Portfolio Optimization without the Self-Financing Assumption

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Abstract
In this paper, we relax the assumption of a self-financing strategy in the dynamic investment models. In so doing we provide smooth solutions and constrained viscosity solutions.

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1. Introduction
The literature on dynamic portfolio optimization is vast. However, previous literature on dynamic investment relied on the assumption of a self-financing strategy; that is, the investor cannot add or withdraw funds during the trading horizon. Examples include [1], [2], [3] and [4] among many others. However, this assumption is somewhat restrictive and sometimes unrealistic.

Moreover, even with the assumption of a self-financing strategy, the previous literature usually provided explicit solutions under the assumption of a logarithmic or power utility function. Therefore, the assumption of a self-financing strategy did not offer a significant simplification of the solutions. Therefore, the self-financing assumption needs to be relaxed.

Consequently, the goal of this paper is to relax the assumption of self-financing strategies. In this paper, we show that the assumption of a self-financing strategy can be relaxed without a significant complication of the optimal solutions. In so doing, we present a stochastic-fac- tor incomplete-markets investment model and provide both smooth solutions and constrained viscosity solutions.

2. The Model
We consider an investment model, which includes a risky asset, a risk-free asset and a random external economic factor (see, for example, [5]). We use a three-dimensional standard Brownian motion \( \{W_{1s}, W_{2s}, W_{3s}, F_{1s}\}_{s \leq T} \) on the probability space \( (\Omega, F_{s}, P) \), where \( \{F_{s}\}_{0 \leq s \leq T} \) is the augmentation of filtration. The risk-free asset price process is \( S_{0} = e^{\int_{0}^{T} r(Y_{s})ds} \), where \( r(Y_{s}) \) is the rate of return and \( Y_{s} \) is the stochastic economic factor.

The dynamics of the risky asset price are given by
\[
\frac{dS_{s}}{S_{s}} = \mu(Y_{s})ds + \sigma(Y_{s})dW_{1s},
\]
where \( \mu(Y_{s}) \) and \( \sigma(Y_{s}) \) are the rate of return and the volatility, respectively. The economic factor process is given by
\[
\frac{dY_{s}}{Y_{s}} = b(Y_{s})ds + \sigma(Y_{s})dW_{2s},
\]
where \( \sigma(Y_{s}) \) is its volatility and \( b(Y_{s}) \) is the rate of return.

The amount of money added to or withdrawn from the investment at time \( s \) is denoted by \( \Phi_{s} \), and its dynamics are given by
\[
\frac{d\Phi_{s}}{\Phi_{s}} = a(Y_{s})ds + \sigma(Y_{s})dW_{3s},
\]
where \( \sigma(Y_{s}) \) is its volatility and \( a(Y_{s})b(Y_{s}) \) is the rate of return.

Thus the wealth process is given by
\[
X_{t}^{x} = x + \int_{0}^{T} \left[ a(Y_{s})ds + \sigma(Y_{s})dW_{1s} + \frac{r(Y_{s})X_{s}^{x}}{\pi_{s}}ds + \left( \mu(Y_{s}) - r(Y_{s})\pi_{s} \right) \right]ds
\]
where \( x \) is the initial wealth, \( \{\pi_{s}, F_{s}\}_{s \leq T} \) is the portfolio process with \( E[\int_{0}^{T} \sigma_{s}^{2}(Y_{s})\pi_{s}^{2}ds] < \infty \).

The investor’s objective is to maximize the expected

utility of the terminal wealth

\[ V(t, x, y) = \sup_{u_t} E \left[ u \left( X^x_T \right) | F_t \right], \]  

where \( V(\cdot) \) is the value function, \( u(\cdot) \) is a differentiable, bounded and concave utility function.

