Inclusion and Argument Properties for Certain Subclasses of Analytic Functions Defined by Using on Extended Multiplier Transformations

Oh Sang Kwon
Department of Mathematics, Kyungsung University, Busan, Korea
E-mail: oskwon@ks.ac.kr
Received March 28, 2011; revised April 27, 2011; accepted May 5, 2011

Abstract

Making use of a multiplier transformation, which is defined by means of the Hadamard product (or convolution), we introduce some new subclasses of analytic functions and investigate their inclusion relationships and argument properties.

Keywords: Subordination, Starlike Functions, Convex Functions, Closed-to-Convex Functions, Multiplier Transformation, Multivalent Functions, Argument Principle

1. Introduction

Let \( A_p \) denote the class of functions \( f \) normalized by
\[
f(z) = z^p + \sum_{k=1}^{\infty} a_k z^{k+p} \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}) \tag{1.1}
\]
which are analytic and \( p \)-valent in the open unit disk
\[U = \{ z : z \in \mathbb{C} \text{and} |z| < 1 \} \]
If \( f \) and \( g \) are analytic in \( U \), we say that \( f \) is subordinate to \( g \), and write
\[f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in U) \]
if there exists a Schwarz function \( \omega(z) \), analytic in \( U \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) in \( z \in U \), such that
\[f(z) = g(\omega(z)) \quad \text{for} \quad z \in U . \]
We denote by \( S_p^\ast(\eta) \) and \( C_p(\eta) \) the subclasses of \( A_p \) consisting of all analytic functions which are, respectively, \( p \)-valent starlike of order \( \eta \) \( (0 \leq \eta < p) \) in \( U \) and \( p \)-valent convex of order \( \eta \) \( (0 \leq \eta < p) \) in \( U \).

Let \( M \) be the class of analytic functions \( \varphi \) with \( \varphi(0) = 1 \), which are convex and univalent in \( U \) and satisfy the following inequality:
\[\Re \{ \varphi(z) \} > 0 \quad (z \in U) \]
Making use of the aforementioned principle of subordination between analytic functions, we define each of the following subclasses of \( A_p \):
\[S_p^\ast(\eta; \varphi) \]
\[:= \{ f : f \in A_p \text{ and } \frac{1}{p - \eta} \left( \frac{zf'(z)}{f(z)} - \eta \right) \prec \varphi(z) \} \quad (0 \leq \eta < p; z \in U; \varphi \in M) \tag{1.2}
\]
\[K_p(\eta; \varphi) \]
\[:= \{ f : f \in A_p \text{ and } \frac{1}{p - \eta} \left( 1 + \frac{zf'(z)}{f(z)} - \eta \right) \prec \varphi(z) \} \quad (0 \leq \eta < p; z \in U; \varphi \in M) \tag{1.3}
\]
For \( m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\} \), we define the multiplier transformation \( J^m(p, \lambda, \ldots) \) of functions \( f \in A_p \) by
\[C_p(\eta, \beta; \varphi, \psi) := \{ f : f \in A_p \text{ and } \exists g \in S_p^\ast(\eta; \varphi) \text{ s.t. } \frac{1}{p - \beta} \left( \frac{zf'(z)}{g(z)} - \beta \right) \prec \psi(z) \} \quad (0 \leq \eta, \beta < p; z \in U; \varphi, \psi \in M) \tag{1.4}
\]
\[ J^m(p, \lambda, l) f(z) = z^p + \sum_{k=1}^{l} \left( \frac{l + \lambda k}{l} \right)^m a_{k+p} z^{k+p} \quad (l > 0; \lambda \geq 0; z \in U) \] (1.5)

Put
\[ \phi^{m}_{p,\lambda,l}(z) = z^p + \sum_{k=1}^{l} \left( \frac{l + \lambda k}{l} \right)^m z^{k+p} \quad (m \in \mathbb{N}; l > 0; \lambda \geq 0; z \in U) \] (1.6)

