Analytical Solution of Two Extended Model Equations for Shallow Water Waves by Adomian’s Decomposition Method

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Abstract

In this paper, we consider two extended model equations for shallow water waves. We use Adomian’s decomposition method (ADM) to solve them. It is proved that this method is a very good tool for shallow water wave equations and the obtained solutions are shown graphically.

Keywords: Adomian’s Decomposition Method, Shallow Water Wave Equation

1. Introduction

Clarkson et al [1] investigated the generalized short water wave (GSWW) equation

\[
\frac{du_t}{dt} + \alpha uu_x - \beta u_x \int u_x dx + u_x = 0
\]

where \( \alpha \) and \( \beta \) are non-zero constants.

Ablowitz et al. [2] studied the specific case \( \alpha = 4 \) and \( \beta = 2 \) where Equation (1) is reduced to

\[
\frac{du_t}{dt} - 4uu_x - 2u_x \int u_x dx + u_x = 0
\]

This equation was introduced as a model equation which reduces to the KdV equation in the long small amplitude limit [2,3]. However, Hirota et al. [3] examined the model equation for shallow water waves

\[
\frac{du_t}{dt} - 3uu_x - 3u_x \int u_x dx + u_x = 0
\]

obtained by substituting \( \alpha = \beta = 3 \) in (1).

Equation (2) can be transformed to the bilinear forms

\[
\left[ D_x \left( D_x - D_x^2 + D_x^3 \right) + \frac{1}{3} D_x \left( D_x + D_x^3 \right) \right] f \cdot f = 0
\]

where the solution of the equation is

\[
u(x,t) = 2(\ln f)_{xx}
\]

where \( f(x,t) \) is given by the perturbation expansion

\[
f(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f_n(x,t)
\]

where \( \epsilon \) is a bookkeeping non-small parameter, and \( f_n(x,t), n = 1,2,\ldots,n \) are unknown functions that will be determined by substituting the last equation into the bilinear form and solving the resulting equations by equating different powers of \( \epsilon \) to zero.

The customary definition of the Hirota’s bilinear operators are given by

\[
\left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x'} \right)^n \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n a(x,t)b(x',t) \bigg| x' = x, t' = t
\]

Some of the properties of the \( D \)-operators are as follows

\[
\frac{D^2_x f \cdot f}{f^2} = \int u_x dx, \quad \frac{D^2_x f \cdot f}{f^2} = u_{xx} + 3u^2, \quad \frac{D^2_x f \cdot f}{f^2} = \ln \left( f^2 \right)_{xx}
\]

\[
\frac{D^2_x f \cdot f}{f^2} = u_{xx} + 15uu_x + 15u^3
\]
where
\[ u(x,t) = 2\left(\ln f(x,t)\right)_{x,t} \tag{11} \]

Also extended model of Equation (2) is obtained by the operator \( D^3_t \) to the bilinear forms (4) and (5)
\[
\left[D_t \left(D_t - D_t D^3 + D_s + D^3_s\right) + \frac{1}{3} D_t \left(D_t + D^3_t\right) \right] f \cdot f = 0
\tag{12}

where \( s \) is an auxiliary variable, and \( f \) satisfies the bilinear equation
\[
D_s \left(D_s + D^3_s\right) f \cdot f = 0
\tag{13}

Using the properties of the D operators given above, and differentiating with respect to \( x \) we obtain the extended model for Equation (2) given by
\[
u_t - u_{xx} - 4uu_x - 2u_x \int^t u_x dx + u_x + u_{xxx} + 6uu_x = 0 \tag{14}
\]

In a like manner, we extend Equation (3) by adding the operator \( D^3 \) to the bilinear forms (6) to obtain
\[
D_t \left(D_t - D_t D^3 + D_s + D^3_s\right) f \cdot f = 0
\tag{15}

Using the properties of the D operators given above we obtain the extended model for Equation (3) given by
\[
u_t - u_{xx} - 3uu_x - 3u_x \int^t u_x dx + u_x + u_{xxx} + 6uu_x = 0 \tag{16}
\]

In this paper, we use the Adomian’s decomposition method (ADM) to obtain the solution of two considered equations above for shallow water waves. Large classes of linear and nonlinear differential equations, both ordinary as well as partial, can be solved by the ADM [4-15]. A reliable modification of ADM has been done by Wazwaz [16].The decomposition method provides an effective procedure for analytical solution of a wide and general class of dynamical systems representing real physical problems [4-14].This method efficiently works for initial-value or boundary-value problems and for linear or nonlinear, ordinary or partial differential equations and even for stochastic systems. Moreover, we have the advantage of a single global method for solving ordinary or partial differential equations as well as many types of other equations.

2. Basic idea of Adomian’s Decomposition Method

We begin with the equation
\[
Lu + R(u) + F(u) = g(t) \tag{17}
\]
where \( L \) is the operator of the highest-ordered derivatives with respect to \( t \) and \( R \) is the remainder of the linear operator. The nonlinear term is represented by \( F(u) \). Thus we get
\[
Lu = g(t) - R(u) - F(u) \tag{18}
\]

The inverse \( L^{-1} \) is assumed an integral operator given by
\[
L^{-1}_c = \int_0^t (\cdot) dt \tag{19}
\]

The operating with the operator \( L^{-1}_c \) on both sides of Equation (18) we have
\[
u = f_0 + L^{-1}_c \left(g(t) - R(u) - F(u)\right) \tag{20}
\]
where \( f_0 \) is the solution of homogeneous equation
\[
Lu = 0 \tag{20}
\]

involving the constants of integration. The integration constants involved in the solution of homogeneous equation (21) are to be determined by the initial or boundary condition according as the problem is initial-value problem or boundary-value problem.

