Comparison between the Laplace Decomposition Method and Adomian Decomposition in Time-Space Fractional Nonlinear Fractional Differential Equations

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Abstract
The aim of this paper is to discuss application of Laplace Decomposition Method with Adomian Decomposition in time-space Fractional Nonlinear Fractional Differential Equations. The approximate solutions result from Laplace Decomposition Method and Adomian decomposition; those two accessions are comfortable to perform and firm when to PDEs. For caption and further representation of the thought, several examples are tool up.

Keywords
Laplace Decomposition Method, Mittag-Leffler Function, Partial Fractional Differential Equation

1. Introduction
Recently, research has shown that numerous phenomena in fluid mechanics, viscoelasticity, biology, physics, engineering and other field from knowledge mastery are successfully modeled by the use of FPDEs.

Researchers developed numerous methods to resolve FODEs, integral equations and fractional partial differential equations of physical interest. The flowing is an illustration of several of the generality used ones; ADM [3] [4] [5] [6], VIM [7], FDM [2], DTM [9], HPM [8]. Among the already mentioned ones, the decomposition method stood up as efficient, easy and accurate in solving a great group of linear and nonlinear ordinary, partial, deterministic or stochastic differential equations. This method is fully appropriate to physically solve problems, some time ago it did not request superfluous linearization, and other bound methods and postulate whom may vary the problem being solved [10] [11]. The
LDM is a numerical algorithm to solve NODEs and PDEs. Khuri [13, 14] developed this method for the sacrificial solution of a class of NODEs. These numerical technique styles ultimately clarify how the Laplace Transform supposedly is utilized to sacrifice the solutions of the nonlinear differential equations by impacting the Decomposition Method that was initially before Adomian.

2. The Definitions of Fractional Calculus and Laplace Transform

**Definition 2.1** Aral function $f(t), t > 0$ is said to be the space $C_\mu, \mu \in \mathbb{R}$, if the there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$, clearly $C_\beta \subset C_\mu$ if $\beta \leq \mu$.

**Definition 2.2** The left sides Riemann-Liouville fractional integral operator of order $\nu \geq 0$ of a function $f(\xi) \in C_\mu, \mu \geq -1$ is defined as [1]-[15]

\[
J^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} f(\xi) d\xi, \quad v, t > 0,
\]

\[
J^\nu f(t) = f(t), \quad v = 0.
\]

**Definition 2.3** The left sided Caputo fractional derivative of $f$, $f \in C_\mu, m \in \mathbb{N} \cup \{0\}$ is defined as [1]-[15]

\[
D^\nu f(t) = \frac{\partial^\nu f(t)}{\partial t^\nu} = \begin{cases}
J^\nu \left[ \frac{\partial^m f(t)}{\partial t^m} \right], & m - 1 < \nu \leq m, m \in \mathbb{N}, \\
\frac{\partial^m f(t)}{\partial t^m}, & v = m.
\end{cases}
\]

Hence, we have the following properties [1]-[15]

1. $J^\nu J^\alpha f(t) = J^{\nu + \alpha} f(t), \quad \alpha, \nu \geq 0.$
2. $J^\nu t^\gamma = \frac{\Gamma(\nu + 1)}{\Gamma(\nu + \gamma + 1)} t^{\nu + \gamma}, \quad \alpha > 0, \gamma > -1, \quad t > 0.$
3. $J^\nu D^\nu f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!}, \quad t > 0, \quad m - 1 < \nu \leq m$

**Definition 2.4** If $m - 1 < \nu \leq m, m \in \mathbb{N}$, then the Laplace transform of the fractional derivative $D^\nu f(t)$ is

\[
L \left[ D^\nu f(t) \right] = s^\nu F(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\nu-k-1}, \quad t > 0,
\]

where $F(s)$ be the Laplace transform of $f(t)$ [1] [2] [3].