The operators \( \phi^{m}_{p,\lambda,l} \) and \( \phi^{m}_{p,\lambda,l} \), are the multiplier transformations introduced and studied earlier by Sarangi and Uralegaddi [16] and Uralegaddi and Somanatha ([1] and [2]), respectively. Corresponding to the function \( \phi^{m}_{p,\lambda,l}(z) \) defined by (1.6), we introduce a function \( \phi^{m}_{p,\lambda,l}(z) \) given by the Hadamard product (or convolution):
\[ \phi^{m}_{p,\lambda,l}(z) * \phi^{m}_{p,\lambda,l}(z) = \frac{z^p}{(1-z)^{p+1}} \quad (\mu > -p) \]

Then, analogous to \( J^m(p, \lambda, l) \), we have a new multiplier transformation
\[ I^m_p(p, \lambda, l): A_p \to A_p \]
as follows:
\[ I^m_p(p, \lambda, l) f(z) = \phi^{m}_{p,\lambda,l}(z) * f(z) \quad (1.7) \]

We note that
\[ I^0_p(p,1,1)f(z) = f(z) \quad \text{and} \quad I^1_p(1,1,2)f(z) = zf'(z) \]

It is easily verified from the above definition of the operator \( I^m_p(p, \lambda, l) \), that
\[ z \left( I^m_p(p, \lambda, l)f(z) \right)' = (\mu + p) I^{m+1}_p(p, \lambda, l)f(z) - \mu I^m_p(p, \lambda, l)f(z) \quad (1.8) \]

and
\[ \lambda z \left( I^m_p(p, \lambda, l)f(z) \right)' = I^m_p(p, \lambda, l)f(z) - (\lambda p - l) I^{m+1}_p(p, \lambda, l)f(z) \quad (1.9) \]

The definition (1.6) of the multiplier transformation \( \phi^{m}_{p,\lambda,l} \) is motivated essentially by the Choi-Saigo-Srivastava operator [3] for analytic functions, which includes a simpler integral operator studied earlier by Noor [7] and others (cf. [4-6]).

Next, by using the operator \( I^m_p(p, \lambda, l) \) defined by (1.7), we introduce the following subclasses of analytic functions:

\[ S^{m,\mu}_{p,\lambda,l}(\eta; \varphi) = \left\{ f : f \in A_p \quad \text{and} \quad I^m_p(p, \lambda, l)f(z) \in S^\eta_{p,\lambda,l}(\eta; \varphi) \right\} \quad (\varphi \in M; \lambda, l, \mu > 0; m \in \mathbb{Z}; 0 \leq \eta < 1) \] (1.10)

\[ K^{m,\mu}_{p,\lambda,l}(\eta; \varphi) = \left\{ f : f \in A_p \quad \text{and} \quad I^m_p(p, \lambda, l)f(z) \in K^\eta_{p,\lambda,l}(\eta; \varphi) \right\} \quad (\varphi \in M; \lambda, l, \mu > 0; m \in \mathbb{N}; 0 \leq \eta < 1) \] (1.11)

\[ C^{m,\mu}_{p,\lambda,l}(\eta, \beta; \varphi, \psi) = \left\{ f : f \in A_p \quad \text{and} \quad I^m_p(p, \lambda, l)f(z) \in C(\eta, \beta; \varphi, \psi) \right\} \quad (\varphi, \psi \in M; \lambda, l, \mu > 0; m \in \mathbb{N}; 0 \leq \eta, \beta < 1) \] (1.12)

We also note that
\[ f(z) \in K^{m,\mu}_{p,\lambda,l}(\eta; \varphi) \Leftrightarrow zf'(z) \in S^{m,\mu}_{p,\lambda,l}(\eta; \varphi) \] (1.13)

In particular, we set
\[ S^{m,\mu}_{p,\lambda,l} \left( \eta, \frac{1 + Az}{1 + Bz} \right) = S^{m,\mu}_{p,\lambda,l}(\eta, A, B) \quad (-1 < B < A \leq 1) \] (1.14)

and
\[ K^{m,\mu}_{p,\lambda,l} \left( \eta, \frac{1 + Az}{1 + Bz} \right) = K^{m,\mu}_{p,\lambda,l}(\eta, A, B) \quad (-1 < B < A \leq 1) \] (1.15)

In the present paper, we investigate some inclusion relationships and argument properties associated with such multivalent functions in the class \( A_p \), as those belonging to the subclasses \( S^{m,\mu}_{p,\lambda,l}(\eta; \varphi) \), \( K^{m,\mu}_{p,\lambda,l}(\eta; \varphi) \) and \( C^{m,\mu}_{p,\lambda,l}(\eta, \beta; \varphi, \psi) \) defined by (1.10), (1.11) and (1.12), respectively.

