An Arbitrary (Fractional) Orders Differential Equation
with Internal Nonlocal and Integral Conditions

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Abstract

In this paper we study the existence of solution for the differential equation of arbitrary (fractional) orders
\[ \frac{dx}{dt} = f(t, D^x x(t)), \quad t \in (0,1) , \] with the general form of internal nonlocal condition
\[ \sum_{k=1}^{m} a_k x(\tau_k) = \beta \sum_{j=1}^{p} b_j x(\eta_j), \]
\[ \tau_k \in (a,c) \subseteq (0,1), \eta_j \in (d,b) \subseteq (0,1), c \leq d . \] The problem with nonlocal integral condition will be studied.

Keywords: Internal Nonlocal Problem, Integral Condition, Fractional Calculus, Existence of Solution, Caratheodory Theorem

1. Introduction

Problems with non-local conditions have been extensively studied by several authors in the last two decades. The reader is referred to ([1-10]), and references therein.

In this work we study the existence of at least one solution for the nonlocal problem of the arbitrary (fractional) order differential equation

\[ \frac{dx(t)}{dt} = f(t, D^x x(t)), \quad t \in (0,1) \quad \text{and} \quad \alpha \in (0,1] \] (1)

with the general nonlocal condition

\[ \sum_{k=1}^{m} a_k x(\tau_k) = \beta \sum_{j=1}^{p} b_j x(\eta_j), \] (2)

where \( \tau_k \in (a,c) \subseteq (0,1), \eta_j \in (d,b) \subseteq (0,1), c \leq d \) and \( \beta \geq 0 \) is parameter.

As an application, we deduce the existence of solution for the nonlocal problem of the differential (1) with the integral condition

\[ \int_{a}^{c} x(s) \, ds = \beta \int_{d}^{b} x(s) \, ds. \] (3)

It must be noticed that the following nonlocal and integral conditions are special cases of our nonlocal and integral conditions

\[ x(\tau) = \beta x(\eta), \quad \tau \in (a,c) \quad \text{and} \quad \eta \in (d,b), \] (4)

\[ \sum_{k=1}^{m} a_k x(\tau_k) = 0, \tau_k \in (a,c), \] (5)

\[ \sum_{k=1}^{m} a_k x(\tau_k) = \beta x(\eta), \quad \tau_k \in (a,c) \quad \text{and} \quad \eta \in (d,b), \] (6)

and

\[ \int_{a}^{c} x(s) \, ds = 0, \quad (a,c). \] (7)

2. Preliminaries

Let \( L^I \) denotes the class of Lebesgue integrable functions on the interval \( I = [a,b] \), with the norm \( \|x\|_L = \int |x(t)| \, dt \) and \( C(I) \) denotes the class of continuous functions on the interval \( I \), with the norm \( \|x\|_C = \sup_{t \in I} |x(t)| \) and \( \Gamma(\cdot) \) denotes the gamma function.

Definition 2.1 The fractional-order integral of the function \( f \in L^I[a,b] \) of order \( \beta \in \mathbb{R}^+ \) is defined by (see [11])

\[ I^\beta_x(t) = \int_{a}^{(t-s)\beta} f(s) \, ds. \]

Definition 2.2 The Caputo fractional-order derivative of
order $\alpha \in (0,1]$ of the absolutely continuous function $f(t)$ is defined by (see [11] and [12])

$$D^\alpha_a f(t) = \frac{d}{dt} I^{1-\alpha}_a f(t).$$

**Definition 2.3** The function $f:[0,1] \times R \to R$ is called $L^\alpha$-Carathödogy if
1) $t \to f(t,x)$ is measurable for each $x \in R$,
2) $x \to f(t,x)$ is continuous for almost all $t \in [0,1]$,
3) there exists $m \in L([0,1],D), D \subset R$ such that $|f| \leq m$.

Now we state Carathödogy Theorem ([13]).

**Theorem 2.1** Let $f:[0,1] \times R \to R$ be $L^\alpha$-Carathödogy, then the initial-value problem

$$\frac{dx(t)}{dt} = f(t,x(t)), \quad x(0) = x_0 \quad (9)$$

has at least one absolutely continuous solution $x \in AC[0,T]$.

Here we generalize Carathödogy theorem for the nonlocal problem (1) - (2).

### 3. Main Results

Consider firstly the fractional-order integral equation

$$y(t) = I^{1-\alpha} f(t,y(t)), \quad (10)$$

**Definition 3.1** The function $y$ is called a solution of the fractional-order integral Equation (10), if $y \in C[0,1]$ and satisfies (10).

**Theorem 3.1** Let $f:[0,1] \times R \to R$ be $L^\alpha$-Carathödogy. Then there exists at least one solution of the fractional-order integral Equation (10).