**Definition 2.5** The Mittag-Leffler function $E_\nu(t)$ with $\nu > 0$ is defined by the following series representation, valid in the whole complex plane [2]

\[
E_\nu(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\nu + 1)}
\]
dered.  

\[ D^\nu u(x,t) + Ru(x,t) + Nu(x,t) = f(x,t), \quad x,t \geq 0, m-1 < \nu < m. \tag{3.1} \]

where \( D^\nu = \frac{\partial^\nu}{\partial t^\nu} \) the Caputo fractional derivative of is order \( \nu, m \in \mathbb{N} \), where \( R \) is a linear operator, \( N \) is a nonlinear function and \( f \) is the source function. The initial and boundary associated with Equation (3.1) are of the from \( u(x,0) = h(x), \quad 0 < \nu \leq 1, \quad t > 0, \tag{3.2} \)

And

\[ \frac{\partial u(x,0)}{\partial t} = k(x), \quad 1 < \nu \leq 2, \quad t > 0, \tag{3.3} \]

According to the Applying of the Laplace transform to Equation (3.1) and the use of linearity Laplace transform, the result is

\[ L[D^\nu u(x,t)] + L[Ru(x,t) + Nu(x,t)] = L[f(x,t)], \tag{3.4} \]

when the property of Laplace transform is used, we get

\[ s^\nu u(x,s) - s^\nu u(x,0) - s^{\nu-1}u_t(x,0) = L[f(x,t)] - L[Ru(x,t) + Nu(x,t)], \tag{3.5} \]

\[ u(x,s) = \frac{h(x)}{s} + \frac{k(x)}{s^2} + \frac{L}{s^\nu} [f(x,t)] - \frac{L}{s^\nu} [Ru(x,t) + Nu(x,t)], \tag{3.6} \]

Stander Laplace decomposition defines the solution \( u(x,t) \) before the series

\[ u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{3.7} \]

The Nonlinear operator is decomposed as follows:

\[ Nu(x,t) = \sum_{n=0}^{\infty} A_n \tag{3.8} \]

See \( A_n \) the Adomian polynomial that are given by

\[ A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ N \left( \sum_{k=0}^{n} \lambda^k u_k \right) \right], \quad n = 1, 2, \cdots. \tag{3.9} \]

Substitution Equation (3.7), Equation (3.8) and Equation (3.9) to Equation (3.6), we have

\[ \sum_{n=0}^{\infty} u_n(x,s) = \frac{h(x)}{s} + \frac{k(x)}{s^2} + \frac{L}{s^\nu} [f(x,t)] - \frac{L}{s^\nu} \left( \sum_{n=0}^{\infty} A_n \right), \tag{3.10} \]

when both sides of Equation (3.10) are matched, the following iterative algorithm is yielded:

\[ u_0(x,s) = \frac{h(x)}{s} + \frac{k(x)}{s^2} + \frac{L}{s^\nu} [f(x,t)] = g(x,s) \tag{3.11} \]

\[ u_i(x,s) = -\frac{L}{s^\nu} [Ru_0(x,t) + A_0], \quad u_2(x,s) = -\frac{L}{s^\nu} [Ru_1(x,t) + A_1] \tag{3.12} \]

Generally, the recursive relation is given as follows
When the inverse Laplace transform Equation (3.13) is applied, the following is obtained
\[ u_0(x,t) = g(x,t) \]  
(3.14)
\[ u_{n+1}(x,t) = -L^{-1}\left[ \frac{L}{s^n}[Ru_n(x,t) + A_n]\right], \quad n \geq 1 \]  
(3.15)
where \( g(x,t) \) is a function that arises from the source term and the prescribed initial condition, the initial solution is important, the choice of Equation (3.15) as the initial solution always leads to noise oscillation during the iteration procedure.

4. Numerical Results

To show the method of coupled fractional nonlinear partial differential equations, three examples are considered in this section.