2. Inclusion Properties

**Lemma 2.1:** Let \( \varphi \) be convex univalent in \( U \) with \( \varphi(0) = 1 \) and \( \Re \{ \beta \varphi(z) + \nu \} > 0 \) \( (\beta, \nu \in \mathbb{C}) \). If \( p \) is analytic in \( U \) with \( p(0) = 1 \), then
\[ p(z) + \frac{z p'(z)}{\beta \varphi(z) + \nu} < \varphi(z) \quad (z \in U) \]
implies that \( p(z) < \varphi(z) \) \( (z \in U) \).

**Theorem 2.2:** Let \( \varphi \in M \) with
\[ \min_{z \in U} \left( \Re \{ \varphi(z) \} \right) > \max \left( \frac{\eta + \mu}{\eta - p}, \frac{\eta - p + 1}{\eta - p} \right) \]

Copyright © 2011 SciRes.
then \( S_{p,a,d}^{m,\mu_1} (\eta; \varphi) \subseteq S_{p,a,d}^{m,\mu} (\eta; \varphi) \subseteq S_{p,a,d}^{m+1,\mu} (\eta; \varphi) \).

**Proof.** First of all, we show that
\[
S_{p,a,d}^{m,\mu_1} (\eta; \varphi) \subseteq S_{p,a,d}^{m,\mu} (\eta; \varphi) \] Let \( f \in S_{p,a,d}^{m,\mu_1} (\eta; \varphi) \) and set
\[
p(z) = \frac{1}{p - \eta} \left( z \left( I_{\mu}^{m} (p, \lambda, l) f(z) \right)' - \eta \right) \tag{2.1}
\]
where the function \( p(z) \) is analytic in \( U \) with \( p(0) = 1 \).

Applying (2.1), we obtain
\[
(\mu + p) I_{\mu_1}^{m} (p, \lambda, l) f(z) = (p - \eta) p(z) + \eta + \mu \tag{2.2}
\]

By logarithmically differentiating both sides of (2.2) and multiplying the resulting equation by \( z \), we have
\[
\frac{1}{p - \eta} \left( z \left( I_{\mu_1}^{m} (p, \lambda, l) f(z) \right)' - \eta \right) = p(z) + \frac{z p'(z)}{(p - \eta) p(z) + \eta + \mu} (z \in U) \tag{2.3}
\]

Since \( \Re \left\{ (p - \eta) \varphi(z) + \eta + \mu \right\} > 0 \), by applying Lemma 2.1 to (2.3), it follows that \( p(z) \prec \varphi(z) \) in \( U \), that is, that \( f(z) \in S_{p,a,d}^{m,\mu} (\eta; \varphi) \).

To prove the second part of Theorem 2.1, let \( f(z) \in S_{p,a,d}^{m,\mu} (\eta; \varphi) \) and put
\[
q(z) = \frac{1}{p - \eta} \left( z \left( I_{\mu_1}^{m} (p, \lambda, l) f(z) \right)' - \eta \right)
\]
where the function \( q(z) \) is analytic in \( U \) with \( q(0) = 1 \).

In precisely the same manner, we can find the result that \( q(z) \prec \varphi(z) \) in \( U \), that is, that \( f(z) \in S_{p,a,d}^{m+1,\mu} (\eta; \varphi) \) under the hypothesis
\[
\Re \left\{ (p - \eta) \varphi(z) + \eta - p + \frac{l}{\lambda} \right\} > 0
\]

**Theorem 2.3:** Let \( \varphi(z) \in M \) with
\[
\min_{z \in U} \left\{ \Re \{ \varphi(z) \} \right\} > \max \left\{ \frac{\eta + \mu}{\eta - p}, \frac{\eta - p + \frac{l}{\lambda}}{\eta - p} \right\}
\]
then \( K_{p,a,d}^{m,\mu_1} (\eta; \varphi) \subseteq K_{p,a,d}^{m,\mu} (\eta; \varphi) \subseteq K_{p,a,d}^{m+1,\mu} (\eta; \varphi) \).

