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Asymptotic Behavior and Stability of Stochastic SIR Model with Variable Diffusion Rates

Xianhua Xie[1], Li Ma[1,2], Jingfei Xu[1*]

[1]Key Laboratory of Jiangxi Province for Numerical Simulation and Emulation Techniques, Gannan Normal University, Ganzhou, China
[2]College of Mathematics and Econometrics, Hunan University, Changsha, China
Email: xxianhua@sina.com.cn, mali1983@hnu.edu.cn, *xujingfei2006@163.com.cn

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Abstract

In this paper, we propose random fluctuation on contact and recovery rates in deterministic SIR model with disease deaths in nonparametric manner and derive a new stochastic SIR model with distributed time delay and general diffusion coefficients. By analysis of the introduced model, we obtain the sufficient conditions for the regularity, existence and uniqueness of a global solution by means of Lyapunov function. Moreover, we also investigate the stochastic asymptotic stability of disease free equilibria and endemic equilibria of this model. Finally, we illustrate our general results by applications.

Keywords

SIR Model, Regularity, Lyapunov Function, Stochastic Asymptotic Stability

1. Introduction

SIR models are the foundation for a large number of compartmental models in mathematical epidemiology which classify the population into three classes: *Susceptible*, *Infected* and *Removed* (see [1]-[19]). Generally, these models admit two types of equilibrium: disease free and endemic equilibrium. If the disease free equilibrium is asymptotically stable, it implies the disease dies out. If the endemic equilibrium is asymptotically stable, it implies the disease persists in the population at the equilibrium level.

In 1976, Hethcote [13] considered the following deterministic SIR model with disease deaths:

*Corresponding author.

$$\begin{cases} S'(t) = -\beta S(t)I(t) + \mu(K - S(t)), \\ I'(t) = \beta S(t)I(t) - (\mu + \alpha)I(t), \\ R'(t) = \alpha I(t) - \mu R(t). \end{cases} \tag{1}$$

In Equation (1), $S(t), I(t), R(t)$ denote the number of the individuals susceptible to the disease, of infected members and of members who have been removed from the population, respectively. The model (1) is based on the following assumptions:

i) The population considered has a constant size *K*, that is, $S(t) + I(t) + R(t) = K$ for all *t*;

ii) Births and deaths occur at equal rates μ in *K*. All the newborns are susceptible. μ is called a daily death removal rate;

iii) β is the daily contact rate, *i.e.*, the average number of contacts per infective per day. A contact of an infective is an interaction which results in infection of the other individual if it is susceptible;

iv) α is the daily recovery removal rate of the infective. Of course, $\beta, \mu, \alpha \in R^+$.

In [8], Beretta and Takeuchi pointed out that when a susceptible vector is infected by a person, there is a time $\tau > 0$ during which the infectious agents develop in the vector and it is only after that time that the infected vector becomes itself infectious, and proposed the following model

$$\begin{cases} S'(t) = -\beta S(t)\int_0^h f(s)I(t-s)\mathrm{d}s + \mu(K - S(t)), \\ I'(t) = \beta S(t)\int_0^h f(s)I(t-s)\mathrm{d}s - (\mu + \alpha + \gamma)I(t), \\ R'(t) = \alpha I(t) - \mu R(t), \end{cases} \tag{2}$$

where $f(s)$ is a non-negative function which is square integrable on $[0, h]$ and satisfies

$$\int_0^h f(s)\mathrm{d}s = 1, \quad \int_0^h sf(s)\mathrm{d}s < +\infty, \tag{3}$$

and the non-negative constant *h* is the time delay, $\beta S(t)\int_0^h f(s)I(t-s)\mathrm{d}s$ can be viewed as the force of infection at time *t*.