**Example 1** Consider the nonlinear time-fractional advection partial differential equation [11].
\[ D^v u(x,t) + u(x,t)u_x(x,t) = x + xt^2, \quad t > 0, 0 < v \leq 1, \]  
(4.1)
The initial condition is subject to:
\[ u(x,0) = 0, \]  
(4.2)
When the Laplace transform is applied to both sides of Equation (4.1) the following is yielded.
\[ s'u(x,s) = s^{-v}u(x,0) + L\left[ x + xt^2 \right] + L\left[ u(x,t)u_x(x,t) \right] \]  
(4.3)
By using the initial condition, the following recurrence relational is yielded
\[ u(x,s) = \frac{x}{s^{v+1}} + \frac{2x}{s^{v+1}} - \frac{1}{s^v}L\left[ u(x,t)u_x(x,t) \right] \]  
(4.4)
Applying the inverse Laplace transform, to both sides of Equation (4.4) yields:
\[ u(x,t) = \frac{x}{\Gamma(v+1)}t^v + \frac{2x}{\Gamma(v+3)}t^{v+2} - L^{-1}\left[ \frac{1}{s^v}L\left[ \sum_{n=0}^{\infty}A_n \right] \right] \]  
(4.5)
The LDM proposes a series solution of the function \( u(x,t) \) which is given series Equation (3.7) and using Equation (3.7) into Equation (4.5)
\[ \sum_{n=0}^{\infty}u_n(x,t) = \frac{x}{\Gamma(v+1)}t^v + \frac{2x}{\Gamma(v+3)}t^{v+2} - L^{-1}\left[ \frac{1}{s^v}L\left[ \sum_{n=0}^{\infty}A_n \right] \right] \]  
(4.6)
In above Equation(4.6) is Adomian polynomials that represents nonlinear terms. The few components of Adomian polynomials are:
\[ A_0 = u_0u_0, \]
\[ A_1 = u_0u_1 + u_0u_1, \]
\[ A_2 = u_0u_2 + u_0u_1 + u_0u_1 \]
\[ A_3 = u_0u_3 + u_0u_2 + u_0u_1 + u_0u_1 \]
\[ \vdots \]
Then following recurrence relations, we have

\[ u_0(x, t) = \frac{x}{\Gamma(v+1)} t^v + \frac{2x}{\Gamma(v+3)} t^{v+2} \]

\[ u_{n+1}(x, t) = -L^{-1} \left[ \frac{1}{s^v} L \left[ \sum_{n=0}^{\infty} A_n \right] \right] \]

Consequently

\[ u_1(x, t) = -L^{-1} \left[ \frac{1}{s^v} L \left[ \sum_{n=0}^{\infty} A_n \right] \right] \]

\[ = \frac{2x \Gamma(2v+1) \Gamma(4v+1)}{\Gamma(v+1) \Gamma(3v+1) \Gamma(5v+1)} t^{3v} \]

\[ + \left[ \frac{8x \Gamma(2v+3)}{\Gamma^2(v+1) \Gamma(3v+3) \Gamma(5v+3)} + \frac{4x \Gamma(2v+3)}{\Gamma(v+1) \Gamma^2(v+3) \Gamma(3v+3)} \right] \frac{\Gamma(4v+3)}{\Gamma(5v+3)} t^{5v+2} \]

\[ + \left[ \frac{8x \Gamma(2v+5)}{\Gamma(v+1) \Gamma^3(v+3) \Gamma(3v+5)} + \frac{16x \Gamma(2v+5)}{\Gamma(v+1) \Gamma^2(v+3) \Gamma(3v+3)} \right] \frac{\Gamma(4v+5)}{\Gamma(5v+5)} t^{6v+4} \]

\[ + \frac{16x \Gamma(2v+5) \Gamma(4v+7)}{\Gamma^3(v+3) \Gamma(3v+5) \Gamma(5v+7)} t^{5v+6} \]

\[ \vdots \]

The third term approximating a solution to Equation (4.1) gives the following:

\[ u(x, t) = \frac{x}{\Gamma(v+1)} t^v + \frac{2x}{\Gamma(v+3)} t^{v+2} + \frac{x \Gamma(2v+1)}{\Gamma^2(v+1) \Gamma(3v+1)} t^{3v} \]

\[ - \frac{4x \Gamma(2v+3)}{\Gamma(v+1) \Gamma(3v+3)} t^{3v+2} \]

\[ - \frac{4x \Gamma(2v+5)}{\Gamma^2(v+3) \Gamma(3v+5)} t^{3v+4} + \ldots \] (4.7)

