A Note on Convergence of a Sequence and its Applications to Geometry of Banach Spaces

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Received January 9, 2011; revised April 22, 2011; accepted April 30, 2011

Abstract


Keywords: Locally Quasi-Nonexpansive, Biased Quasi-Nonexpansive, Conditionally Biased Quasi-Nonexpansive, Drop, Super Drop

1. Introduction

In the last four decades of the last century, there have been a multitude of results on fixed points of nonexpansive and quasi-nonexpansive mappings in Banach spaces (e.g., [5-7, 9-11]).

Our aim in this note is to point out several obscure places in the results of Ahmed and Zeyada [J. Math. Anal. Appl. 274 (2002) 458-465]. In order to rectify and improve the results of Ahmed and Zeyada, we introduce the concepts of locally quasi-nonexpansive, biased quasi-nonexpansive and conditionally biased quasi-nonexpansive of a mapping w.r.t. a sequence in metric spaces.

Let \( X \) be a metric space and \( D \) a nonempty subset of \( X \). Let \( T \) be a mapping of \( D \) into \( X \) and let \( F(T) \) be the set of all fixed points of \( T \). For a given \( x_0 \in D \), the sequence of iterate \( \{x_n\} \) is determined by

\[ x_n = T(x_{n-1}) = T^n(x_0), n = 1, 2, 3 \ldots \] (I)

Let \( X \) be a normed space, \( \lambda \in (0,1) \) and \( \mu \in (0,1) \), the sequence of iterates \( \{x_n\} \) are defined by

\[ x_n = T_\lambda(x_{n-1}) = T_\lambda^n(x_0), \]

\[ T_\lambda = \lambda I + (1-\lambda)T, n = 1, 2, 3 \ldots \] (II)

\[ x_n = T_{\lambda,\mu}(x_{n-1}) = T_{\lambda,\mu}^n(x_0), \]

\[ T_{\lambda,\mu} = (1-\lambda)I + \lambda T[(1-\mu)I + \mu T], \] (III)

The iteration scheme (I) is called Teoplitz iteration and the iteration scheme (II) was introduced by Mann [12] while the iteration scheme (III) was introduced by Ishikawa [9].

The concept of quasi-nonexpansive mapping was initiated by Tricomi in 1941 for real functions. It was further studied by Diaz and Metcalf [5] and Doston [6, 7] for mappings in Banach spaces. Recently, this concept was given by Kirk [10] in metric spaces as follows:

Definition 1.1. The mapping \( T \) is said to be quasi-nonexpansive if for each \( x \in D \) and for every \( p \in F(T) \), \( d(T(x), p) \leq d(x, p) \). A mapping \( T \) is conditionally quasi-nonexpansive if it is quasi-nonexpansive whenever \( F(T) \neq \emptyset \).

We now introduce the following definition:

Definition 1.2. The mapping \( T \) is said to be locally quasi-nonexpansive at \( p \in F(T) \) if for each \( x \in D \), \( d(T(x), p) \leq d(x, p) \).

Obviously, quasi-nonexpansive locally quasi-nonexpansive at each \( p \in F(T) \) but the reverse implication...
may not be true. To this end, we observe the following example.

**Example 1.1.** Let \( X = \{0, 1\} \) and \( D = \left[ \frac{3}{4}, 0 \right] \) be endowed with the Euclidean metric \( d \). Define the mapping \( T : D \to X \) by \( T(x) = \frac{3}{2} x^2 \) for each \( x \in D \). Then we observe that \( F(T) = \left\{ \frac{2}{3} \right\} \) for all \( x \in D \) and \( p = 0 \) \( \in F(T) \), we have that

\[
d(T(x), p) = \frac{3}{2} x^2 - 0 \leq |x - 0| = d(x, p),
\]

i.e., \( T \) is locally quasi-nonexpansive at \( p = 0 \in F(T) \). However, one can easily see that \( T \) is not locally quasi-nonexpansive at \( p = \frac{2}{3} \in F(T) \). Indeed, for all \( x \in \left( 0, \frac{2}{3} \right) \) and \( p = \frac{2}{3} \in F(T) \) we have

\[
d(T(x), p) = \frac{3}{2} x^2 - \frac{2}{3} \geq |x - \frac{2}{3}| = d(x, p).
\]

Hence we conclude that \( T \) is not quasi-nonexpansive, although it is locally quasi-nonexpansive at \( p = 0 \in F(T) \).

The concept of asymptotic regularity was formally introduced by Browder and Petryshyn [3] for mappings in Hilbert spaces. Recently, it was defined by Kirk [11] in metric spaces as follows:

**Definition 1.3.** The mapping \( T \) is said to be asymptotically regular if \( \lim_{n \to \infty} d(T^n(x), T^{n+1}(x)) = 0 \) for each \( x \in D \).

### 2. Main Results

Let \( \mathbb{N} \) denote the set of all positive integers and \( \omega = \mathbb{N} \cup \{0\} \). Ahmed and Zeyada [1] introduced the following:

**Definition 2.1.** The mapping \( T \) is said to be quasi-nonexpansive w.r.t. a sequence \( \{x_n\} \) if for all \( n \in \omega \) and for each \( p \in F(T) \), \( d(x_n, p) \leq d(x_{n+1}, p) \).