In fact, all infectious diseases are subject to randomness in terms of the nature of transmission. Recently, Tornatore *et al.* [16] investigate the dynamics of system (2) by perturbing the functional contact rates and modified (2) as:

$$\begin{cases} \mathrm{d}S(t) = \left[-\beta S(t)\int_0^h f(s)I(t-s)\mathrm{d}s - \mu S(t) + \mu\right]\mathrm{d}t - \sigma S(t)\int_0^h f(s)I(t-s)\mathrm{d}s\mathrm{d}W(t) \\ \mathrm{d}I(t) = \left[\beta S(t)\int_0^h f(s)I(t-s)\mathrm{d}s - (\mu + \alpha)I(t)\right] + \sigma S(t)\int_0^h f(s)I(t-s)\mathrm{d}s\mathrm{d}W(t) \\ \mathrm{d}R(t) = \alpha I(t) - \mu R(t), \end{cases} \tag{4}$$

where σ is a positive constant and *W* is a real Wiener process defined on a stochastic basis $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, P)$. They only proved the stability of disease-free equilibrium under some given condition. Along these clues, we propose a stochastic SIR model with deaths and varying contact and recovery rates, where the introduced model covers general diffusion coefficients (functional contact and recovery rates).

In order to make the SIR system (2) more realistic, we consider the case of $S(t)+I(t)+R(t)\le K$ and we perturbed the deterministic system (2) by a white noise and obtained a stochastic counterpart by replacing the rates β by $\beta+F_1\big(S(t),I(t),R(t)\big)\frac{\mathrm{d}W_1(t)}{\mathrm{d}t}$ and α by $\alpha+F_2\big(S(t),I(t),R(t)\big)\frac{\mathrm{d}W_2(t)}{\mathrm{d}t}$, and hence we modify the SIR system (2) as the following model:

$$\begin{cases}\mathrm{d}S(t)=\left[-\beta S(t)\int_0^h f(s)I(t-s)\mathrm{d}s+\mu\big(K-S(t)\big)\right]\mathrm{d}t-S(t)\int_0^h f(s)I(t-s)\mathrm{d}s\\ \qquad\times F_1\big(S(t),I(t),R(t)\big)\mathrm{d}W_1(t),\\ \mathrm{d}I(t)=\left[\beta S(t)\int_0^h f(s)I(t-s)\mathrm{d}s-(\mu+\gamma+\alpha)I(t)\right]+S(t)\int_0^h f(s)I(t-s)\mathrm{d}s\\ \qquad\times F_1\big(S(t),I(t),R(t)\big)\mathrm{d}W_1(t)-\int_0^h f(s)I(t-s)\mathrm{d}sF_2\big(S(t),I(t),R(t)\big)\mathrm{d}W_2(t)\\ \mathrm{d}R(t)=\alpha I(t)-\mu R(t)+\int_0^h f(s)I(t-s)\mathrm{d}sF_2\big(S(t),I(t),R(t)\big)dW_2(t),\end{cases}\tag{5}$$

where α,β,μ,K have the same meaning as model (2), $W_i(t)(i=1,2)$ are real Wiener processes and *i.i.d* which defined on a filtered complete probability space $\big(\Omega,\mathfrak{F},(\mathfrak{F}_t)_{t\ge 0},P\big)$. Here, we introduce other two new general stochastic terms: functions $F_i(i=1,2)$ which are locally Lipschitz continuous defined on

$$\mathbb{D}=\big\{(S,I,R)\in\mathbb{R}^3:S\ge 0,I\ge 0,R\ge 0,S+I+R\le K\big\}.$$

Besides, there are deaths due to disease the total population size may vary in time so that we always assume the total population size is less than K in the context, where K represents a carrying capacity. Note that if we consider the population size is a constant K and the disease-related death rate $\gamma=0$, besides we also take $F_1(t)=F_2(t)=\sigma$ (where σ is a positive constant), the system (3) becomes the model which has been discussed in [16]. In [16], Tornatore proved the stability of disease-free equilibrium under some restricted conditions. However, they didn't consider the dynamics of the endemic equilibrium. It is of great importance from a theoretical point of view to investigate the stability of the endemic equilibrium.

In this paper, we mainly study the stochastic SIR model (5) with distributed delay which has more general diffusion coefficients than model (4)'s. By means of averaged Itô formula and Lyapunov function, we obtain the sufficient conditions for the regularity, existence and uniqueness of a global solution. Furthermore, we also investigate the stochastic asymptotic stability of disease free equilibria and the dynamics of endemic equilibria which has not been discussed in [16].