**Example 4.2** Consider the nonlinear time-fractional hyperbolic equation [16].

\[ D^\mu u_i(x, t) = \frac{\partial}{\partial x} \left[ u(x, t) u_i(x, t) \right], \quad t > 0, 1 < v \leq 2 \] (4.8)

The initial condition is subject to:

\[ u(x, t) = x^2, \quad u_i(x, t) = -2x^2 \] (4.9)

When the Laplace transform is applied to both sides of Equation (4.8) the following is yielded.
By using initial conditions, the following recurrence relational is found

\[ u(x, s) = \frac{x^2}{s} - \frac{2x^2}{s^2} + L\left[ \frac{\partial}{\partial x}[u(x, t)u_s(x, t)] \right] \]  

(4.11)

When the inverse Laplace transform, is applied, to both sides of Equation (4.11) the following is yielded:

\[ u(x, t) = x^2 - 2x^2t^2 + L^{-1}\left[ L\left[ \frac{\partial}{\partial x}[u(x, t)u_s(x, t)] \right] \right] \]  

(4.12)

The LDM proposes a series solution of the function \( u(x, t) \) which is given series Equation (3.6) and using Equation (3.6) into Equation (4.12)

\[ \sum_{n=0}^{\infty} u_n(x, t) = x^2 - 2x^2t^2 + L^{-1}\left[ L\left[ \frac{\partial}{\partial x}\left( \sum_{n=0}^{\infty} A_n \right) \right] \right] \]  

(4.13)

In above Equation (4.13) Adomian polynomials represent nonlinear terms \( N = uu_s \). The few components of the decomposition series are derived as follows:

\[ u_0(x, t) = x^2 - 2x^2t \]

\[ u_1(x, t) = 6x^2\left( \frac{t^v}{\Gamma(v+1)} - \frac{4t^{v+1}}{\Gamma(v+2)} + \frac{8t^{v+2}}{\Gamma(v+3)} \right) \]

\[ u_2(x, t) = 72x^2\left( \frac{t^{2v}}{\Gamma(3v+1)} - 4\frac{t^{2v+1}}{\Gamma(2v+2)} + 2\frac{\Gamma(v+1)t^{2v+1}}{\Gamma(2v+4)\Gamma(v+1)} - 16\frac{\Gamma(v+4)t^{2v+3}}{\Gamma(2v+4)\Gamma(v+3)} \right) + 72x^2\left( \frac{1}{\Gamma(2v+3)} + 8\frac{\Gamma(v+3)}{\Gamma(2v+3)\Gamma(v+2)} \right) \]

**Example 4.3**

Consider a system of nonlinear coupled with partial differential equation [16].

\[ D^u(x, y, t) + v_yw_y - v_yw_z = -u \]  

(4.14)

\[ D^v(x, y, t) + w_xu_y + u_xw_y = v \]  

(4.15)

\[ D^w(x, y, t) + u_yv_y + u_yv_z = w \]  

(4.16)

The initial condition is subject to:

\[ u(x, y, 0) = e^{xy} \]  

(4.17)

\[ v(x, y, 0) = e^{-xy} \]  

(4.18)

\[ w(x, y, 0) = e^{-xy} \]  

(4.19)

When the Laplace transform is applied the both sides of [Equation (4.14), Equation (4.15) and Equation (4.16)] through using the initial condition, the fol-
lowing recurrence relational is yielded
\[
\begin{align*}
 u(x, y, s) &= \frac{e^{x+y}}{s} - \frac{L}{s^a} [v_y w_y - v_y w_y - u] \\
 v(x, y, s) &= \frac{e^{-x-y}}{s} + \frac{L}{s^b} [v - w_y u_y - u, w_y] \\
 w(x, y, s) &= \frac{e^{x-y}}{s} + \frac{L}{s^d} [w - u_y v_y - u_y v_y]
\end{align*}
\] (4.20)

(4.21)

(4.22)

Through applying inverse Laplace transform to both sides of [Equation (4.20), Equation (4.21) and Equation (4.22)] we can yields.