**Proof.** Applying (1.11) and Theorem 2.2, we observe that
\[
f(z) \in K_{p,a,d}^{m,\mu_1} (\eta; \varphi) \Leftrightarrow z f'(z) \in S_{p,a,d}^{m,\mu_1} (\eta; \varphi)
\]

\[
\Rightarrow z f'(z) \in S_{p,a,d}^{m,\mu} (\eta; \varphi) \Leftrightarrow f(z) \in K_{p,a,d}^{m,\mu} (\eta; \varphi)
\]
and
\[
f(z) \in K_{p,a,d}^{m,\mu} (\eta; \varphi) \Leftrightarrow z f'(z) \in S_{p,a,d}^{m+1,\mu} (\eta; \varphi)
\]

\[
\Rightarrow z f'(z) \in S_{p,a,d}^{m+1,\mu} (\eta; \varphi) \Leftrightarrow f(z) \in K_{p,a,d}^{m+1,\mu} (\eta; \varphi)
\]
which evidently prove Theorem 2.3.

By setting
\[
\varphi(z) = \frac{1 + Az}{1 + Bz} (-1 < B < A \leq 1; z \in U)
\]
in Theorems 2.2 and 2.3, we deduce the following corollary.

**Corollary 2.4:** Suppose that
\[
1 - A > \max \left\{ \frac{\eta + \mu}{\eta - p}, \frac{\eta - p + \frac{l}{\lambda}}{\eta - p} \right\}
\]

Then, for the function classes defined by (1.12) and (1.13),
\[
S_{p,a,d}^{m,\mu_1} (\eta; A,B) \subseteq S_{p,a,d}^{m,\mu} (\eta; A,B) \subseteq S_{p,a,d}^{m+1,\mu} (\eta; A,B)
\]

and
\[
K_{p,a,d}^{m,\mu_1} (\eta; A,B) \subseteq K_{p,a,d}^{m,\mu} (\eta; A,B) \subseteq K_{p,a,d}^{m+1,\mu} (\eta; A,B)
\]

**Theorem 2.5:** Let \( \varphi, \psi \in M \) with
\[
\min_{z \in U} \left\{ \Re \{ \varphi(z) \} \right\} > \max \left\{ \frac{\eta + \mu}{\eta - p}, \frac{\eta - p + \frac{l}{\lambda}}{\eta - p} \right\}
\]
then
\[
C_{p,a,d}^{m,\mu_1} (\eta; \psi) \subseteq C_{p,a,d}^{m,\mu} (\eta; \psi) \subseteq C_{p,a,d}^{m+1,\mu} (\eta; \psi)
\]

**Proof.** We begin by proving that
\[
C_{p,a,d}^{m,\mu_1} (\eta; \beta; \psi, \varphi) \subseteq C_{p,a,d}^{m,\mu} (\eta; \beta; \psi, \varphi), \text{ which is the first inclusion relationship asserted by Theorem 2.5.}
\]

Let \( f(z) \in C_{p,a,d}^{m,\mu_1} (\eta; \beta; \psi, \varphi) \). Then there exists a function \( k(z) \in S_{p}^{m} (\eta; \varphi) \) such that
\[
\frac{1}{p - \beta} \left( z \left( I_{\mu_1}^{m} (p, \lambda, l) f(z) \right)' - \beta \right) \prec \psi(z) (z \in U)
\]
Choose the function \( g(z) \) such that
\[
I_{\mu_1}^{m} (p, \lambda, l) g(z) = k(z) \in S_{p}^{m} (\eta; \varphi)
\]
Then \( g(z) \in S_{\mu,\lambda}^{n,\lambda}(\eta;\varphi) \subset S_{\mu,\lambda}^{n,\lambda}(\eta;\varphi) \), and
\[
\frac{1}{p-\beta} \left( \frac{z(I_{\mu,1}^n(p,\lambda,l) f(z))'}{I_{\mu,1}^n(p,\lambda,l) g(z)} - \beta \right) < \psi(z) \quad (z \in U)
\] (2.4)

\[
p(z) = \frac{z(I_{\mu}^n(p,\lambda,l) f(z))'}{I_{\mu}^n(p,\lambda,l) g(z)} - \beta
\] (2.5)

where the function \( p(z) \) is analytic in \( U \) with \( p(0) = 1 \).

