A Study on the Conversion of a Semigroup to a Semilattice

Bahman Tabatabaie, Seyed Mostafa Zebarjad
Department of Mathematics, Shiraz University, Shiraz, Iran
E-mail: s_zebarjad@yahoo.com

Received January 19, 2011; revised March 15, 2011; accepted March 25, 2011

Abstract

The main aim of the current research has been concentrated to clarify the condition for converting the inverse semigroups such as S to a semilattice. For this purpose a property the so-called $E^*$–unitary has been defined and it has been tried to prove that each inverse semigroups limited with $E^*$–unitary show the specification of a semilattice.

Keywords: Semigroup, Semilattice, $E^*$–unitary

1. Introduction

1.1. Literature Survey

Literature survey done by the authors show that a special class of semigroups possessing is formed by the $E^*$–unitary inverse semigroups, sometimes also called 0–$E^*$–unitary, which was defined by Szendrei [1] and has been intensely studied in the semigroup literature. See, for example, Kellendonk’s topological groupoid is Hausdorff when $S$ is $E^*$–unitary [2], and the related class of $E^*$–unitary inverse semigroups have also been shown to provide Hausdorff groupoids [3]. In the current research the authors try to prove that each inverse semigroups limited with $E^*$–unitary show the specification of a semilattice. For this purpose, firstly we present elementary concepts as follows.

1.2. Preliminary Definitions and Propositions

A groupoid is a set $G$ together with a subset $G^2 \subseteq G \times G$, a product map $(a, b) \mapsto ab$.

From $G^2$ to $G$, and an inverse map $a \mapsto a^{-1}$ (so that $(a^{-1})^{-1} = a$) from $G$ onto $G$ such that:

1) if $(a, b), (b, c) \in G^2$, then $(ab, c), (a, bc) \in G^2$ and $(ab)c = a(bc)$.

2) $(b, b^{-1}) \in G^2$ for all $b \in G$, and if $(a, b) \in G^2$ then $a^{-1}(ab) = b$ and $(ab)b^{-1} = a$.

Note that $G^2$ is nothing but the set of all pairs $(x, y)$ in $G \times G$ for which $xy$ is defined, and $G^2$ is called the set of composable pairs of the groupoid $G$ [3].

If $x \in G, d(x) = x^{-1}x$ is the domain of $x$ and $r(x) = xx^{-1}$ its range. The pair $(x, y)$ is composable if and only if the range of $y$ is the domain of $x$. $G^0 = d(G) = r(G)$ is the unit space of $G$, its elements are units in sense that $xd(x) = x$ and $r(x) = x$ [4].

By an inverse semigroup we mean a semigroup $S$ such that for each $a$ in $S$, there exists a unique element $a'$ in $S$ with the following properties:

$aaa' = a$, and $a'a'a = a'$

It is well known that the correspondence $a \mapsto a'$ is an involutive anti-homomorphism, i.e., $(ab)^* = b^*a^*$ for all $a$ and $b$ in $S$. It is very common to denote it by $E(S)$, the set of all idempotent elements of $S$, it means that $a^2 = a$ for all $a$ in $E(S)$. It is clear that $a' = a$ for all $a$ in $E(S)$.

A very important example of an inverse semigroup is given by $S = I(X)$ the set of all partial one-to-one maps on a set $X$. So each element of $I(X)$ is a bijection form a subset $U$ of $X$ onto another subset $V$ of $X$. The set $I(X)$ is a semigroup where the multiplication rule is given by composition of partial maps with the largest possible domain.

For example, if $\theta_1, \theta_2 \in I(X)$ with $\theta_1 : U_1 \rightarrow V_1$ and $\theta_2 : U_2 \rightarrow V_2$, then

$\theta_1 \theta_2 : \theta_1^{-1}(V_2 \cap U_1) \rightarrow \theta_2((V_2 \cap U_1)$

is given by:

$\theta_1(\theta_2(a)) = \theta_2(\theta(a)).$
The element $\theta^t_0$ is taken to be $\theta^{-1}$. It is easily checked that $I(X)$ is an inverse semigroup [3,5].

We recall that a relation $\leq$ on a set $X$ is called a partial ordering of $X$ if for all $a, b, c \in X$:

1) $a \leq a$

2) $a \leq b$ and $b \leq a$ implies $a = b$

3) $a \leq b$ and $b \leq c$ implies $a \leq c$.