The following lemma was quoted by Ahmed and Zeyada [1] without proof.

**Lemma A.** If \( T \) is quasi-nonexpansive, then \( T \) is quasi-nonexpansive w.r.t. a sequence \( \{T^n x_0\} \) (respectively, \( \{T^n_0 x_0\} \) ) for each \( x_0 \in D \).

**Remark 2.1.** We notice that the above lemma is valid if \( \{T^n x_0\} \in D \) for each \( n \in \omega \) and a given \( x_0 \in D \) (or \( D \) is \( T \)-invariant). So the correct version of Lemma A should be read as follows:

**Lemma 2.1.** If \( T \) is quasi-nonexpansive and for a given \( x_0 \in D \) and each \( n \in \omega \), \( \{T^n x_0\} \in D \), then \( T \) is quasi-nonexpansive w.r.t. a sequence \( \{T^n x_0\} \) (respectively, \( \{T^n_0 x_0\} \) ) for each \( x_0 \in D \).

Further, they claimed that the reverse implication in Lemma A may not be true in their Example 2.1. We again notice that there are several obscure places in this example. We now quote Example 2.1 of Ahmed and Zeyada [1] in the following:

**Example A.** Let \( X = \{0, 1\} \) and \( D = \left[ \frac{4}{5}, 0 \right] \) be endowed with the Euclidean metric \( d \). Define the mapping \( T : D \to X \) by \( T(x) = 2x^2 \) for each \( x \in D \).

For a given \( x_0 = \frac{1}{4} \in D \) we have

\[
d(T^{n+1}(x_0), p) = \left( \frac{1}{2} \right)^{2^n} - 0 \leq \left( \frac{1}{2} \right)^{2^n} - 0 = d(T^n(x_0), p)\]

where \( T^n(1/4) = (1/2)^{2^n} \in D \forall n \in \mathbb{N} \cup \{0\} \) and \( F(T) = \{0\} \), i.e., \( T \) is quasi-nonexpansive w.r.t. a sequence \( T^n(1/4) \). Furthermore, the map \( T \) is quasi-nonexpansive w.r.t. a sequence \( \{T^n_0(1/2)\} \) and \( \{T^n_{0,1}(1/2)\} \). They found that \( T \) is neither conditionally quasi-nonexpansive nor quasi-nonexpansive, for \( x = \frac{3}{4} \in D \) and \( p = 0 \in F(T), d(3/4, 0) > d(3/4, 0) \) and \( D \) is not closed.

**Remark 2.2.** We notice that the following claims made in Example A were false:

1. \( T : D \to X \) is a mapping. In fact,

\[
T(D) = \left[ \frac{0, 32}{25} \right] \ni \{0, 1\} = X.
\]

2. \( F(T) = \{0\} \). In fact, \( F(T) = \left[ \frac{0, 1}{2} \right] \).

3. \( T \) is quasi-nonexpansive w.r.t. a sequence \( \{T^n(1/4)\} \).

4. \( T \) is quasi-nonexpansive w.r.t. a sequence \( \{T^n_0(1/2)\} \) and \( \{T^n_{0,1}(1/2)\} \).

However, (i) can be rectified by taking \( X \) as \( \left[ \frac{0, 32}{25} \right] \) or any superset of \( \left[ \frac{0, 32}{25} \right] \) in \( [0, \infty) \) Even if this correction is made we find that the remaining statements 2) - 4) will remain false. Consequently, the claim of Ahmed and Zeyada [1] that the reverse implication in Lemma A may not be true seems false.

We now introduce the following definition.

**Definition 2.2.** The mapping \( T \) is said to be locally quasi-nonexpansive at \( p \in F(T) \) w.r.t. a sequence \( \{x_n\} \)
if for all $n \in \omega$, $d(x_{n,1},p) \leq d(x_n,p)$.

Obviously, locally quasi-nonexpansiveness at $p \in F(T)$ ⇒ locally quasi-nonexpansiveness at $p \in F(T)$ w.r.t. a sequence $\{x_n\}$.

We now state the following lemma without proof.

**Lemma 2.2.** If $T$ is quasi-nonexpansive w.r.t. a sequence $\{x_n\}$ then $T$ is locally quasi-nonexpansive at each $p \in F(T)$ w.r.t. the sequence $\{x_n\}$.

The reverse implication in Lemma 2.2 may not be true as shown in the following example:

**Example 2.1.** Let $X = [0,1]$ and $D = \left[ 0, \frac{2}{3} \right]$ be endowed with the Euclidean metric $d$. Define the mapping $T : D \to X$ by $T(x) = 2x^2$ for each $x \in D$. Then we observe that $F(T) = \left\{ 0, \frac{1}{2} \right\}$. For a given $x_0 = \frac{1}{e} \in D$ and $p = 0 \in F(T)$ we have that

\[
d(T^{n+1}(x_0),p) = d(T^n(x_0),p),
\]

where $T^n \left( \frac{1}{4} \right) = \left( \frac{1}{2} \right)^{2^n}$, i.e., $T$ is locally quasi-nonexpansive at $p = 0 \in F(T)$ w.r.t. a sequence $\{T^n \left( \frac{1}{4} \right)\}$. However, one can easily see that $T$ is not locally quasi-nonexpansive at $p = \frac{1}{2} \in F(T)$ w.r.t. the sequence $\{T^n \left( \frac{1}{4} \right)\}$. Indeed, we have

\[
d(T^{n+1}(x_0),p) = d(T^n(x_0),p)
\]

for all $n \in \omega$. Consequently, $T$ is neither quasi-nonexpansive nor quasi-nonexpansive w.r.t. the sequence $\{T^n \left( \frac{1}{4} \right)\}$.