The ADM assumes that the unknown function \( u(x,t) \) can be expressed by an infinite series of the form
\[
u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{22}
\]
and the nonlinear operator \( F(u) \) can be decomposed by an infinite series of polynomials given by
\[
F(u) = \sum_{n=0}^{\infty} A_n \tag{23}
\]

where \( u_n(x,t) \) will be determined recurrently, and \( A_n \) are the so-called polynomials of \( u_0, u_1, \cdots, u_n \) defined by
\[
A_n = \frac{1}{n!} \frac{d^n}{dt^n} \left[F\left(\sum_{n=0}^{\infty} u^n\right)\right]_{|t=0}, n = 0,1,2,\cdots \tag{24}
\]

3. ADM Implement for First Extended Model of Shallow Water Wave Equation

We consider the application of ADM to first extended model of shallow water wave equation. If Equation (14) is dealt with this method, it is formed as
\[
L u = L_{xx} u + 4uu_x + 2L_x u \int^t L_x u dx
\tag{25}
\]
where
\[
L_{xx} = \frac{\partial^3}{\partial x^2 \partial t}, \quad L_x = \frac{\partial}{\partial x}, \quad L_{xx} = \frac{\partial^3}{\partial x^2 \partial t}, \quad L_{xx} = \frac{\partial^3}{\partial x^3}
\tag{26}
\]

If the invertible operator \( L^{-1}_c \) is applied to Equation (25), then
Substituting from Equation (29) in Equation (28), we find
\[ u(x,t) = u(x,0) + L^{-1}_t \left( \sum_{n=0}^{\infty} u_n(x,t) \right) \]
\[ + 4 \sum_{n=0}^{\infty} u_n(x,t) L_t \left( \sum_{n=0}^{\infty} u_n(x,t) \right) \]
\[ + 2 L_t \left( \sum_{n=0}^{\infty} u_n(x,t) \right) \int L_t \left( \sum_{n=0}^{\infty} u_n(x,t) \right) dx \]
\[ - L_t \left( \sum_{n=0}^{\infty} u_n(x,t) \right) - L_{\text{ext}} \left( \sum_{n=0}^{\infty} u_n(x,t) \right) \]
\[ - 6 \left( \sum_{n=0}^{\infty} u_n(x,t) \right) L_t \left( \sum_{n=0}^{\infty} u_n(x,t) \right) \]

is obtained by this
\[ u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{29} \]

Substituting from Equation (29) in Equation (28), we find
\[ \sum_{n=0}^{\infty} u_n(x,t) = u(x,0) + L^{-1}_t \left( \sum_{n=0}^{\infty} u_n(x,t) \right) \]
\[ + 4 \left( \sum_{n=0}^{\infty} u_n(x,t) \right) L_t \left( \sum_{n=0}^{\infty} u_n(x,t) \right) \]
\[ + 2 L_t \left( \sum_{n=0}^{\infty} u_n(x,t) \right) \int L_t \left( \sum_{n=0}^{\infty} u_n(x,t) \right) dx \]
\[ - L_t \left( \sum_{n=0}^{\infty} u_n(x,t) \right) - L_{\text{ext}} \left( \sum_{n=0}^{\infty} u_n(x,t) \right) \]
\[ - 6 \left( \sum_{n=0}^{\infty} u_n(x,t) \right) L_t \left( \sum_{n=0}^{\infty} u_n(x,t) \right) \]

is found.

According to Equation (19) approximate solution can be obtained as follows:
\[ u_0(x,t) = \frac{(c-1) \sec h^2 \left( \frac{1}{2} \frac{c-1}{c+1} x \right)}{2c+2} \tag{31} \]
\[ u_1(x,t) = \frac{\sinh \left( \frac{1}{2} \frac{c-1}{c+1} x \right) \left( \frac{c-1}{c+1} \right) c(t-1)}{\cosh \left( \frac{1}{2} \frac{c-1}{c+1} x \right) \left( c+1 \right)^2} \tag{32} \]
\[ u_2(x,t) = \int_0^t \left( \sum_{n=0}^{\infty} u_n(x,t) + 4u_t L_u + 2L_u \int L_u dx \right) dt \tag{33} \]

Thus the approximate solution for first extended model of shallow water wave equation is obtained as
\[ u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) \tag{34} \]
is found.

According to Equation (19) approximate solution can be obtained as follows:

\[ u_0(x,t) = \frac{(c-1) \sec \left( \frac{1}{2} \sqrt{\frac{c-1}{c+1}} x \right)}{2c+2} \]  
\[ u_1(x,t) = \frac{\sinh \left( \frac{1}{2} \sqrt{\frac{c-1}{c+1}} x \right) \left( \frac{c-1}{c+1} \right) c \frac{c-1}{c+1} \right)}{c+1} \]  
\[ u_2(x,t) = \int_0^t \left( L_u u_1 + 3u_1 u_1 + 3u_1 u_1 \right) \right) L_u dx \]  
\[ -L_u u_1 - L_u x u_1 - 6u_t L_u u_1 \right) dt \]  

Thus the approximate solution for second extended model of shallow water wave equation is obtained as

\[ u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) \]  

The terms \( u_0(x,t), u_1(x,t), u_2(x,t) \) in Equation (44), obtained from Equations (41), (42), (43).

5. Conclusions

In this paper, Adomian’s decomposition method have been successfully applied to find the solution of two extended model equations for shallow water. The obtained results were showed graphically it is proved that Adomian’s decomposition method is a powerful method for solving these equations. In our work; we used the Maple Package to calculate the functions obtained from the Adomian’s decomposition method.

6. References


[8] K. Abbaoui and Y. Cherruault, “The Decomposition met-