**Proof.** Let

$$M = \text{Max} \left\{ I^\alpha_a m(t) : t \in (0,1), \alpha \geq 0 \text{ and } \beta \in (0,1) \right\},$$

then

$$\left| I^\beta_a f(t,y(t)) \right| \leq I^{\min(\beta,1)}_a \left| f(s,y(s)) \right| ds$$

$$\leq I^{\min(\beta,1)}_a m(s) ds \leq M, \quad \alpha \geq 0.$$ 

Define the sequence $\{y_n(t)\}$ by

$$y_{n+1}(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(1-\alpha)} f(s,y_n(s)) ds, \quad t \in [0,1]$$

which can be written in the operator form

$$y_{n+1}(t) = I^{1-\alpha} I^\beta f(t,y_n(t)).$$

Then

$$|y_{n+1}(t)| \leq I^{1-\alpha} \left| I^\beta f(t,y_n(t)) \right| \leq M \int_0^t \frac{(t-s)^{\alpha-\beta}}{\Gamma(1-\alpha-\beta)} ds$$

$$\leq M \frac{(t)^{1-\beta}}{\Gamma(2-\alpha-\beta)} \leq \frac{M}{\Gamma(2-\alpha-\beta)}$$

For $t_1,t_2 \in [0,1]$ such that $t_1 < t_2$, then

$$y_{n+1}(t_2) - y_{n+1}(t_1) = \int_{t_1}^{t_2} \frac{(t-s)^{\alpha-1}}{\Gamma(1-\alpha)} f(s,y_n(s)) ds$$

$$- \int_{t_1}^{t_2} \frac{(t-s)^{\alpha-1}}{\Gamma(1-\alpha)} f(s,y_n(s)) ds$$

$$= \int_{t_1}^{t_2} \frac{(t-s)^{\alpha-1}}{\Gamma(1-\alpha)} f(s,y_n(s)) ds$$

$$- \int_{t_1}^{t_2} \frac{(t-s)^{\alpha-1}}{\Gamma(1-\alpha)} f(s,y_n(s)) ds$$

$$\leq \frac{(t_2-t_1)^{\alpha-1}}{\Gamma(1-\alpha)} f(s,y_n(s)) ds$$

$$+ \frac{(t_2-t_1)^{\alpha-1}}{\Gamma(1-\alpha)} f(s,y_n(s)) ds$$

$$- \frac{(t_2-t_1)^{\alpha-1}}{\Gamma(1-\alpha)} f(s,y_n(s)) ds.$$

Therefore

$$|y_{n+1}(t_2) - y_{n+1}(t_1)| \leq \int_{t_1}^{t_2} \frac{(t-s)^{\alpha-1}}{\Gamma(1-\alpha)} m(s) ds$$

$$\leq \int_{t_1}^{t_2} \frac{(t-s)^{\alpha-1}}{\Gamma(1-\alpha)} m(\theta) d\theta \leq M \int_{t_1}^{t_2} \frac{(t-s)^{\alpha-1}}{\Gamma(1-\alpha-\beta)} ds$$

$$\leq M \frac{(t_2-t_1)^{1-\alpha}}{\Gamma(2-\alpha-\beta)}.$$

Hence

$$|t_2 - t_1| < \delta \Rightarrow |y_{n+1}(t_2) - y_{n+1}(t_1)| < \epsilon(\delta)$$

and $\{y_n(t)\}$ is a sequence of equi-continuous and uniformly bounded functions. By Arzela-Ascoli Theorem, ([14] and [15]) there exists a subsequence $\{y_{n_k}(t)\}$ of continuous functions which converges uniformly to a continuous function $y$ as $k \to \infty$.

Now we show that this limit function is the required solution.

Since

$$|f(s,y_{n_k}(s))| \leq m(s) \in L^1,$$

and $f(s,y_{n_k}(s))$ is continuous in the second argument,
in Equation (13), we get
\[ y(t) = I^{1-a} (t^a f(t, y(t))) ds = y(t), \]
which proves the existence of at least one solution
\[ y \in C[0,1] \] of the fractional-order functional integral
Equation (10).

For the existence of solution for the nonlocal problem
(1) - (2) we have the following theorem.

**Theorem 3.2** Let the assumptions of Theorem 3.1 are satisfied. Then nonlocal problem (1) - (2) has at least one
solution \( x \in AC[0,1]. \)

**Proof.** Consider the nonlocal problem (1) - (2).

Let \( y(t) = D^a x(t), \) then
\[ y(t) = I^{1-a} \frac{dx(t)}{dt}, \quad (11) \]
\[ y(t) = I^{1-a} f(t, y(t)) \quad (12) \]
and \( y \) is the solution of the fractional-order integral
Equation (10).