We now introduce the following:

**Definition 2.3.** The mapping $T : D \to X$ is said to be biased quasi-nonexpansive (b.q.n) w.r.t. a sequence $\{x_n\} \subset X$ if for all $n \in \omega$ and at each $p \in \text{cond}(F(T))$, $d(x_{n+1},p) \leq d(x_n,p)$ where

\[
\text{cond}(F(T)) = \left\{ p \in F(T) : \limsup_{n \to \infty} d(x_n,p) \leq \liminf_{n \to \infty} d(x_n,F(T)) \right\}
\]

A mapping $T$ is conditionally biased quasi-nonexpansive (c.b.q.n) w.r.t. a sequence $\{x_n\}$ if $\text{cond}(F(T)) \neq \emptyset$.

**Remark 2.3.** We observe that the following implications are obvious:

(a) Conditionally biased quasi-nonexpansiveness w.r.t. a sequence $\{x_n\}$ ⇒ biased quasi-nonexpansiveness w.r.t. a sequence $\{x_n\}$ but the reverse implication may not be true (Indeed, any mapping $T : D \to X$ for which $\text{cond}(F(T)) \neq \emptyset$ is a biased quasi-nonexpansive w.r.t. a sequence $\{x_n\}$ but not conditionally biased quasi-nonexpansive w.r.t. a sequence $\{x_n\}$. However, under certain conditions a biased quasi-nonexpansive map w.r.t. a sequence $\{x_n\}$ may be a conditional biased quasi-nonexpansive w.r.t. a sequence $\{x_n\}$ (see Lemma 2.6 below).

(b) If $T$ is conditionally biased quasi-nonexpansive w.r.t. a sequence $\{x_n\}$ and $\text{cond}(F(T)) = F(T) \neq \emptyset$ then $T$ is locally quasi-nonexpansive at each $p \in F(T)$ w.r.t. a sequence $\{x_n\}$.

(c) If $T$ is biased quasi-nonexpansive w.r.t. a sequence $\{x_n\}$ and $\emptyset \neq \text{cond}(F(T)) \neq F(T)$ then $T$ is locally quasi-nonexpansive at each $p \in \text{cond}(F(T))$ w.r.t. a sequence $\{x_n\}$.

(d) Quasi-nonexpansiveness ⇒ locally quasi-nonexpansiveness at $p \in F(T)$ ⇒ locally quasi-nonexpansiveness at $p \in F(T)$ w.r.t. a sequence $\{x_n\}$.

In Example 2.1 above, we observe that

1) for $p = 0 \in F(T)$, we have

\[
\limsup_{n \to \infty} d(x_n,p) = \limsup_{n \to \infty} \left( \frac{1}{2} \right)^{2^n + 1} - \frac{1}{2} = 0
\]

\[
= \lim_{n \to \infty} \left( \frac{1}{2} \right)^{2^n + 1} = 0
\]

2) for $p = \frac{1}{2} \in F(T)$, we have

\[
\limsup_{n \to \infty} d(x_n,p) = \limsup_{n \to \infty} \left( \frac{1}{2} \right)^{2^n + 1} - \frac{1}{2} = \frac{1}{2}
\]

\[
= \lim_{n \to \infty} \left( \frac{1}{2} \right)^{2^n + 1} - \frac{1}{2} = \frac{1}{2}
\]

\[
\liminf_{n \to \infty} d(x_n,F(T)) = \liminf_{n \to \infty} \left( \frac{1}{2} \right)^{2^n + 1} = \lim_{n \to \infty} \left( \frac{1}{2} \right)^{2^n + 1} = 0
\]
Here \( \text{cond}(F(T)) \neq \{0\} \) and in view of (*) and (**), it is evident that \( T \) is conditionally biased quasi-nonexpansive (c.b.q.n.) w.r.t. a sequence \( \left\{ T^n \left( \frac{1}{4} \right) \right\} \) and hence it is biased quasi-nonexpansive (b.q.n.) w.r.t. a sequence \( \left\{ T^n \left( \frac{1}{4} \right) \right\} \).

We now show in the following example that \( \text{cond}(F(T)) \) need not be a singleton set.