\[
\begin{align*}
 u(x, y, t) &= e^{x+y} - L^{-1} \left[ \frac{L}{s^a} [v_y w_y - v_y w_y - u] \right] \\
 v(x, y, t) &= e^{-x-y} + L^{-1} \left[ \frac{L}{s^b} [v - w_y u_y - u, w_y] \right] \\
 w(x, y, t) &= e^{x-y} + L^{-1} \left[ \frac{L}{s^d} [w - u_y v_y - u_y v_y] \right]
\end{align*}
\] (4.23)

(4.24)

(4.25)

The LADM assumes a series solution of the function \( u(x, t) \) which is given series Equation (3.6) and through using Equation (3.6) into [Equation (4.23), Equation (4.24) and Equation (4.25)] yielding.

\[
\begin{align*}
 \sum_{n=0}^{\infty} u_n (x, y, t) &= e^{x+y} - L^{-1} \left[ \sum_{n=0}^{\infty} F_n (v, w) - \sum_{n=0}^{\infty} G_n (v, w) - u_n \right] \\
 \sum_{n=0}^{\infty} v_n (x, y, t) &= e^{-x-y} + L^{-1} \left[ \sum_{n=0}^{\infty} H_n (u, w) - \sum_{n=0}^{\infty} I_n (u, w) \right] \\
 \sum_{n=0}^{\infty} w_n (x, y, t) &= e^{x-y} + L^{-1} \left[ \sum_{n=0}^{\infty} J_n (u, v) - \sum_{n=0}^{\infty} K_n (u, v) \right]
\end{align*}
\] (4.26)

(4.27)

(4.28)

where \( F_n (v, w), G_n (v, w), H_n (u, w), I_n (u, v), J_n (v, w) \) and \( K_n (v, w) \) are Adomian polynomials they represent nonlinearities arising in above system [Equation (4.26), Equation (4.27) and Equation (4.28)] of nonlinear coupled partial differential equations. The components of above Adomian polynomials are given below

\[
\begin{align*}
 F_0 (v, w) &= v_0 w_0 \\
 F_1 (v, w) &= v_1 w_0 + v_0 w_1 \\
 F_2 (v, w) &= v_0 w_2 + v_1 w_1 + v_2 w_0 \\
 & \vdots \\
 F_n (v, w) &= \sum_{i=0}^{n} v_i w_{n-i} \\
 G_0 (v, w) &= v_0 w_0 \\
 G_1 (v, w) &= v_1 w_0 + v_0 w_1 \\
 G_2 (v, w) &= v_0 w_2 + v_1 w_1 + v_2 w_0 \\
 & \vdots \\
 G_n (v, w) &= \sum_{i=0}^{n} v_i w_{n-i}
\end{align*}
\] (4.29)
The following recursive relation is obtained:

\[ H_0(v, w) = w_0 u_{0y} \]
\[ H_1(v, w) = w_1 u_{0y} + w_0 u_{1y} \]
\[ H_2(v, w) = w_0 u_{2y} + w_1 u_{1y} + w_2 u_{0y} \]
\[ \vdots \]
\[ H_n(v, w) = \sum_{i=0}^{n} w_i u_{n-i} \]

\[ I_0(v, w) = w_0 u_{0x} \]
\[ I_1(v, w) = w_1 u_{0x} + w_0 u_{2x} \]
\[ I_2(v, w) = w_0 u_{2x} + w_1 u_{1x} + w_2 u_{0x} \]
\[ \vdots \]
\[ I_n(v, w) = \sum_{i=0}^{n} w_i u_{n-i} \]

\[ J_0(v, w) = u_0 v_{0y} \]
\[ J_1(v, w) = u_1 v_{0y} + u_0 v_{1y} \]
\[ J_2(v, w) = u_0 v_{2y} + u_1 v_{1y} + u_2 v_{0y} \]
\[ \vdots \]
\[ J_n(v, w) = \sum_{i=0}^{n} u_i v_{n-i} \]

\[ K_0(v, w) = u_0 v_{0x} \]
\[ K_1(v, w) = u_1 v_{0x} + u_0 v_{1x} \]
\[ K_2(v, w) = u_0 v_{2x} + u_1 v_{1x} + u_2 v_{0x} \]
\[ \vdots \]
\[ K_n(v, w) = \sum_{i=0}^{n} u_i v_{n-i} \]