The remaining parts of the paper are organized as follows: In Section 2, we will give some basic concepts and conclusions. In Section 3, we employ the averaged Itô formula to obtain the regularity, existence and uniqueness of the global solution of SIR model (5). In Section 4, we derive the sufficient condition to ensure the global stochastic asymptotic stability of disease free equilibrium in SIR model (5), besides we also consider the stochastic asymptotic stability of endemic equilibrium in Section 5. Finally, we illustrate our general results by applications.

2. Some Preliminary Definition and Lemmas

At first, we recall the notation of regularity of continuous time stochastic processes as introduced in [10]. Let $\mathbb{D} \subset \mathbb{R}^d \left(d \geq 1\right)$ be a fixed closed domain. For Simplicity, we only consider deterministic domains $\mathbb{D} \subset \mathbb{R}^3$ in this exposition.

Definition 1. *A continuous time stochastic process* $\left\{X\left(t\right), t \geq 0\right\}$ *is called regular on* $\mathbb{D}$ (*or invariant with respect to* $\mathbb{D}$) *if*

$$\forall t \geq 0: \quad P\left(X\left(t\right) \in \mathbb{D}\right) = 1,$$

otherwise irregular with respect to $\mathbb{D}$ (or not invariant with respect to $\mathbb{D}$).

Consider the d-dimensional stochastic differential equation of the form

$$\mathrm{d}X\left(t\right) = f\left(X\left(t\right), t\right)\mathrm{d}t + g\left(X\left(t\right), t\right)dW\left(t\right), \tag{6}$$

with an initial value $X\left(t_0\right) = X_0, t_0 \leq t \leq T < \infty$ where $f : \mathbb{R}^d \times \left[t_0, T\right] \to \mathbb{R}^d$ and $f : \mathbb{R}^d \times \left[t_0, T\right] \to \mathbb{R}^{d \times m}\left(m \geq 1\right)$ are Borel measurable, $W = \left\{W\left(t\right)\right\}_{t \geq t_0}$ is an $\mathbb{R}^m$-valued random variable.

Definition 2. *The infinitesimal generator* $\mathcal{L}$ *associated with the SDE* (6) *is given by*

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i\left(x, t\right)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{m}\left(g\left(x, t\right)g^{\mathrm{T}}\left(x, t\right)\right)_{ij}\frac{\partial^2}{\partial x_i \partial x_j}.$$

Lemma 2.1. (*Regularity Theorem* [10]) *Let* $\mathbb{D}$ *and* $\mathbb{D}_n$ *be open sets in* $\mathbb{R}^d$ *with*

$$\mathbb{D}_n \subseteq \mathbb{D}_{n+1}, \quad \mathbb{D}_n \subseteq \mathbb{D}, \quad \text{and} \quad \mathbb{D} = \bigcup_n \mathbb{D}_n,$$

and suppose f and g satisfy the existence and uniqueness conditions for solutions of (6) on each set $\left\{\left(t, x\right) : t > t_0, x \in \mathbb{D}_n\right\}$. Suppose there is a nonnegative continuous function $V : \mathbb{D} \times \left[t_0, T\right] \to \mathbb{R}^+$ with continuous partial derivatives and satisfying $\mathcal{L}V \leq cV$ for some positive constant c and $t > t_0$, $x \in \mathbb{D}$. Moreover, if

$$\inf_{t > t_0, x \in \mathbb{D} \setminus \mathbb{D}_n} V\left(x, t\right) \to \infty \quad \text{as} \quad n \to \infty,$$

then $P\left(X_0 \in \mathbb{D}\right) = 1$ for any X_0 independent of $\sigma\left(W\right)$, that is to say the stochastic process $\left\{X\left(t\right), t \geq 0\right\}$ is called regular on $\mathbb{D}$. Regularity on $\mathbb{D}$ implies boundedness, uniqueness, continuity and Markov property of the strong solution process X of SDE (6) with $X\left(0\right) = X_0$, and $X\left(t\right) \in \mathbb{D}$ for all $t > 0$ (a.s.).