**Example 2.2.** Let \( X = [0, 2] \) and \( D = [0, 1) \cup (1, 2] \) be endowed with the Euclidean metric \( d \). Define the mapping \( T : D \to X \) by \( Tx = +\sqrt{x} \) for \( x \in [0, 1) \cup (1, 2] \) and \( T(x) = 2 \) for \( x = 2 \). Clearly, \( F(T) = \{0, 2\} \). Consider the sequence \( \{x_n\} = \{1\} \) in \( X \) then we observe that

1. for \( p = 0 \in F(T) \), we have
   \[
   \lim_{n \to \infty} d(x_n, p) = \lim_{n \to \infty} 1 = 1;
   \]
2. for \( p = 2 \in F(T) \), we have
   \[
   \lim_{n \to \infty} d(x_n, p) = \lim_{n \to \infty} 1 = 1;
   \]

and
   \[
   \lim_{n \to \infty} d(x_n, F(T)) = \lim_{n \to \infty} 1 = 1.
   \]

Thus we have \( \text{cond}(F(T)) = \{0, 2\} \) and it is evident that \( T \) is conditionally biased quasi nonexpansive (c.b.q.n.) w.r.t. the sequence \( \{x_n\} = \{1\} \) in \( X \), and hence it is biased quasi-nonexpansive (b.q.n.) w.r.t. the sequence \( \{x_n\} = \{1\} \) in \( X \).

However, interested reader can check that if we consider the sequence \( \{x_n\} \) such that \( x_n \to 1 \) then \( \text{cond}(F(T)) = \{2\} \). Further, we observe that for \( p = 2 \in \text{cond}(F(T)) \) and for all \( n \in \mathbb{N} \) we have
   \[
   d(x_n, p) \leq d(x_n, p)
   \]

Thus, \( T \) is conditionally biased quasi-nonexpansive (c.b.q.n.) w.r.t. the sequence \( \{x_n\} \) in \( X \).

On the other hand, if we consider the sequence \( \{x_n\} \) such that \( x_n \to 1 \) then \( \text{cond}(F(T)) = \{0\} \) and \( T \) is conditionally biased quasi-nonexpansive (c.b.q.n.) w.r.t. the sequence \( \{x_n\} \) in \( X \).

**Remark 2.4.** Example 2.2 above also shows that \( \text{cond}(F(T)) \) is a closed set even though \( T \) is discontinuous at \( p = 2 \).

We need the following lemmas to prove our main theorem:

**Lemma 2.3.** Let \( T \) be locally quasinonexpansive at \( p \in F(T) \) w.r.t. \( \{x_n\} \) and \( \lim_{n \to \infty} d(x_n, F(T)) = 0 \). Then \( \{x_n\} \) is a Cauchy sequence.

The proof is straightforward and hence is omitted.

**Proof.** Since \( \lim_{n \to \infty} d(x_n, F(T)) = 0 \) then for any given \( \varepsilon > 0 \) there exists \( n_1 \in \mathbb{N} \) such that for each \( n \geq n_1 \),
   \[
   d(x_n, F(T)) < \frac{\varepsilon}{2}
   \]
   So, there exists \( q \in F(T) \) such that
   \[
   d(x_n, q) < \frac{\varepsilon}{2},
   \]
   for all \( n \geq n_1 \).

Thus, for any \( m, n \geq n_1 \) we have
   \[
   d(x_n, x_m) \leq d(x_n, q) + d(x_m, q) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad q \in F(T),
   \]
   Hence \( \{x_n\} \) is a Cauchy sequence.

**Lemma 2.4.** Let \( T \) be conditionally biased quasi-nonexpansive w.r.t. \( \{x_n\} \), and \( \lim_{n \to \infty} d(x_n, F(T)) = 0 \). Then:
1. \( \{x_n\} \) converges to a point \( p \) in \( \text{cond}(F(T)) \) and \( T \) is locally quasi-nonexpansive at \( p \in \text{cond}(F(T)) \) w.r.t. \( \{x_n\} \).
2. \( \{x_n\} \) is a Cauchy sequence.

**Proof.** 1) Since \( T \) is conditionally biased quasi-nonexpansive w.r.t. \( \{x_n\} \), it follows that \( \text{cond}(F(T)) \neq \emptyset \). As \( \lim_{n \to \infty} d(x_n, F(T)) = 0 \) we have that \( \lim sup d(x_n, p) = 0 \) for some \( p \in \text{cond}(F(T)) \). So, we have \( \lim_{n \to \infty} d(x_n, p) = 0 \) for some \( p \in \text{cond}(F(T)) \); i.e.,
   \[
   \{x_n\} \text{ converges to a point } p \text{ in } \text{cond}(F(T)) \text{ and } T \text{ is locally quasi-nonexpansive at } p \in \text{cond}(F(T)) \text{ w.r.t. } \{x_n\}.
   \]
2) From \( \lim_{n \to \infty} d(x_n, p) = 0 \) it follows that for any given \( \varepsilon > 0 \) there exists \( n_1 \in \mathbb{N} \) such that for each \( n \geq n_1, d(x_n, p) < \frac{\varepsilon}{2} \). Thus, for any \( m, n \geq n_1 \), we have
   \[
   d(x_n, x_m) \leq d(x_n, q) + d(x_m, q) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad q \in F(T),
   \]
   Hence \( \{x_n\} \) is a Cauchy sequence.

The following lemma follows easily.

**Lemma 2.5.** Let \( T \) be biased quasi-nonexpansive w.r.t. \( \{x_n\} \), and let \( \{x_n\} \) converges to a point \( p \) in \( F(T) \). Then:
1. \( \{x_n\} \) converges to a point \( p \) in \( \text{cond}(F(T)) \) and \( T \) is conditionally biased quasi-nonexpansive w.r.t. \( \{x_n\} \).
2. \( \{x_n\} \) is a Cauchy sequence.