Under regularity conditions, the value function is differentiable and thus satisfies the Hamilton-Jacobi-Bellman PDE

\[ V_t + \left[ r(y) x + a(y) \right] V_x + b(y)V_y \\
+ \frac{1}{2} \sigma_y^2(y) V_{xx} + \rho_{2, 2} \sigma_y(y) \sigma_z(y) V_{yx} + \frac{1}{2} \sigma_z^2(y) V_{yy} \\
+ \sup_{\pi} \left( \mu(y) - r(y) \right) V_x \\
+ \frac{1}{2} \pi^2 \sigma_y^2(y) + \rho_{1, 2} \sigma_y(y) \sigma_z(y) \pi_x \right] V_{xx} \\
+ \rho_{1, 2} \sigma_1(y) \sigma_y(y) \pi_x V_{xy} \right] V_{yy} \right] = 0, \]

\[ V(T, x, y) = u(x), \]  \hspace{1cm} (6)

where \( \rho_{ij} \) is the correlation coefficient between the Brownian motions. Hence, the optimal solution is

\[ \pi^*_i = \frac{-\left( \mu(y) - r(y) \right) V_x + \rho_{2, 2} \sigma_y(y) \sigma_z(y) V_{yx}}{\sigma_y^2(y) V_{xx}} \]

\[ -\rho_{1, 2} \sigma_1(y) \sigma_y(y), \]  \hspace{1cm} (7)

Similar to the previous literature, an explicit solution can be obtained for specific forms of utility such as a logarithmic utility function.

### 3. Viscosity Solutions

We can apply the constrained viscosity solutions to (6), given the HJB is degenerate elliptic and monotone increasing in \( V \) (see, for example, [6]).

Consider this HJB

\[ H \left( x, V(x), V_x(x), V_{xx}(x) \right) = 0, x \in \Omega, \]

\[ V(x) = g(x), x \in \partial \Omega, \]  \hspace{1cm} (8)

where \( \Omega \) is a bounded open set.

**Definition 1** A continuous function \( V(x) \) is a viscosity subsolution of (6) if

\[ H \left( x, V(x), P, X \right) \leq 0, \forall P \in \mathbb{D} V(x), \]

\[ \forall x \in \Omega \]  \hspace{1cm} (9)

A continuous function \( V(x) \) is a viscosity supersolution of (6) if

\[ H \left( x, V(x), P, X \right) \geq 0, \forall P \in \mathbb{D} V(x), \]

\[ \forall x \in \Omega, \]  \hspace{1cm} (10)

where

\[ \mathbb{D} V(x) = \left\{ P : \limsup_{_{y \to x}} \frac{V(y) - V(x) - \langle P, y - x \rangle}{|y - x|} \leq 0 \right\}, \]  \hspace{1cm} (11)

\[ \mathbb{D} V(x) = \left\{ P : \liminf_{_{y \to x}} \frac{V(y) - V(x) - \langle P, y - x \rangle}{|y - x|} \geq 0 \right\}, \]  \hspace{1cm} (12)

are the super-differential and sub-differential, respectively; and

\[ J^2 V(x) = \{(P, X) : \]

\[ \limsup_{_{y \to x}} \frac{V(y) - V(x) - \langle P, y - x \rangle - \frac{1}{2} \langle X(y - x), y - x \rangle}{|y - x|^2} \leq 0 \}, \]  \hspace{1cm} (13)

\[ J^2 V(x) = \{(P, X) : \]

\[ \liminf_{_{y \to x}} \frac{V(y) - V(x) - \langle P, y - x \rangle - \frac{1}{2} \langle X(y - x), y - x \rangle}{|y - x|^2} \geq 0 \}, \]  \hspace{1cm} (14)

are the super-perfect and subject, respectively. A function \( V(x) \) is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

**Proposition 1** \( V(x) \) is the unique constrained viscosity solution of (6).

**Proof** Let \( V \in C(\overline{\Omega}) \) and let \( s(V) \) and \( i(V) \) be the upper and lower semicontinuous envelopes of \( V \), respectively, where

\[ s(V) = \sup \{ u(x) : u_t \leq u \leq u_t \}, \]

\[ i(V) = \inf \{ u(x) : u_t \leq u \leq u_t \}, \]

where \( u_t \) and \( u_t \) are sub-solution and super-solution, respectively.

Thus \( s(V) \) is \( \text{USC}(\overline{\Omega}) \) and \( i(V) \) is \( \text{LSC}(\overline{\Omega}) \) are a viscosity subsolution and supersolution, respectively. At the boundary we have

\[ V(x) = s(V) = i(V), \]  \hspace{1cm} (15)

by the comparison principle

\[ \forall x \in \Omega, \]

and by definition \( s(V) \geq i(V) \) and

\[ \forall x \in \Omega. \]

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thus

\[ V(x) = s(V) = i(V) \text{ in } \Omega \]

is the unique viscosity solution.

4. References