Definition 3. *The equilibrium solution* x^* *of the SDE* (6) *is stochastically stable* (*stable in probability*) *if for every* $\epsilon > 0$ *and* $s \geq t_0$

$$\lim_{X_0 \to x^*} P\left(\sup_{t_0 \le s < \infty} \left\| X_{s,X_0}(t) \right\| \ge \epsilon\right) = 0,$$

where $X_{s,X_0}(t)$ denotes the solution of (6) satisfying $X(s) = X_0$ at time $t \ge s$.

Definition 4. *The equilibrium solution* x^* *of the SDE* (6) *is stochastically asymptotically stable* (*stable in probability*) *if it is stochastically stable and*

$$\lim_{X_0 \to x^*} P\left(\lim_{t \to \infty} X_{s,X_0}(t) = x^*\right) = 1.$$

Definition 5. *The equilibrium solution* x^* *of the SDE* (6) *is said to be globally stochastically asymptotically stable* (*stable in probability*) *if it is stochastically stable and for every* X_0 *and every* s

$$P\left(\lim_{t \to \infty} X_{s,X_0}(t) = x^*\right) = 1.$$

Lemma 2.2. ([4]) *Assume that* f *and* g *satisfy locally Lipschitz-continuous and satisfy linear growth condition and they have continuous coefficients with respect to t.*

1) Suppose that there exists a positive definite function $V \in C^{2,1}(U_h \times [t_0, \infty))$, where $U_h = \left\{x \in \mathbb{R}^d : \left\| x - x^* \right\| < h\right\}$ for $h > 0$, such that

$$\mathcal{L}V(x,t) \le 0, \quad \forall t \ge t_0, \quad x \in U_h.$$

Then the equilibrium solution x^* of (6) is stochastically stable.

2) In addition, if V is descreesent (that is to say there exists a positive definite function V_1 such that $V(x,t) \le V_1(x)$ for all $x \in U_h$) and $\mathcal{L}V(x,t)$ is negative definite, then the equilibrium solution x^* is stochastically asymptotically stable.

3) If the assumptions of part (2) hold for a radially unbounded function $V \in C^{2,1}(\mathbb{R}^d \times [t_0, \infty))$ defined everywhere then the equilibrium solution x^* is globally stochastically asymptotically stable.

3. Existence, Uniqueness and Regularity of Stochastic SIR Model Solution

Theorem 3.1. *Let* $(S(t_0), I(t_0), R(t_0)) = (S_0, I_0, R_0) \in D = \left\{(S,I,R) \in \mathbb{R}^3, t \ge 0 : S \ge 0, I \ge 0, R \ge 0, S + I + R \le K\right\}$, *and* (S_0, I_0, R_0) *be independent of σ-algebra* $\sigma(W(t), t \ge 0)$. *Then, under the condition* (*A*) *or* (*B*)

a) $\int_0^h f(s) I(t-s) \mathrm{d}s \le L I(t) \ (L > 0)$;

b) $\int_0^h f(s) I(t-s) \mathrm{d}s \le I(t) + R(t)$;

the stochastic process $\left\{(S(t), I(t), R(t)), t \ge t_0\right\}$ governed by Equation (5) is regular on $\mathbb{D}$; *i.e.* we have $P\left((S(t), I(t), R(t)) \in \mathbb{D}\right) = 1$ for all $t \ge 0$. Moreover, regularity on $\mathbb{D}$ implies stochastic SIR model (5) admits a a unique, continuous-time, Markovian global solution process $\left\{(S(t), I(t), R(t)), t \ge t_0\right\}$.