We now state our main theorem in the present paper.

**Theorem 2.1.** Let \( F(T) \) be a nonempty closed set.

Then
1. \( \lim_{n \to \infty} d(x_n, F(T)) = 0 \) if \( \{x_n\} \) converges to a point \( p \) in \( F(T) \);
2. \( \{x_n\} \) converges to a point in \( F(T) \) if...
lim d(x_n, F(T)) = 0, \ T \text{ is locally quasi-nonexpansive at } p \in F(T) \text{ w.r.t. } \{x_n\} \text{ and } X \text{ is complete.}

Proof. 1) Since F(T) is closed, p \in F(T) and the mapping \ x \mapsto d(x, F(T)) \text{ is continuous (see [1, p. 13]), then}
\lim d(x_n, F(T)) = d\left(\lim x_n, F(T)\right) = d(p, F(T)) = 0

2) From Lemma 2.3, \{x_n\} is a Cauchy sequence. Since X is complete, then \{x_n\} converges to a point, say q in X. Since F(T) is closed, then
\lim d(x_n, F(T)) = d\left(\lim x_n, F(T)\right) = d(p, F(T))
implies that q \in F(T).

As consequences of Theorem 2.1, we have the following:

**Corollary 2.1.** Let F(T) a nonempty closed set and for a given x_0 \in D and each n \in \omega, \{T^nx_0\} \in D Then
1) lim d(T^n x_0, F(T)) = 0 if \{T^n x_0\} converges to a point p \in F(T);
2) \{T^n x_0\} converges to a point in F(T) if,
\lim d(T^n x_0, F(T)) = 0, \ T \text{ is locally quasi-nonexpansive at } p \in F(T) \text{ w.r.t. } \{T^n x_0\} \text{ and } X \text{ is complete.}

**Corollary 2.2.** Let X be a normed linear space, F(T) a nonempty closed set and for a given x_0 \in D and each n \in \omega, \{T^nx_0\} \in D .

1) If the sequence \{T^nx_0\} converges to a point p in F(T), then
\lim d(T^nx_0, F(T)) = 0
2) If \lim_{n \to \infty} d(T^nx_0, F(T)) = 0 \ T \text{ is locally quasi-nonexpansive at } p \in F(T) \text{ w.r.t. } \{T^nx_0\} \text{ and } X \text{ is complete, then} \{T^nx_0\} \text{ converges to a point p in } F(T)

**Corollary 2.3.** Let X be a normed linear space, F(T) a nonempty closed set and for a given x_0 \in D and each n \in \omega, \{T^nx_0\} \in D . Then
1) lim d(T^nx_0, x, F(T)) = 0 if the sequence \{T^nx_0\} converges to a point p in F(T);
2) \{T^nx_0\} converges to a point p in F(T) if,
\lim d(T^nx_0, F(T)) = 0, \ T \text{ is locally quasi-nonexpansive at } p \in F(T) \text{ w.r.t. } \{T^nx_0\} \text{ and } X \text{ is complete.}

Note that the continuity of T implies that F(T) is closed but the converse need not be true. To effect this consider the following example.

**Example 2.3.** Let X = [0, \infty) and D = [0, 1) be endowed with the Euclidean metric d. Define the mapping T : D \to X by T(x) = x if x \in [0, \frac{1}{2}] and T(x) = 3x^2 if x \in (\frac{1}{2}, 1]
Obviously, F(T) = [0, 1/2] is a nonempty closed but T is not continuous at x = 1/2.

**Remark 2.5.** (a) In order to support the above fact Ahmed and Zeyada [1] stated wrongly in their Example 2.2, where X = [0, 1), D = [0, 1/4) \cup (1/2, 5/6), T(x) = x.
If X \in [0, 1/4) and \ T(x) = x/2 if x \in (1/2, 5/6) that T is not continuous. In fact, we observe that in this example T is continuous.

(b) From Lemma 2.1, Examples 2.1 and 2.3, the continuity of T implies that F(T) is closed but the converse may not be true; then we have that Corollaries 2.1, 2.2 and 2.3 are improvement of Theorem 1.1 in [13, p. 462], Theorem 1.1’ in [13, p. 469], and Theorem 3.1 in [8, p. 98], respectively.

(c) Since every quasi-nonexpansive map w.r.t. a sequence \{x_n\} is locally quasi-nonexpansive at each p \in F(T) w.r.t. a sequence \{x_n\}, but the converse may not be true; we have that Theorem 2.1, Corollaries 2.1, 2.2 and 2.3 are improvement of corresponding Theorem 2.1, Corollary 2.1, 2.2 and 2.3 of Ahmed and Zeyada [1].

(d) By considering the closedness of F(T) in lieu of the continuity of T and \ T : D \to X instead of T : X \to X we have that our Corollary 2.1 improves Proposition 1.1 of Kirk [10, p. 168].

(e) The closedness condition of D in Theorem 1.1 and 1.1’ of Petryshyn and Williamson [12, p. 462, 469] and Theorem 3.1 in [8, p. 98] is superfluous.