Operating by \( I^a \) on both sides of Equation (11), we obtain
\[ I^a y(t) = I \frac{dx(t)}{dt} = x(t) - x(0) \Rightarrow \]
\[ x(t) = x(0) + I^a y(t). \quad (14) \]

Let \( t = t_k \) in Equation (13), we get
\[ \sum_{k=1}^{m} a_k x(t_k) = \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{a-1}}{\Gamma(a)} y(s) ds + x(0) \sum_{k=1}^{m} a_k. \]
And let \( t = \eta_j \) in Equation (13), we get
\[ \sum_{j=1}^{p} b_j x(\eta_j) = \sum_{j=1}^{p} b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{a-1}}{\Gamma(a)} y(s) ds + x(0) \sum_{j=1}^{p} b_j. \]

From Equation (2), we get
\[ \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{a-1}}{\Gamma(a)} y(s) ds + x(0) \sum_{k=1}^{m} a_k \]
\[ = \beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{a-1}}{\Gamma(a)} y(s) ds + x(0) \beta \sum_{j=1}^{p} b_j. \]

Then we get
\[ x(0) = A \left( \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{a-1}}{\Gamma(a)} y(s) ds \right) \]
\[ - \beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{a-1}}{\Gamma(a)} y(s) ds \]
and
\[ x(t) = A \left( \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{a-1}}{\Gamma(a)} y(s) ds \right) \]
\[ - \beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{a-1}}{\Gamma(a)} y(s) ds \quad (15) \]
where
\[ A = \left( \beta \sum_{j=1}^{p} b_j - \sum_{k=1}^{m} a_k \right)^{-1} \]
which, by Theorem 3.1, has at least one solution
\( x \in AC(0,1). \)

Now, from Equation (15), we have
\[ x(0) = \lim_{t \to 0^+} x(t) = A \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{a-1}}{\Gamma(a)} y(s) ds \]
\[ - \beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{a-1}}{\Gamma(a)} y(s) ds \]
and
\[ x(1) = \lim_{t \to 1^-} x(t) = A \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{a-1}}{\Gamma(a)} y(s) ds \]
\[ - \beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{a-1}}{\Gamma(a)} y(s) ds + \int_0^{(1-s)^{\alpha-1}} \frac{1}{\Gamma(\alpha)} y(s) ds \]
from which we deduce that Equation (15) has at least one solution
\( x \in AC[0,1]. \)

To complete the proof, differentiating (15), we obtain
\[ \frac{dx}{dt} = y(t) = f(t, D^a x(t)). \]

Also from (15) we can prove that the solution satisfies
the nonlocal condition (2).

### 4. Nonlocal Integral Condition

Let \( x \in AC[0,1]. \) be the solution of the nonlocal problem
(1) - (2).

Let \( a_k = t_{k-1} - t_k, t_k \in (t_{k-1}, t_k), a = t_0 < t_1 < \cdots < t_m = c \)
and \( b_j = s_j - s_{j-1}, \eta_j \in (s_{j-1}, s_j), d = s_0 < s_1 < s_2, \ldots < s_p = b \)
then the nonlocal condition (2) will be
\[
\sum_{k=1}^{p} (t_k - t_{k-1}) x(t_k) = \beta \sum_{j=1}^{p} (s_j - s_{j-1}) x(\eta_j).
\]

From the continuity of the solution \( x \) of the nonlocal problem (1) - (2) we can obtain
\[
\lim_{m \to \infty} \sum_{k=1}^{m} (t_k - t_{k-1}) x(t_k) = \beta \lim_{m \to \infty} \sum_{j=1}^{m} (s_j - s_{j-1}) x(\eta_j).
\]
and the nonlocal condition (2) transformed to the integral one
\[
\int_{a}^{c} x(s) \, ds = \beta \int_{a}^{c} x(s) \, ds. \quad (16)
\]

Now, we have the following theorem.

**Theorem 4.1** Let the assumptions of Theorem 3.2 are satisfied. Then there exist at least one solution \( x \in AC[0,1] \) of the nonlocal problem with integral condition,
\[
x'(t) = f(t, D^n x(t)), \quad t \in (0,1),
\]
\[
\int_{a}^{c} x(s) \, ds = \beta \int_{a}^{c} x(s) \, ds, \quad \beta (b-d) \neq (c-a).
\]

Letting \( \beta = 0 \) in (16), the we can easily prove the following corollary.

**Theorem 4.2** Let the assumptions 1) - 2) are satisfied. Then the nonlocal problem
\[
x'(t) = f(t, D^n x(t)), \quad t \in (0,1),
\]
\[
\int_{a}^{c} x(s) \, ds = 0, \quad (a,c) \subset (0,1)
\]
has at least one solution \( x \in AC[0,1] \).

### 5. References