Now let
\[
1 = \frac{z(I_{\mu}^n(p,\lambda,l) f(z))'}{I_{\mu}^n(p,\lambda,l) g(z)} - \beta
\]

Using (1.9), we find that
\[
q(z) = \frac{1}{p-\eta} \left( \frac{z(I_{\mu}^n(p,\lambda,l) f(z))'}{I_{\mu}^n(p,\lambda,l) g(z)} - \eta \right)
\] (2.6)

where \( q(z) \prec \varphi(z) \) in \( U \) with the assumption that \( \varphi \in M \). By (2.5),
\[
z\left( I_{\mu}^n(p,\lambda,l) f(z) \right)' = (p-\beta) p(z) + \beta
\] (2.7)

Differentiating both side of (2.7) with respect to \( z \) and multiplying by \( z \), we obtain
\[
1 = \frac{1}{p-\beta} \left( \frac{z(I_{\mu,1}^n(p,\lambda,l) f(z))'}{I_{\mu,1}^n(p,\lambda,l) g(z)} - \beta \right)
\]

\[
= \frac{1}{p-\beta} \left( \frac{(\mu+1)z(I_{\mu}^n(p,\lambda,l) f(z))' + z^2(I_{\mu}^n(p,\lambda,l) f(z))''}{z(I_{\mu}^n(p,\lambda,l) g(z)) + \mu I_{\mu}^n(p,\lambda,l) g(z)} - \beta \right)
\]

\[
= \frac{1}{p-\beta} \left( \frac{(\mu+1)z(I_{\mu}^n(p,\lambda,l) f(z))' + z^2(I_{\mu}^n(p,\lambda,l) f(z))''}{z(I_{\mu}^n(p,\lambda,l) g(z)) + \mu} \right)
\]

Since \( g(z) \in S_{\mu,\lambda}^{n,\lambda}(\eta;\varphi) \), then we set
\[
q(z) = \frac{1}{p-\eta} \left( \frac{z(I_{\mu}^n(p,\lambda,l) f(z))'}{I_{\mu}^n(p,\lambda,l) g(z)} - \eta \right)
\]

(2.6)

\[
z\left( I_{\mu}^n(p,\lambda,l) f(z) \right)' = (p-\beta) q(z) + \eta - 1 + \frac{(p-\beta) p(z)}{(p-\beta) p(z) + \beta}
\] (2.8)

Hence
\[
z^2\left( I_{\mu}^n(p,\lambda,l) f(z) \right)' = \frac{z^2(I_{\mu}^n(p,\lambda,l) f(z))'}{I_{\mu}^n(p,\lambda,l) g(z)} = \frac{(p-\beta) p(z) + \beta}{(p-\beta) p(z) + \beta}
\]

Computing the above equations, we can obtain
\[
q(z) = \frac{1}{p-\eta} \left( \frac{z(I_{\mu}^n(p,\lambda,l) f(z))'}{I_{\mu}^n(p,\lambda,l) g(z)} - \eta \right)
\]

(2.6)

\[
q(z) = \frac{1}{p-\eta} \left( \frac{z(I_{\mu}^n(p,\lambda,l) f(z))'}{I_{\mu}^n(p,\lambda,l) g(z)} - \eta \right)
\]

(2.6)
3. Argument Properties

**Lemma 3.1:** Let \( \phi \) be convex univalent in \( U \) and \( \omega \) be analytic in \( U \) with \( \text{Re}\{\omega(z)\} \geq 0 \). If \( p(z) \) is analytic in \( U \) and \( p(0) = \phi(0) \), then \( p(z) + \omega(z)zp'(z) < \phi(z) \) \((z \in U)\)

implies that \( p(z) < \phi(z) \) \((z \in U)\).