Proof. First we consider the result under the condition of (A). Denote drift term

$$b(S,I,R)=\begin{pmatrix} -\beta S\int_0^h f(s)T(t-s)\mathrm{d}s+\mu(K-S) \\ \beta S\int_0^h f(s)I(t-s)\mathrm{d}s-(\alpha+\mu+\gamma)I \\ \alpha I-\mu R \end{pmatrix}$$

and the diffusion term

$$B(S,I,R)=\begin{pmatrix} -SF_1\int_0^h f(s)I(t-s)\mathrm{d}s & 0 & 0 \\ S\int_0^h f(s)I(t-s)\mathrm{d}s & -IF_2 & 0 \\ 0 & IF_2 & 0 \end{pmatrix}.$$

Let open domains

$$\mathbb{D}_n:=\left\{(S,I,R):\mathrm{e}^{-n}<S<K-\mathrm{e}^{-n},\mathrm{e}^{-n}<I<K-\mathrm{e}^{-n},\mathrm{e}^{-n}<R<K-\mathrm{e}^{-n},S+I+R\le K,n\in\mathbb{N}\right\}.$$

Since Equation (5) is well-defined on $\mathbb{D}$ and $\mathbb{D}_n$, and the coefficients $b(S,I,R)$, $B(S,I,R)$ are locally Lipschitz-continuous and satisfy linear growth condition on $\mathbb{D}$, then there exists a unique, bounded and Markovian solution up to random time $\tau(\mathbb{D})$ (or $\tau(\mathbb{D}_n)$), where $\tau(\mathbb{D})$ (or $\tau(\mathbb{D}_n)$ represents the random time of the first exit of stochastic process $(S(t),I(t),R(t))$ from the domain $\mathbb{D}$ (or $\mathbb{D}_n$), started in $(S(t),I(t),R(t))=(S(s),I(s),R(s))=(S_0,I_0,R_0)\in\mathbb{D}$ (or $(S_0,I_0,R_0)\in\mathbb{D}_n$) at the initial time $s\in[t_0,\infty)$. To ensure the solution regular, we only prove that $P(\tau(\mathbb{D})=\infty)=1$. a.s. Now, we use function $V\in C^2(\mathbb{D})$ defined on $\mathbb{D}$ via

$$V(S,I,R)=S-\ln S+I-\ln I+(K-S)-\ln(K-S)+(K-R)-\ln(K-R),$$

and assume that $EV(S,I,R)<\infty$. For $(S,I,R)\in\mathbb{D}$, we have $V(S,I,R)\ge 4$ and for $(S,I,R)\in\mathbb{D}\setminus\mathbb{D}_n$, we have

$$\inf_{(S,I,R)\in\mathbb{D}\setminus\mathbb{D}_n}V(S,I,R)>2n+2,\quad\text{for } n\in\mathbb{N}. \tag{7}$$

Define $\mathcal{L}$ as infinitesimal generator as in Definition 2, then calculate