The following example is of great importance to us. Define $e \leq f$ ($e, f \in E(S)$) to mean $ef = fe = e$. It is clear that $\leq$ is a partial ordering of $E(S)$. We shall call $\leq$ the natural partial ordering of $E(S)$.

An element $b$ of a partially ordered set $X$ is called an upper bound of a subset $Y$ of $X$, if $y \leq b$ for each $y \in Y$. An upper bound of $Y$ is called a least upper bound or join of $Y$, if $b \leq c$ for every upper bound $c$ of $Y$. If $Y$ has a join in $X$, it is clearly unique. Lower bound and greatest lower bound or meet can be defined similarly.

A partially ordered set $X$ is called a semilattice if every two elements subset $\{a, b\}$ of $X$ has a join and a meet in $X$; it implies that every finite subset of $X$ has both a join and a meet. The join (or meet) of $a$ and $b$ is called the meet (or join) of $a$ and $b$.

Definition 1.1 Suppose that $S$ is an inverse semigroup and $X$ can be assumed that as a locally compact Hausdorff topological space.

An action of $S$ on $X$ is a semigroup homomorphism as follows:

$$\alpha : S \rightarrow I(X)$$

$$a \mapsto \alpha_a$$

such that

1) for every $a \in S$ there is a continuous $\alpha_a$ with open domain in $X$.

2) the union of the domains of all the $\alpha_a$ coincides with $X$.

Proposition 1.2 Let $S$ be an inverse semigroup, $\alpha$ an action of $S$ on a set $X$ and $a \in S$, then

$$\alpha_a \circ \alpha_a = \alpha_a$$

and $\alpha_a \circ \alpha_a = \alpha_a$.

Proof: Since $\alpha$ is an action of $S$ on $X$ then $\alpha : S \rightarrow I(X)$ is a semigroup homomorphism, so for every $a \in S$ we have $\alpha(a)\alpha(a')\alpha(a) = \alpha(a)$, then $\alpha_a \circ \alpha_a = \alpha_a$, and simillarly $\alpha_a \circ \alpha_a = \alpha_a$.

With regard to the above text one may conclude that $\alpha_a = \alpha_a^{-1}$, and if $e \in E(S)$, so $\alpha_e$ is the identity map on its domain.

Since the range of each $\alpha_a$ coincides with the domain of $\alpha_a = \alpha_a^{-1}$, therefore it can be open as well as its domain. Also it can be mentioned that $\alpha_a^{-1}$, is continuous, so $\alpha_a$ is necessarily a homeomorphism onto its range.

For every $e \in E(S)$ the domain (and range) of $\alpha_e$ can be denoted by $E_e$, it means:

$$\alpha_e : E_e \rightarrow E_e.$$ 

It is clear to show that the domains of both $\alpha_e$ and $\alpha_e^{-1}$ are the same, and implies that the domain of $\alpha_e$ is $E_{\alpha_1}$. Likewise the range of $\alpha_e$ is given by $E_{\alpha_\lambda}$. Thus $\alpha_e : E_{\alpha_\lambda} \rightarrow E_{\alpha_{\lambda^*}}$ is a homeomorphism for every $a \in S$.

Briefly if $e$ and $f$ are in $E(S)$ then we have $\alpha_e \circ \alpha_f = \alpha_{ef}$ and $E_e \cap E_f = E_{ef}$.

Proposition 1.3 For each $a \in S$ and $e \in E(S)$ we have:

$$\alpha_e (E_e \cap E_{\alpha_1}) = E_{\alpha_e}.$$ 

Proof: Since N. Sieben [6], R. Exel [7] and Lawson [8] proved it, the authors use their result.

Definition 1.4 Let $\Sigma$ be the subset of $S \times X$ given by:

$$\Sigma = \{(ab) \in S \times X : b \in E_{\alpha_{ab}}\}$$

and for every $(a_1, b_1)$ and $(a_2, b_2)$ in $\Sigma$ we will say that $(a_1, b_1) \sim (a_2, b_2)$ if $b_1 = b_2$ and there exists an idempotent $e$ in $E(S)$ such that $b_1 \in E_e$, and $a_1 e = a_2 e$.

It is clearly that the relation $\sim$ is an equivalence relation on $\Sigma$. The equivalence class of $(a, b)$ will be denoted by $[a, b]$.