**Lemma 3.2:** Let \( p \) be analytic in \( U \) with \( p(0) = 1 \) and \( p(z) \neq 0 \) for all \( z \in U \). If there exist two points \( z_1, z_2 \in U \) such that

\[
-\frac{\pi}{2} \alpha_1 = \arg\{p(z_1)\} < \arg\{p(z)\} < \arg\{p(z_2)\} = \frac{\pi}{2} \alpha_2
\]

(3.1)

for some \( \alpha_1 \) and \( \alpha_2 \) \((\alpha_1, \alpha_2 > 0)\) and for all \( z \)

\[
\delta_1 = \alpha_1 + 2\tan\left\{\frac{(\alpha_1 + \alpha_2)(1 - |b|)\cos\left(\frac{\pi}{2} t_1\right)}{2\left[\frac{(p - \eta)(1 + A)}{1 + B} + \eta + \mu\right](1 + |b|) + (\alpha_1 + \alpha_2)(1 - |b|)\sin\left(\frac{\pi}{2} t_1\right)}\right\}
\]

and

\[
\delta_2 = \alpha_2 + 2\tan^{-1}\left\{\frac{(\alpha_1 + \alpha_2)(1 - |b|)\cos\left(\frac{\pi}{2} t_1\right)}{2\left[\frac{(p - \eta)(1 + A)}{1 + B} + \eta + \mu\right](1 + |b|) + (\alpha_1 + \alpha_2)(1 - |b|)\sin\left(\frac{\pi}{2} t_1\right)}\right\}
\]

where \( b \) is given by (3.2), and

\[
t_1 = t_1(\lambda) = \frac{2}{\pi}\cos^{-1}\left(\frac{(p - \eta)(A - B)}{(p - \eta)(1 - AB) + (\eta + \mu)(1 - B^2)}\right)
\]

(3.3)

**Proof.** Let \( p(z) = \frac{1}{p - \gamma}\left[z\left(I_{\mu_1}^w(\eta, p, \lambda, l)g(z)\right)\right] + \gamma\left(I_{\mu_1}^w(\eta, p, \lambda, l)g(z)\right)\)

Then \( p(z) \) is analytic in \( U \) with \( p(0) = 1 \). By using (1.9), we obtain

\[
(p - \gamma)p(z)^m + \gamma \left(I_{\mu_1}^w(\eta, p, \lambda, l)g(z)\right) = (\mu + p)I_{\mu_1}^w(\eta, p, \lambda, l)f(z) - \mu I_{\mu_1}^w(\eta, p, \lambda, l)f(z)
\]

(3.4)

Differentiating both sides of the above equation and multiplying the resulting equation by \( z \), we find that

\[
(p - \gamma)p(z)\left[I_{\mu_1}^w(\eta, p, \lambda, l)g(z)\right] + \gamma \left((p - \gamma)p(z) + \gamma\right)\left[I_{\mu_1}^w(\eta, p, \lambda, l)g(z)\right] = (\mu + p)\left[I_{\mu_1}^w(\eta, p, \lambda, l)f(z)\right] - \mu \left[I_{\mu_1}^w(\eta, p, \lambda, l)f(z)\right]
\]

Since \( g(z) \in S_{\mu_1}^{w, \nu_1}(\eta, p, A, B) \), by Corollary 2.4, it follows that \( g(z) \in S_{\mu_1}^{w, \nu_1}(\eta, p, A, B) \).

Next we let \( q(z) = \frac{1}{p - \eta}\left[z\left(I_{\mu_1}^w(\eta, p, \lambda, l)g(z)\right)\right] + \eta\left(I_{\mu_1}^w(\eta, p, \lambda, l)g(z)\right)\).
Then, using (1.9), we have

\[(\mu + p) \frac{I''(p, \lambda, l) g(z)}{I''(p, \lambda, l) g(z)} = (p - \eta) q(z) + \eta + \mu\]  \hspace{1cm} (3.5)

From (3.4) and (3.5), we obtain

\[\frac{1}{p - \gamma} \left( z \frac{I''(p, \lambda, l) f(z)}{I''(p, \lambda, l) g(z)} - \gamma \right) = p(z) + \frac{z p'(z)}{(p - \eta) q(z) + \eta + \mu}\]

Furthermore, by using a known result, we have

\[\left| q(z) - \frac{1 - A B}{1 - B^2} \right| < \frac{A - B}{1 - B^2}\]  \hspace{1cm} (3.6)