$$\begin{aligned}
\mathcal{L}V(S,I,R)&=\left(-\beta S\int_0^h f(s)I(t-s)\mathrm{d}s+\mu(K-S)\right)\frac{\partial V}{\partial S}\\
&+\left(\beta S\int_0^h f(s)I(t-s)\mathrm{d}s+(\alpha I-\mu R)\frac{\partial V}{\partial R}\right.\\
&+\frac{1}{2}S^2\left(\int_0^h f(s)I(t-s)\mathrm{d}s\right)^2F_1^2(S,I,R)\left(\frac{\partial^2V}{\partial S^2}-2\frac{\partial^2V}{\partial S\partial I}+\frac{\partial^2V}{\partial I^2}\right)\\
&\left.+\frac{1}{2}I^2F_2^2(S,I,R)\left(\frac{\partial^2V}{\partial I^2}-2\frac{\partial^2V}{\partial I\partial R}+\frac{\partial^2V}{\partial R^2}\right)-(\alpha+\gamma+\mu)I\right)\frac{\partial V}{\partial I}\\
&=-\beta S\int_0^h f(s)I(t-s)\mathrm{d}s+\mu(K-S)\left(\frac{1}{K-S}-\frac{1}{S}\right)\\
&+\beta S\int_0^h f(s)I(t-s)\mathrm{d}s\left(1-\frac{1}{I}\right)+(\alpha I-\mu R)\left(\frac{1}{K-R}-1\right)\\
&+\frac{1}{2}S^2\left(\int_0^h f(s)I(t-s)\mathrm{d}s\right)^2F_1^2(S,I,R)\left(\frac{1}{(K-S)^2}+\frac{1}{S^2}+\frac{1}{I^2}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}I^2F_2^2(S,I,R)\left(\frac{1}{I^2}+\frac{1}{(K-R)^2}\right)-(\alpha+\gamma+\mu)I\left(1-\frac{1}{I}\right)\\
&=\frac{-\beta S\int_0^h f(s)I(t-s)\mathrm{d}s}{K-S}+\mu+\beta\int_0^h f(s)I(t-s)\mathrm{d}s-\frac{\mu(K-S)}{S}\\
&+\beta S\int_0^h f(s)I(t-s)\mathrm{d}s-\frac{1}{I}\beta S\int_0^h f(s)I(t-s)\mathrm{d}s+\alpha+\gamma+\mu\\
&+\frac{\alpha I}{K-R}-\alpha I-\frac{\mu R}{K-R}+\mu R+\frac{1}{2}\left(\int_0^h f(s)I(t-s)\mathrm{d}s\right)^2F_1^2(S,I,R)\\
&+\frac{1}{2}S^2\frac{\left(\int_0^h f(s)I(t-s)\mathrm{d}s\right)^2F_1^2(S,I,R)}{I^2}\\
&+\frac{1}{2}\frac{\left(\int_0^h f(s)I(t-s)\mathrm{d}s\right)^2F_2^2(S,I,R)}{(K-R)^2}-(\alpha+\gamma+\mu)I\\
&+\frac{1}{2}S^2\frac{\left(\int_0^h f(s)I(t-s)\mathrm{d}s\right)^2F_1^2(S,I,R)}{(K-S)^2}\\
&+\frac{1}{2}\frac{\left(\int_0^h f(s)I(t-s)\mathrm{d}s\right)^2F_2^2(S,I,R)}{I^2}.
\end{aligned}
$$

In view of the condition (A) that $\int_0^h f(s)I(t-s)\mathrm{d}s\le LI(t)$, and hence we have

$$
\begin{aligned}
\mathcal{L}V(S,I,R)&\le 3\mu+\beta\int_0^h f(s)I(t-s)\mathrm{d}s+\beta S\int_0^h f(s)I(t-s)\mathrm{d}s+2\alpha\\
&\quad+\gamma+\mu R+\sup_{(S,I,R)\in\mathbb{D}}\left\{\frac{3}{2}LK^2F_1^2(S,I,R)+LF_2^2(S,I,R)\right\}\\
&\le 3\mu+\beta LI+\beta SI+2\alpha+\gamma+\mu R\\
&\quad+\sup_{(S,I,R)\in\mathbb{D}}\left\{\frac{3}{2}LK^2F_1^2(S,I,R)+LF_2^2(S,I,R)\right\}.
\end{aligned}
$$

If we take

$$
0<c\le\frac{1}{4}\left[3\mu+\beta LI+\beta SI+2\alpha+\gamma+\mu R+\sup_{(S,I,R)\in\mathbb{D}}\left\{\frac{3}{2}LK^2F_1^2(S,I,R)+LF_2^2(S,I,R)\right\}\right],
$$

therefore $\mathcal{L}V(S,I,R)\le cV(S,I,R)$ due to $V(S,I,R)\ge 4$ for $(S,I,R)\in\mathbb{D}$.

In what follows, to show that $P(\tau(\mathbb{D})=\infty)=1$, *i.e.*, $P(\tau(\mathbb{D})<t)=0$. Now, introduce a new function $W\in C^{1,2}([s,\infty)\times\mathbb{D})$ by $W(t,(S,I,R))=\mathrm{e}^{-c(t-s)}V(S,I,R)$, where c is defined as above. And hence

$$
\mathcal{L}W(t,(S,I,R))=\mathrm{e}^{-c(t-s)}\left[-cV(S,I,R)+\mathcal{L}V(S,I,R)\right]\le 0,
$$