Let $G = \{[a, b] : a \in S, b \in X\}$ and put

$$G^2 = \{[[a_1, b_1], [a_2, b_2]] \in G \times G : b_1 = \alpha_{a_1}(b_2)\}$$

And for every $[[a_1, b_1], [a_2, b_2]] \in G^2$ define:

$$[[a_1, b_1], [a_2, b_2]] = [a_1 a_2, b_2]$$

$$[[a_1, b_1]]^{-1} = [a_1^{-1}, \alpha_{a_1}(b_1)]$$

it is easy to see that $G$ is a groupoid [3] and the unit space $G^{(0)}$ of $G$ naturally identifies with $X$ under the correspondence

$$[e, b] \in G^{(0)} \mapsto b \in X,$$

where $e$ is any idempotent such that $e \in E_e$. We show $G$ semigroup as $G(a, x, X)$.

We would now like to give $G$ is a topology. Let $a \in S$ and $U$ be an open subset of $E_{\alpha_1}$ we define $\psi(a, U)$ as follows:

$$\psi(a, U) = \{[a, b] \in G : b \in U\}$$

The collection of all $\psi(a, U)$ is the basis of a topology on $G$, and also the multiplication and inversion operations on $G$ are continuous, therefore $G$ is a topological groupoid.
2. Main Results

Recall from [2] that an inverse semigroup $S$ is naturally equipped with a partial order defined by:

$$a \leq b \iff a = ba^*a \forall a \in S$$

**Proposition 2.1** Assume that $S$ is an inverse semigroup which is a semilattice. Suppose that $a$ is an action of $S$ on a locally compact Hausdorff space $X$, such that for each $a \in S$, the domain $E_{s,a}$ of $a_s$ is closed. Then $G = G(a,S,X)$ is Hausdorff.

**Proof:** Suppose $[a,c]$ and $[b,d]$ are two distinct elements of $G(a,S,X)$. The aim is to find two disjoint open subsets $T_1$ and $T_2$ of $G(a,S,X)$ such that:

$$[a,c] \in T_1, [b,d] \in T_2, T_1 \cap T_2 = \emptyset$$

We consider two cases:

Case 1): If $(c \neq d)$.
Since $X$ is Hausdorff space then

$$\exists F_1, F_2 \subseteq X \text{ (open), } c \in F_1, d \in F_2, F_1 \cap F_2 = \emptyset$$

Now let $T_1 = \wp (a,F_1 \cap E_{s,a})$ and $T_2 = \wp (b,F_2 \cap E_{s,b})$

Since $T_1$ and $T_2$ are open set and

$$T_1 = \{(a,k) \in G : k \in F_1 \cap E_{s,a} \},$$

$$T_2 = \{(b,k) \in G : k \in F_2 \cap E_{s,b} \},$$

It is clearly that:

$$[a,c] \in T_1, [b,d] \in T_2 \text{ and } T_1 \cap T_2 = \emptyset$$

Case 2): If $(c = d)$.
Since $S$ is a semilattice let $h = a \land b$ so

$$h \leq a \to h = ah'h$$

$$h \leq b \to h = bh'h$$

Then referring to what proposed in Definition 1.4, $e \in E_{s,b}$, $E_{s,b}$ is closed then $T_2 = X \setminus E_{s,b}$ can be open and $c \in T_2$.

Now we can set $T$ as $T_2 \cap E_{s,b} \cap E_{s,b}$. But we know that $\psi (a,T) = \{(a,k) : k \in T \}$ and it is clear that $[a,c] \in \psi (a,T), [b,c] \in \psi (b,T)$.

To do so it is enough to prove that $\psi (a,T) \cap \psi (b,T) = \emptyset$.

Suppose that $[l,k] \in \psi (a,T) \cap \psi (b,T)$ then:

$$[l,k] \in \psi (a,T) \to [l,k] = [a,k] \to (l,k) \to (a,k) \to \exists e \in E(S), k \in E_s, ae = le$$

$$[l,k] \in \psi (b,T) \to [l,k] = [b,k] \to (l,k) \to (b,k) \to \exists f \in E(S), k \in E_s, bf = le$$

Since $\psi \in E(S)$ and $ef = fe$, $k \in E_s \cap E_{s'}$, it can be replaced $e$ and $f$ with $ef$ and finally we have:

$$aef = lef \iff lef = ef = bef$$

Therefore we can find an element $e \in E(S)$ such that $k \in E_s, ae = le, le = be$. So $(le') = (le) = aef'le' = le'le' = le$, then $le \leq a$, and similary $le \leq b$, since $h = a \land b$ thus $le \leq h$, then $le = le'h'$, hence $le" = le'h" \leq h"h$, and finally

$$k \in E_{r\epsilon \cap E_\epsilon} \subseteq E_{\epsilon^\epsilon}$$

But $k \in T$ which is a contradiction.