Thus, from (3.6), we obtain

\[(p - \eta) q(z) + \eta + \mu = \rho \exp \left( \frac{i \pi}{2} \phi \right)\]

\[\arg \left\{ p(z) + \frac{z p'(z)}{(p - \eta) q(z) + \eta + \mu} \right\} = -\frac{\pi}{2} \alpha + \arg \left\{ 1 - \left( \frac{\alpha + \alpha_z}{2} \right) m \left( \rho \exp \left( \frac{i \pi}{2} \phi \right) \right)^{-1} \right\} \leq -\frac{\pi}{2} \alpha - \tan^{-1} \left( \frac{2 \rho + (\alpha + \alpha_z) m \cos \left( \frac{\pi}{2} (1 - \phi) \right)}{(\alpha + \alpha_z) m \cos \left( \frac{\pi}{2} (1 - \phi) \right)} \right)\]

\[\leq -\frac{\pi}{2} \alpha - \tan^{-1} \left( \frac{(\alpha + \alpha_z) (1 - |b|) \cos \left( \frac{\pi}{2} t_1 \right)}{2 \left( \frac{(p - \eta)(1 + A)}{1 + B} + \eta + \mu \right) (1 + |b|) + (\alpha + \alpha_z) (1 - |b|) \cos \left( \frac{\pi}{2} t_1 \right)} \right) = -\frac{\pi}{2} \delta_1\]

and

\[\arg \left\{ p(z) + \frac{z p'(z)}{(p - \eta) q(z) + \eta + \mu} \right\} \geq \frac{\pi}{2} \alpha + \tan^{-1} \left( \frac{(\alpha + \alpha_z) (1 - |b|) \cos \left( \frac{\pi}{2} t_1 \right)}{2 \left( \frac{(p - \eta)(1 + A)}{1 + B} + \eta + \mu \right) (1 + |b|) + (\alpha + \alpha_z) (1 - |b|) \cos \left( \frac{\pi}{2} t_1 \right)} \right) = \frac{\pi}{2} \delta_2\]

which would obviously contradict the assertion of Theorem 3.3. We thus complete the proof of Theorem 3.3.

If we let \( \delta_1 = \delta_2 \) in Theorem 3.5, we easily obtain the following consequence.

**Corollary 3.4:** Let \( f \in A_p \). 0 < \( \delta \leq 1 \). 0 < \( \gamma < p \). If

\[\left| \arg \left( \frac{z \frac{I''(p, \lambda, l) f(z)}{I''(p, \lambda, l) g(z)} - \gamma}{\frac{I''(p, \lambda, l) f(z)}{I''(p, \lambda, l) g(z)} - \gamma} \right) \right| < \frac{\pi}{2} \alpha\]

where, in terms of \( t_1 \) given by (3.3).

\[\frac{(p - \eta)(1 - A)}{1 - B} + \eta + \mu < \rho < \frac{(p - \eta)(1 - A)}{1 - B} + \eta + \mu - t_1 < \phi < t_1\]

We note that \( p \) is analytic in \( U \) with \( p(0) = 1 \). Let \( \omega = h(z) \) be the function which maps \( U \) onto the angular domain

\[\left\{ \omega: -\frac{\pi}{2} \delta_1 < \arg(\omega) < \frac{\pi}{2} \delta_2 \right\} \quad \text{with} \quad h(0) = 1\]

Applying Lemma 3.1 for this function \( h \) with

\[\omega(z) = \frac{1}{(p - \eta) q(z) + \eta + \mu}\]

we see that \( \Re \{ p(z) \} > 0 \) ( \( z \in U \) ), and hence \( p(z) \neq 0 \) ( \( z \in U \) ). By using Lemma 3.2, if there exist two points \( z_1, z_2 \in U \) such that the condition (3.1) is satisfied, then we obtain (3.2) under the constraint (3.2). And we obtain

\[\arg \left\{ p(z) + \frac{z p'(z)}{(p - \eta) q(z) + \eta + \mu} \right\} \geq \frac{\pi}{2} \alpha + \tan^{-1} \left( \frac{(\alpha + \alpha_z) (1 - |b|) \cos \left( \frac{\pi}{2} t_1 \right)}{2 \left( \frac{(p - \eta)(1 + A)}{1 + B} + \eta + \mu \right) (1 + |b|) + (\alpha + \alpha_z) (1 - |b|) \cos \left( \frac{\pi}{2} t_1 \right)} \right) = \frac{\pi}{2} \delta_2\]

for some \( g \in S^{\alpha, \mu \lambda}_{p, \lambda, l} (\eta, p, A, B) \), then