since $\mathcal{L}V(S,I,R)\le cV(S,I,R)$. Denote $\tau_n:=\min\{\tau(\mathbb{D}_n),t\}$ and apply averaged Itô formula, we have

$$
\begin{aligned}
&E\left[\mathrm{e}^{c(t-\tau_n)}V(S(\tau_n),I(\tau_n),R(\tau_n))\right]\\
&=E\left[\mathrm{e}^{c(t-s)}\mathrm{e}^{-c(\tau_n-s)}V(S(\tau_n),I(\tau_n),R(\tau_n))\right]\\
&=E\left[\mathrm{e}^{c(t-s)}W\left[\tau_n,(S(\tau_n),I(\tau_n),R(\tau_n))\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{e}^{c(t-s)} E\left[W\left[\tau_n,\left(S\left(\tau_n\right),I\left(\tau_n\right),R\left(\tau_n\right)\right)\right]\right] \\
&= \mathrm{e}^{c(t-s)} \left[EW\left[s,\left(S(s),I(s),R(s)\right)\right]\right. \\
&\quad \left.+ E\int_s^{\tau_n} \mathcal{L}W\left(x,\left(S(x),I(x),R(x)\right)\right)\mathrm{d}x\right] \\
&\le \mathrm{e}^{c(t-s)} EW\left[s,\left(S(s),I(s),R(s)\right)\right] \\
&= \mathrm{e}^{c(t-s)} EV\left[\left(S(s),I(s),R(s)\right)\right] \\
&\le \mathrm{e}^{c(t-s)} EV\left(S_0,I_0,R_0\right).
\end{aligned}
$$

Using this fact and Equation (7), one can estimates

$$
\begin{aligned}
0 &\le P\left(\tau(\mathbb{D})<t\right) \le P\left(\tau\left(\mathbb{D}_n\right)<t\right) = P\left(\tau_n<t\right) = E\left(I_{\tau_n}<t\right) \\
&\le E\left[\mathrm{e}^{c(t-\tau_n)} \frac{V\left(S\left(\tau\left(\mathbb{D}_n\right)\right),I\left(\tau\left(\mathbb{D}_n\right)\right),R\left(\tau\left(\mathbb{D}_n\right)\right)\right)}{\inf_{(S,I,R)\in\mathbb{D}\setminus\mathbb{D}_n} V(S,I,R) I_{\tau_n<t}}\right] \\
&\le \mathrm{e}^{c(t-\tau_n)} \frac{EV\left(S_0,I_0,R_0\right)}{\inf_{(S,I,R)\in\mathbb{D}\setminus\mathbb{D}_n} V(S,I,R)} \\
&\le \mathrm{e}^{c(t-\tau_n)} \frac{EV\left(S_0,I_0,R_0\right)}{2n+2} \to 0 \quad \text{as } n\to\infty,
\end{aligned}
$$

for all fixed $t\in[s,+\infty)$, because of the appearance of the function $I(.)$. Thus $P\left(\tau(\mathbb{D})<t\right)=P\left(\tau\left(\mathbb{D}_n\right)<t\right)=0$ for $\left(S_0,I_0,R_0\right)\in\mathbb{D}$ and $t\ge t_0$, that is, $P\left(\tau\left(\mathbb{D}_n\right)=\infty\right)=1$.

Then it proves the regularity and the global existence of the solution $\left(S(t),I(t),R(t)\right)\in\mathbb{D}$ and by means of Lemma 2.1 under the condition of (A), we also derive the uniqueness and continuity of the solution.

Similarly the above discussions, we only need to take the function $V(S,I,R)$ as $S-\ln S+(K-S)-\ln(K-S)$, and we can also obtain the same results under the condition of (B). Here, we omits the details.