The following recursive relation is obtained:

\[ u_0(x, y, t) = e^{x+y}, \ u_{n+1}(x, y, t) = \frac{1}{L} \int \left[ \frac{L}{s^a} \sum_{n=0}^{\infty} F_n(v, w) - \sum_{n=0}^{\infty} G_n(v, w) - U_n \right] \]  

\[ v_0(x, y, t) = e^{x+y}, \ v_{n+1}(x, y, t) = \frac{1}{L} \int \left[ \frac{L}{s^b} \left[ v_n - \sum_{n=0}^{\infty} H_n(u, w) - \sum_{n=0}^{\infty} I_n(u, w) \right] \right] \]  

\[ w_0(x, y, t) = e^{x+y}, \ w_{n+1}(x, y, t) = \frac{1}{L} \int \left[ \frac{L}{s^c} \left[ w_n - \sum_{n=0}^{\infty} J_n(u, v) - \sum_{n=0}^{\infty} K_n(u, v) \right] \right] \]  

\[ u_1(x, y, t) = \frac{1}{L} \int \left[ \frac{L}{s^a} [F_0(v, w) - G_0(v, w) - U_0] \right] \]
\[ = \frac{1}{L} \int \left[ \frac{L}{s^a} [v_0 w_0 - v_0 w_0 - U_0] \right] \]
\[ = \frac{1}{L} \int \left[ \frac{L}{s^a} [-e^{x+y}] \right] = -\frac{e^{x+y}}{\Gamma(v+1)} t \]
\[ v_1(x, y, t) = L^{-1} \left[ \frac{L}{s^\beta} \left[ v_0 - H_0(u, w) - I_0(u, w) \right] \right] = L^{-1} \left[ \frac{L}{s^\beta} \left[ e^{-y} - u_{0y}w_{0y} - u_{0y}w_{0y} \right] \right] = L^{-1} \left[ \frac{L}{s^\beta} \left[ e^{-y} \right] \right] = \frac{e^{-y}}{\Gamma(\beta + 1)} t^\beta \]

\[ w_1(x, y, t) = L^{-1} \left[ \frac{L}{s^\alpha} \left[ w_0 - J_0(u, v) - K_0(u, v) \right] \right] = L^{-1} \left[ \frac{L}{s^\alpha} \left[ e^{-y} - u_{0y}v_{0y} - u_{0y}v_{0y} \right] \right] = L^{-1} \left[ \frac{L}{s^\alpha} \left[ e^{-y} \right] \right] = \frac{e^{-y}}{\Gamma(\beta + 1)} t^\beta \]

\[ u_2(x, y, t) = L^{-1} \left[ \frac{L}{s^\alpha} \left[ F_1(v, w) - G_1(v, w) - u_1 \right] \right] = L^{-1} \left[ \frac{L}{s^\alpha} \left[ e^{xy} \int^t \right] \right] = \frac{e^{xy}}{\Gamma(\alpha + 1)} t^\alpha \]

\[ v_2(x, y, t) = L^{-1} \left[ \frac{L}{s^\alpha} \left[ v_2 - H_2(u, w) - I_2(u, w) \right] \right] = L^{-1} \left[ \frac{L}{s^\alpha} \left[ e^{xy} \int^t \right] \right] = \frac{e^{xy}}{\Gamma(\beta + 1)} t^\beta \]

\[ w_2(x, y, t) = L^{-1} \left[ \frac{L}{s^\alpha} \left[ w_1 - J_1(u, v) - K_1(u, v) \right] \right] = L^{-1} \left[ \frac{L}{s^\alpha} \left[ e^{xy} \int^t \right] \right] = \frac{e^{xy}}{\Gamma(\mu + 1)} t^\mu \]

So the expected solutions are as follows:

\[ u(x, y, t) = \sum_{n=0}^{\infty} u_n(x, y, t) = e^{xy} + \frac{e^{xy}}{\Gamma(\alpha + 1)} t^\alpha + \frac{e^{xy}}{\Gamma(2\alpha + 1)} t^{2\alpha} + \cdots \]

\[ = e^{xy} \left[ 1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \cdots \right] \quad (4.38) \]

\[ = e^{xy} \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} \]