\[\arg \left( \frac{z \frac{I''(p, \lambda, l) f(z)}{I''(p, \lambda, l) g(z)} - \gamma}{\frac{I''(p, \lambda, l) f(z)}{I''(p, \lambda, l) g(z)} - \gamma} \right) < \frac{\pi}{2} \alpha\]

where \( \alpha \) is the solutions for the following equation:
$$\delta = \alpha + \frac{2}{\pi} \tan^{-1} \left( \frac{\alpha (1-|\beta|) \cos \left( \frac{\pi}{2} t_1 \right)}{(p-\eta)(1+A) + \eta + \mu} \right)$$

for some $g \in S_{\rho,\lambda}^{\eta,\mu} (\eta; p; A, B)$, then

$$\frac{\pi}{2} \delta < \arg \left( \frac{L_{\nu}^m (p, \lambda, l) f(z)}{L_{\nu}^m (p, \lambda, l) g(z)} - \gamma \right) < \frac{\pi}{2} \delta$$

Theorem 3.5: Let $f \in A_{\nu}$. $0 \leq \delta_1, \delta_2 \leq 1$. $0 < \gamma < p$. If

$$-\frac{\pi}{2} \delta_1 < \arg \left( \frac{z (L_{\nu}^m (p, \lambda, l) f(z))}{L_{\nu}^m (p, \lambda, l) g(z)} - \gamma \right) < \frac{\pi}{2} \delta_2$$

where $\alpha_1, \alpha_2$ are the solutions for the following equations:

$$\delta_1 = \alpha_1 + \frac{2}{\pi} \tan^{-1} \left( \frac{(\alpha_1 + \alpha_2)(1-|\beta|) \cos \left( \frac{\pi}{2} t_1 \right)}{2 \left( \frac{(p-\eta)(1+A)}{1+B} + \eta - p + \frac{1}{l} \right) (1+|\beta|) + (\alpha_1 + \alpha_2)(1-|\beta|) \cos \left( \frac{\pi}{2} t_1 \right)} \right)$$

and

$$\delta_2 = \alpha_2 + \frac{2}{\pi} \tan^{-1} \left( \frac{(\alpha_1 + \alpha_2)(1-|\beta|) \cos \left( \frac{\pi}{2} t_1 \right)}{2 \left( \frac{(p-\eta)(1-A)}{1-B} + \eta - p + \frac{1}{l} \right) (1+|\beta|) + (\alpha_1 + \alpha_2)(1-|\beta|) \cos \left( \frac{\pi}{2} t_1 \right)} \right)$$

$$t_1 = t_1 (\lambda)$$

$$= \frac{2}{\pi} \cos^{-1} \left( \frac{(p-\eta)(A-B)}{(p-\eta)(1-AB) + (\eta - p + \frac{1}{l})(1-B^2)} \right)$$

for some $g \in S_{\rho,\lambda}^{\eta,\mu} (\eta; p; A, B)$, then

$$\frac{\pi}{2} \alpha < \arg \left( \frac{z (L_{\nu}^{m+1} (p, \lambda, l) f(z))}{L_{\nu}^{m+1} (p, \lambda, l) g(z)} - \gamma \right) < \frac{\pi}{2} \alpha$$

(3.8)

If we let $\delta_1 = \delta_2$ in Theorem 3.5, we easily obtain the following consequence.

Corollary 3.6: Let $f \in A_{\rho}$. $0 < \delta \leq 1$. $0 < \gamma < p$. If $\alpha$ is the solutions for the following equation:

$$\delta = \alpha + \frac{2}{\pi} \tan^{-1} \left( \frac{\alpha (1-|\beta|) \cos \left( \frac{\pi}{2} t_1 \right)}{(p-\eta)(1+A) + \eta - p + \frac{1}{l} \right) (1+|\beta|) + (1-|\beta|) \cos \left( \frac{\pi}{2} t_1 \right) \right)$$
\( b \) is given by (1.17), and
\[
t_i = t_i \left( \lambda \right) = \frac{2}{\pi} \cos^{-1} \left( \frac{(p - \eta)(A - B)}{(p - \eta)(1 - AB) + (\eta - p + \frac{1}{\lambda})(1 - B^2)} \right)
\]
(3.9)

4. Acknowledgements

The research was supported by Kyungsung University Research Grants in 2011.

5. References