**Definition 2.2** A zero in an inverse semigroup $S$ is an element $0 \in S$ such that:

$$oa = a0 = 0 \forall a \in S$$

**Definition 2.3** An inverse semigroup $S$ with zero is said to be $E^*-$unitary if for every $e,a \in S$ one has that $e^* \neq e \leq a \Rightarrow a^* = a$.

In other words, if an element dominates a nonzero idempotent then that element itself is an idempotent.

**Proposition 2.4** If $S$ is an $E^*-$unitary inverse semigroup and $a,b$ belong to the defined semigroup $S$ such that $a^*a = b^*b$ and $ae = be$ for some nonzero idempotent $e \leq a^*a$ then $a = b$.

**Proof:** We define $x = aea^*$. So $x$ is nonzero idempotent because:

$$e \leq a^*a \Rightarrow e = (a^*a)^*(a^*a)e = a^*aa^*a$$

Then $e = a^*aa^*a$ (because of the ability of idempotent elements for being commute) and we have

$$ba'x = ba^*aa^* = bb'bea^* = bea^* = x.$$ 

Therefore, we have $x \leq ba'$. Since $S$ is an $E^*-$unitary which implies that $ba'$ is idempotent. Then $ba' = (ba')^* = ab'$ so $ab'$ is idempotent as well. But, we have

$$bb' = bb'bb' = ba'ab' = ab'ba = aa'aa' = aa.$$ 

Setting $y = ba'$, we have that

$$y' = ba'ba' = b'aa'a = b'a'aa' = b'a'a = b'b = b'bb'b = b'b'$$

Also $y = a'a'$, while

$$b = bb'bb = by' , \text{ and } a = aa'a = ay'y,$$

So it is enough to prove that $y' = ay'$. We have

$$ay' = ab'ba' = ab' = ba'bb'a = bb'ab' = by'.$$

In what follows we give the main result of this paper.

**Theorem 2.5** In condition that $S$ is an $E^*-$unitary inverse semigroup with zero, then can be appeared as a semilattice.

**Proof:** For proving the above theorem it is necessary to show that $a \land b$ exists for every $a,b \in S$. If there is not nonzero $h \in S$ such that $h \leq a,b$, it is obvious that
For doing this we can assume that there is a nonzero $h \in S$ in which $h^* h = h = bh^* h = bkh^* h = yh^* h$

Using the proposition (2.4) $x = y$ will be achieved and so

$$ab^* h = y^* y = bb^*$$

and finally

$$ab^* b = ba^*$$

By applying the above argument to $a^*, b^*, h^*$ and knowing that $h^* \neq 0$ and $h^* \leq a^*, b^*$ we have

$$a^* b^* b^* h = b^* a^*$$

so

$$\left(a^* b^* b^*\right)^* = \left(b^* a^*\right)^*$$

and therefore Equation (1) can be modified to

$$bb^* a = aa^* b$$

We have that $h \leq a, b$ then $h = ah^* h$ and $h = bh^* h$ , then we can show that

$$b^* a h^* h = b^* h h = h^* h$$

Since $S$ is a $E^*$-unitary and $b^* a$ is dominated by $h^* h$ , we have $\left(b^* a\right)^* = b^* a$. By applying the same reasoning to $a^*, b^*$ and $h^*$, $\left(b^* a\right)^* = ba^*$ can be a result.

Thus

$$\left(b^* a\right)^* = b^* a$$
$$\left(b^* a\right)^* = ba^*$$

and hence $ab^* b = ba^* b = bb^* a$

$$ab^* b = bb^* a$$

By combination of Equations (1) to (3), Equation (4) will be appeared.

$$ab^* b = ba^* a = bb^* a = aa^* b$$

At the end we try to prove that $ab^* b$ can satisfy the following condition

$$h \leq ab^* b \leq a, b$$

for every $h \in S$ such that $h \leq a, b$.

It is clear that $ab^* b \leq a, b$ and as defined before $k = a^* ab^* b$, then we have $h^* k \leq k$, and so

$$h = ab^* b = ah^* h = aa^* ab^* b h = ah^* bk = \left(ab^* b\right) h^* h$$

Finally $h \leq ab^* b$. It means that $ab^* b$ is the join of $a$ and $b$ and this is the proof of theorem.

3. References