The solution in a closed form is given as

\[ = e^{xy} E_{a,j} \left( t^a \right) \quad (4.39) \]

\[ v(x, y, t) = \sum_{n=0}^{\infty} v_n(x, y, t) = e^{xy} + \frac{e^{xy}}{\Gamma(\beta + 1)} t^\beta + \frac{e^{xy}}{\Gamma(2\beta + 1)} t^{2\beta} + \cdots \]

\[ = e^{xy} \left[ 1 + \frac{t^\beta}{\Gamma(\beta + 1)} + \frac{t^{2\beta}}{\Gamma(2\beta + 1)} + \cdots \right] \quad (4.40) \]

\[ = e^{xy} \sum_{k=0}^{\infty} \frac{t^{k\beta}}{\Gamma(k\beta + 1)} \]
The solution in a closed form is given as

$$w(x, y, t) = \sum_{\mu=0}^{\infty} w_\mu(x, y, t)$$

$$= e^{-\alpha \cdot x} E_{\beta, 1} \left( t^\mu \right)$$

$$= e^{-\alpha \cdot x} + \frac{e^{-\alpha \cdot y}}{\Gamma(\mu + 1)} t^\mu + \frac{e^{-\alpha \cdot y}}{\Gamma(2\mu + 1)} t^{2\mu} + \ldots$$

(4.42)

The solution in a closed structure is given as

$$= e^{-\alpha \cdot x} E_{\beta, 1} \left( t^\mu \right)$$

(4.43)

5. Conclusion

In this treatise, Laplace decomposition method has been successfully used to solve the approximate solution of nonlinear PFDEs (see Table 1 and Table 2).

**Table 1.** Numerical values for Equation (4.1) [11].

<table>
<thead>
<tr>
<th>t</th>
<th>x</th>
<th>v = 0.5</th>
<th>v = 0.75</th>
<th>v = 1</th>
<th>exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.250</td>
<td>0.11276</td>
<td>0.112844</td>
<td>0.07878</td>
<td>0.07878</td>
<td>0.050000</td>
</tr>
<tr>
<td>0.500</td>
<td>0.22553</td>
<td>0.225688</td>
<td>0.15756</td>
<td>0.157574</td>
<td>0.100000</td>
</tr>
<tr>
<td>0.750</td>
<td>0.33829</td>
<td>0.311249</td>
<td>0.23635</td>
<td>0.236361</td>
<td>0.150000</td>
</tr>
<tr>
<td>1.000</td>
<td>0.45106</td>
<td>0.451375</td>
<td>0.31513</td>
<td>0.315148</td>
<td>0.200000</td>
</tr>
<tr>
<td>0.250</td>
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<td>0.164004</td>
<td>0.128775</td>
<td>0.128941</td>
<td>0.100011</td>
</tr>
<tr>
<td>0.500</td>
<td>0.324523</td>
<td>0.328008</td>
<td>0.257550</td>
<td>0.257550</td>
<td>0.200022</td>
</tr>
<tr>
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<td>0.492011</td>
<td>0.386326</td>
<td>0.386821</td>
<td>0.300033</td>
</tr>
<tr>
<td>1.000</td>
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<td>0.656015</td>
<td>0.515101</td>
<td>0.515762</td>
<td>0.400045</td>
</tr>
</tbody>
</table>

**Table 2.** Numerical values for Equation (4.8) [16].

<table>
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<th>x</th>
<th>v = 0.5</th>
<th>v = 0.75</th>
<th>v = 1</th>
<th>exact</th>
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</thead>
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<td>0.043395</td>
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<tr>
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<td>0.237132</td>
<td>0.194804</td>
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<td>0.173611</td>
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<tr>
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<td>0.779219</td>
<td>0.694321</td>
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</tr>
<tr>
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<td>0.031566</td>
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</tr>
<tr>
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<tr>
<td>0.75</td>
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<tr>
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<td>0.699968</td>
<td>0.505071</td>
<td>0.510204</td>
<td></td>
</tr>
</tbody>
</table>

DOI: 10.4236/am.2018.94032
The solutions of these examples of methods are utilized for the solution of high-order initial value problems.

References