(f) The convexity condition of D in Theorem 1.1’ of Petryshyn and Williamson [12, p. 469] is superfluous because the author assumed in their theorem that \{T^nx_0\} \in D for each n \in \omega and a given x_0 \in D in condition (1.3).

**Theorem 2.2.** Let \cond(F(T)) be a nonempty closed set. Then \{x_n\} converges to a point in \cond(F(T)) if \lim inf d(x_n, \cond(F(T))) = 0, \ T \text{ is condionally biased quasi-nonexpansive w.r.t. } \{x_n\} \text{ and } X \text{ is complete.}

Proof. Since \cond(F(T)) \subseteq F(T) we have that
lim inf d(x_n, \cond(F(T))) = 0 \text{ implies} \lim inf d(x_n, F(T)) = 0 \text{ Now using the technique of the proof of Theorem 2.1 the conclusion follows from Lemma 2.3.}

The following results follows easily from Lemma 2.5.

**Theorem 2.3.** Let F(T) be a nonempty closed set. Then \{x_n\} converges to a point in \cond(F(T)) if \{x_n\} converges to a point p in F(T), T is biased quasi-nonexpansive w.r.t. \{x_n\} and X is complete.
Theorem 2.4. Let $X$ be a complete metric space and let $\text{cond}(F(T))$ be a nonempty closed set. Assume that

1) $T$ is biased quasi-nonexpansive w.r.t. $\{x_n\}$;
2) $\lim_{n \to \infty} d(x_n, x_{n+1}) = 0$ or $\{x_n\}$ is a Cauchy sequence;
3) if the sequence $\{y_n\}$ satisfies $\lim_{n \to \infty} d(y_n, y_{n+1}) = 0$ then

$$\liminf_{n \to \infty} d\left(y_n, \text{cond}(F(T))\right) = 0$$
or
$$\limsup_{n \to \infty} d\left(y_n, \text{cond}(F(T))\right) = 0.$$

Then $\{x_n\}$ converges to a point in $\text{cond}(F(T))$.

Proof. Since $\text{cond}(F(T)) \neq \emptyset$ it follows from (i) that $T$ is condionally biased quasi-nonexpansive w.r.t. $\{x_n\}$ and the sequence $\{d(x_n, \text{cond}(F(T)))\}$ is monotonically decreasing and bounded from below by zero. Then $\liminf_{n \to \infty} d\left(x_n, \text{cond}(F(T))\right)$ exists.

From 2) and 3), we have that

$$\limsup_{n \to \infty} d\left(x_n, \text{cond}(F(T))\right) = 0.$$

Then $\lim_{n \to \infty} d\left(x_n, \text{cond}(F(T))\right) = 0$. Therefore, by Theorem 2.2, the sequence $\{x_n\}$ converges to a point in $\text{cond}(F(T))$.

As consequences of Theorem 2.4, we obtain the following:

Corollary 2.4. Let $X$ be a complete metric space and let $\text{cond}(F(T))$ be a nonempty closed set. Assume that

1) $T$ is biased quasi-nonexpansive w.r.t. $\{x_n\}$;
2) $T$ is asymptoticc regular at $x_0 \in D$ (or $\{T^n(x_0)\}$ is a Cauchy sequence);
3) if the sequence $\{y_n\}$ satisfies $\lim_{n \to \infty} d(y_n, y_{n+1}) = 0$ then

$$\liminf_{n \to \infty} d\left(y_n, \text{cond}(F(T))\right) = 0$$
or
$$\limsup_{n \to \infty} d\left(y_n, \text{cond}(F(T))\right) = 0.$$

Then $\{T^n(x_0)\}$ converges to a point in $\text{cond}(F(T))$.

Corollary 2.5. Let $X$ be a Banach space and let $\text{cond}(F(T))$ be a nonempty closed set. Assume that

1) $T$ is biased quasi-nonexpansive w.r.t. $\{x_n\}$;
2) $T$ is asymptoticc regular at $x_0 \in D$ (or $\{T^n(x_0)\}$ is a Cauchy sequence);
3) if the sequence $\{y_n\}$ satisfies $\lim_{n \to \infty} \|y_n - T_n y_n\| = 0$, then

$$\liminf_{n \to \infty} d\left(y_n, \text{cond}(F(T))\right) = 0$$
or
$$\limsup_{n \to \infty} d\left(y_n, \text{cond}(F(T))\right) = 0.$$

Then $\{T^n(x_0)\}$ converges to a point in $\text{cond}(F(T))$.

Remark 2.6. From Lemmas 2.1 and 2.2, Examples 2.1 and 2.3, Remark 2.3, the continuity of $T$ implies that $F(T)$ is closed but the converse may not be true; we obtain that Corollary 2.4 include Theorem 1.2 in [12, p. 464] and Theorem 3.2 in [7, p. 99] as special cases.

As another consequence of Theorem 2.1, we establish the following theorem:

Theorem 2.5. Let $X$ be a complete metric space and let $\text{cond}(F(T))$ be a nonempty closed set. Assume that

1) $T$ is biased quasi-nonexpansive w.r.t. $\{x_n\}$;
2) for every $x \in D - \text{cond}(F(T))$ there exists $p_x \in \text{cond}(F(T))$ such that $d(x_{n-1}, p_x) < d(x_n, p_x)$;
3) the sequence $\{x_n\}$ contains a subsequence $\{x_{n_j}\}$ converging to $x' \in D$.

