\alpha^i = (0, 1, 0, -1, \theta, 1, 0)$ and $S^i = 0 \in 2Z$. Under $s_2, \alpha^0$ becomes $\alpha^i = (-\theta, 0, 1, \theta, 0)$ and $S^i = 0 \in 2Z$. Under $\pi, \alpha^0$ becomes $\alpha^i = \alpha^i = (0, 1, -1, \theta, 0)$ and $S^i = 0 \in 2Z$. Hence for $i = 1$ the statement is true.

Suppose that the statement is true for $i$. We will prove that it is true for $i + 1$. Let us recall (14), that is

$$A = \alpha^0 + \alpha^i + \alpha^j + \alpha^k = 1 $$

for every $i$

Observe that the conditions $S^i \in 2Z, A_i$ imply that $\alpha^0 \in 2Z$. In the same manner $S^i \in 2Z, A_i$ imply that $\alpha^0 \in 2Z$. Under $s_0$, the vector $\alpha'$ becomes $\alpha^i = 0, \alpha^i = 0, \alpha^i = 0, \alpha^i = 0$ and for the sums $S^i$ and $S^i$ we obtain: $S^i = 2a_0, S^i = 2 - 2a_0$. So if $S^i \in 2Z$ then $S^i \in 2Z$ and if $S^i \in 2Z$ then $S^i \in 2Z$.

Next, if $\alpha^i \in Z, S^i \in 2Z \{ \alpha^i \in Z, S^i \in 2Z \}$ then $\alpha^i \in Z \{ \alpha^i \in Z, S^i \in 2Z \}$. Similarly, if $\alpha^i \in Z, S^i \in 2Z \{ \alpha^i \in Z, S^i \in 2Z \}$ then $\alpha^i \in Z$

from $A \{ \alpha^i \in Z, S^i \in 2Z \}$. Hence the statement is true.

We leave the proof of the statement for $i + 1$ applying on $\alpha'$ the rest of the transformations of (33) as an easy exercise similar to the case of $s_0$.

The following corollary, by virtue of orbit turns out to be a natural generalization of g1. g2 For values of the parameters satisfying orbit the Painlevé $V$ equation (2) is not integrable.

The proof follows from orbit, g1 and the well-known fact that the Bäcklund transformations are birational canonical transformations on $q, p$ and $t$ [5].

If we recall that $\theta = \alpha_2 - \alpha_0 - 1, \kappa_e = \alpha_1, \kappa_0 = \alpha_1$, (13), then $\alpha_2 - \alpha_0 - 1, \alpha_1, \alpha_2$ show how the initial $\theta, \kappa_e, \kappa_0$ change under the Bäcklund transformation group (33). Hence as a corollary we obtain: M Assume that $\kappa_e + \kappa_0 = \pm \theta + m$ where $m$ is even and at least one $\kappa_j \in Z, j = \infty, 0$. Then the fifth Painlevé equation (2) is not integrable.

5. Concluding Remarks

We prove non-integrability of one parameters’ family of the fifth Painlevé equation as a Hamiltonian system. The main tool to identify obstruction to complete integrability of this Hamiltonian system is Ziglin-Morales-Ramis-Simó theory reducing the question to differential Galois theory. We explicitly compute formal and analytic...
invariants of the second variational equation (in fact of the part of it) by a method based on the asymptotic analysis of its irregular singularity at zero. From these results we compute the Galois group of our differential equation.

We consider here only the case \( \kappa = 0, \kappa_0 = -\theta \) with \( \theta \) an arbitrary complex parameter. It is tempting to use the methods of Galois theory for \( P_1 \) with other values of the parameters, as well as, for other Painlevé transcendents \( P_{III} \) and \( P_{IV} \), where the variational equations along each particular solution will have an irregular singularity at zero. We can hope that in the case of Abelian differential Galois group of the first variational equation and one irregular singularity at 0, the reducibility of the second variational equation, considered as a linear homogeneous scalar differential equation, could be an efficient tool to write down the corresponding solution space expressly and therefore to compute formal monodromies, exponential tori and Stokes multipliers.

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7. References