Then $\{x_{n_j}\}$ converges to a point in $\text{cond}(F(T))$.

Proof. Since $\text{cond}(F(T)) \neq \emptyset$ it follows from (i) that $T$ is condionally biased quasi-nonexpansive w.r.t. $\{x_n\}$ and the sequence $\{d(x_n, \text{cond}(F(T)))\}$ is monotonically decreasing and bounded from below by zero. Then $\lim_{n \to \infty} d\left(x_n, \text{cond}(F(T))\right) = d\left(x_{n_j}, \text{cond}(F(T))\right)$ converging to $x' \in D$.

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be a Banach space, and let

Thus there exists \( p_x \in \text{cond}(F(T)) \) such that

This is a contradiction. So, \( x^* \in \text{cond}(F(T)) \).

**Corollary 2.7.** Let \( X \) be a complete metric space, \( \text{cond}(F(T)) \) a nonempty closed set and for a given \( x_0 \in D \) and each \( n \in \omega \), \( \{T^n x_0 \} \in D \). Assume that

1) \( T \) is biased quasi-nonexpansive w.r.t. \( \{T^n x_0 \} \);
2) for every \( x \in D - \text{cond}(F(T)) \) there exists \( p_x \in \text{cond}(F(T)) \) such that

3) the sequence \( \{T^n x_0\} \) contains a subsequence \( \{T^{n_i} x_0\} \) converging to \( x^* \in D \).

Then \( \{T^n x_0\} \) converges to a point in \( \text{cond}(F(T)) \).

**Corollary 2.8.** Let \( X \) be a Banach space, \( \text{cond}(F(T)) \) a nonempty closed set and for a given \( x_0 \in D \) and each \( n \in \omega \), \( \{T^n x_0 \} \in D \) Assume that

1) \( T \) is biased quasi-nonexpansive w.r.t. \( \{T^n x_0 \} \);
2) for every \( x \in D - \text{cond}(F(T)) \) there exists \( p_x \in \text{cond}(F(T)) \) such that

3) the sequence \( \{T^n x_0\} \) contains a subsequence \( \{T^{n_i} x_0\} \) converging to \( x^* \in D \).

Then \( \{T^n x_0\} \) converges to a point in \( \text{cond}(F(T)) \).

**Corollary 2.9.** Let \( X \) be a Banach space, \( \text{cond}(F(T)) \) a nonempty closed set and for a given \( x_0 \in D \) and each \( n \in \omega \), \( \{T^n x_0 \} \in D \) Assume that

1) \( T \) is biased quasi-nonexpansive w.r.t. \( \{T^n x_0 \} \);
2) for every \( x \in D - \text{cond}(F(T)) \) there exists \( p_x \in \text{cond}(F(T)) \) such that

3) the sequence \( \{T^n x_0\} \) contains a subsequence \( \{T^{n_i} x_0\} \) converging to \( x^* \in D \).

Then \( \{T^n x_0\} \) converges to a point in \( \text{cond}(F(T)) \).

Throughout this section, let \( R \) denote the set of real numbers. Let \( K = K(z,r) \) be a closed ball in a Banach space \( X \). For a sequence \( \{x_n\}_{n=0} \cup K \) converging to \( x \) we define

\[
\lim D_a = SD(x,K)
\]

where

\[
D_a = \text{conv}(\{x_n\} \cup K)
\]

and

\[
D_{a+1} = \text{conv}(\{x_n\} \cup D_a) \forall n \in \omega
\]

and \( SD(x,K) \) is called a super drop.

Clearly, for a constant sequence \( \{x_n\} = \{x\} \) converging to \( x \) we have \( D_{a+1} = D_a \forall n \in \omega \) so that \( D(x,K) = \text{conv}(\{x\} \cup K) \) and is called a drop. Thus the concept of a drop is a special case of super drop. It is also clear that if \( y \in D(x,K) \) then \( D(y,K) \subseteq D(x,K) \) and if \( z = 0 \) then \( \|z\| = \|x\| \).

Recall that a function \( \varphi : X \to R \) is called a lower semicontinuous whenever \( \{x \in X : \varphi(x) \leq a\} \) is closed for each \( a \in R \).

Caristi [4] proved the following:

**Theorem A.** Let \( (X,d) \) be complete and \( \varphi : X \to R \) a lower semicontinuous function with a finite lower bound. Let \( T : X \to X \) be any function such that \( d(x,T(x)) \leq \varphi(x) - \varphi(T(x)) \) for each \( x \in X \). Then \( T \) has a fixed point.

We now state and prove some applications of our main results in section 2 to geometry of Banach Spaces.

**Theorem 3.1.** Let \( C \) be a closed subset of a Banach space \( X \) let \( z \in X - C \) and let \( K = K(z,r) \) be a closed ball of radius \( r < d(z,C) = R \) Let \( x \) be an arbitrary element of \( C \) let \( \{x_n\} \) be a sequence in \( C \) converging to \( x \) and let \( T : C \to X \) be any continuous function defined implicitly by \( T(x) \in C \cap SD(x,K) \) for each \( x \in C \) in the sense that \( T(x_n) \in C \cap D_n \) for each \( n \in \omega \). Then...
1) \( \lim_{n \to \infty} d(x_n, F(T)) = 0 \) if \( \{x_n\} \) converges to a point \( p \) in \( F(T) \);

2) \( \{x_n\} \) converges to a point in \( F(T) \) if \( \lim_{n \to \infty} d(x_n, F(T)) = 0 \), \( T \) is locally quasi-nonexpansive at \( p \in F(T) \) w.r.t. \( \{x_n\} \).

Proof. Without loss of generality we may assume that \( z = 0 \). Let \( \{x_n\} \) be a sequence in \( X \) with \( y \in X \) and a sequence \( \{y_n\} \) converging to \( y \), we shall estimate \( \|y - T(y)\| \) on \( X \).

For given \( y \in X \) and the corresponding sequence \( \{y_n\} \) there is a sequence \( \{b_n\} \) in \( X \) with \( T(y_n) = b_n + (1-t)y_n \) for \( 0 < t < 1 \) Now \( \|T(y_n)\| \leq \|b_n\| + (1-t)\|y_n\| \), we have

\[
\ell \|y_n - b_n\| \leq \|y_n - T(y_n)\| + (1-t)\|y_n\| \leq \|y_n - T(y_n)\| + (\eta + r) \leq \frac{\eta + r}{R - \eta} \|y_n - T(y_n)\|
\]

Thus,

\[
\|y_n - T(y_n)\| \leq \ell \|y_n - b_n\| \leq \ell \|y_n - y\| \leq \ell \|x - y\| \leq \ell \|x - y\| \leq \frac{\eta + r}{R - r} \|y\| \text{ for } \forall x, y \in X \text{ and } \Phi(y) = \frac{\eta + r}{R - r} \|y\| \text{ then } X \text{ is complete as a metric space and } \Phi: X \to R \text{ is a continuous function. So } \Phi \text{ is a lower-semicontinuous function. Also, the above inequality takes the form } d(y_n, T(y_n)) \leq \Phi(y_n) - \Phi(T(y_n)).
\]

Proceeding to the limit as \( n \to \infty \) we obtain

\[
d(y, T(y)) \leq \Phi(y) - \Phi(T(y)) \text{ for each } y \in X.
\]

Therefore, applying the theorem of Caristi we obtain that \( T \) has a fixed point \( p = p(x) \) for each \( x \in C \).

Since drop is a special case of super drop, we have the following:

**Corollary 3.1.** Let \( C \) be a closed subset of a Banach space \( X \) let \( x \in X \) and \( K = K(z, r) \) be a closed ball of radius \( r < d(z, C) = R \) Let \( x \) be an arbitrary element of \( C \), and let \( T: C \to X \) be any (not necessarily continuous) function defined implicitly by \( T(x) \in C \cap D(x, K) \) for each \( x \in C \). Then

\[
1) \lim_{n \to \infty} d(x_n, F(T)) = 0 \text{ if } \{x_n\} \text{ converges to a point } p \text{ in } F(T);
\]

\[
2) \{x_n\} \text{ converges to a point in } F(T) \text{ if } \lim_{n \to \infty} d(x_n, F(T)) = 0, \text{ } T \text{ is locally quasi-nonexpansive at } p \in F(T) \text{ w.r.t. } \{x_n\}.
\]

We now prove the following result for biased quasi-nonexpansive mapping w.r.t a sequence \( \{x_n\} \).

**Theorem 3.2.** Let \( C \) be a closed subset of a Banach space \( X \) let \( z \in X-C \) and let \( \bar{K} = K(z, r) \) be a closed ball of radius \( r < d(z, C) = R \). Let \( x \) be an arbitrary element of \( C \), \( \{x_n\} \) a sequence in \( C \) converging to \( x \), and let \( T: C \to X \) be any continuous function defined implicitly by \( T(x) \in C \cap D(x, K) \) for each \( x \in C \) in the sense that \( T(x_n) \in C \cap D_{x_n} \) for each \( n \in \omega \). If \( \{x_n\} \) converges to a point in \( F(T) \), \( T \) is biased quasi-nonexpansive w.r.t. \( \{x_n\} \) then \( \{x_n\} \) converges to a point in \( \text{cond}(F(T)) \).

Proof. Using Theorem 2.3, instead of Theorem 2.1 the conclusion follows on the lines of the proof technique of Theorem 3.1.

As a consequence of Theorem 3.2, we obtain the following:

**Corollary 3.2.** Let \( C \) be a closed subset of a Banach space \( X \) let \( z \in X-C \) and let \( \bar{K} = K(z, r) \) be a closed ball of radius \( r < d(z, C) = R \). Let \( x \) be an arbitrary element of \( C \), and let \( T: C \to X \) be any (not necessarily continuous) function defined implicitly by \( T(x) \in C \cap D(x, K) \) for each \( x \in C \). If \( \{x_n\} \) converges to a point in \( F(T) \), \( T \) is biased quasi-nonexpansive w.r.t. \( \{x_n\} \) then \( \{x_n\} \) converges to a point in \( \text{cond}(F(T)) \).

**Open Question.** To what extent can the continuity hypothesis on \( T \) be muted in Theorems 3.1 and 3.2?

4. References