This completes the proof of Theorem 3.1. □

Remark 3.1. *Because $I=S=R=0$ are undefined in the domain $\mathbb{D}$. In what follows, we distinguish three cases to investigate the solution of these special situations.*

1) If $S=0$, then the system (5) will reduce to

$$
\begin{cases}
\mathrm{d}I(t)=\left(\mu K-(\alpha+\gamma+\mu)I(t)\right)\mathrm{d}t-\int_0^h f(s)I(t-s)\mathrm{d}sF\left(I(t),R(t)\right)\mathrm{d}W, \\
\mathrm{d}R(t)=\left(\alpha I(t)-\mu R(t)\right)\mathrm{d}t+\int_0^h f(s)I(t-s)\mathrm{d}sF\left(I(t),R(t)\right)\mathrm{d}W,
\end{cases} \tag{8}
$$

with intial condition $\left(I_0,R_0\right)\in D_1=\left\{(I,R): I>0,R>0,I+R\le K\right\}$. By using the similar analysis, we know that the above SDE is regular which implies there exists a unique global solution on D_1;

2) If $I=0$, then the system (5) will reduce to an ODE

$$
\begin{cases}
\mathrm{d}S(t)=\mu\left(K-S(t)\right)\mathrm{d}t, \\
\mathrm{d}R(t)=-\mu R(t)\mathrm{d}t,
\end{cases} \tag{9}
$$

with intial condition $(S_0, R_0) \in D_2 = \{(S,R): S>0, R>0, S+R \le K\}$. By using the theory of ODE, we know that the above ODE is regular which implies there exists a unique global solution on D_2;

3) If $R=0$, then the system (5) will become

$$\begin{cases} \mathrm{d}S(t) = \left(-\beta S(t)\int_0^h f(s) I(t-s)\mathrm{d}s + \mu\left(K - S(t)\right)\right)\mathrm{d}t \\ \qquad\qquad -\int_0^h f(s) I(t-s) ds \times F\left(S(t), I(t)\right)\mathrm{d}W, \\ \mathrm{d}I(t) = \left(\beta S(t)\int_0^h f(s) I(t-s)\mathrm{d}s - (\alpha+\mu) I(t)\right)\mathrm{d}t \\ \qquad\qquad + S(t)\int_0^h f(s) I(t-s)\mathrm{d}s \times F\left(S(t), I(t)\right)\mathrm{d}W, \end{cases} \tag{10}$$

with intial condition $(S_0, I_0) \in D_3 = \{(S,I): S>0, I>0, S+I \le K\}$. *By using the similar analysis, we know that the above SDE is regular which implies there exists a unique global solution on* D_3.

4. Global Stochastic Asymptotic Stability of Disease Free Equilibrium

Theorem 4.1. *Assume that* $\int_0^h f(s) I(t-s)\mathrm{d}s \le \frac{\alpha+\gamma+\mu}{K\beta} I(t)$ *for all fixed* $t \in [s, \infty)$, *then the disease free equilibrium solution* $(S_1, I_1, R_1) = (K, 0, 0)$ *of* Equation (5) *is globally stochastically stable on* $\mathbb{D}$.

Proof. Notice that the assumption $\int_0^h f(s) I(t-s)\mathrm{d}s \le \frac{\alpha+\gamma+\mu}{K\beta} I(t)$ for all fixed $t \in [s, \infty)$, and hence one can estimates that there exists a positive constant C which satisfies $K\beta\int_0^h f(s) I(t-s)\mathrm{d}s \le C \le (\alpha+\gamma+\mu) I(t)$ for all fixed $t \in [s, \infty)$. Introduce a Lyapunov function

$$V(S, I, R) = \frac{1}{2}(S+I+R-K)^2 + CI.$$

Just note that the infinitesimal generator $\mathcal{L}$ satisfies

$$\begin{aligned} \mathcal{L}V(S,I,R) &= \left(-\beta S\int_0^h f(s) I(t-s)\mathrm{d}s + \mu(K-S)\right)(S+I+R-K) \\ &\quad + \left(\beta S\int_0^h f(s) I(t-s)\mathrm{d}s - (\alpha+\gamma+\mu) I\right)(S+I+R-K) \\ &\quad + (\alpha I - \mu R)(S+I+R-K) + C\left(\beta S\int_0^h f(s) I(t-s)\mathrm{d}s - (\alpha+\gamma+\mu) I\right) \\ &= -\mu(K-S-I-R)^2 + (K-S-I-R)\gamma I + C\beta S\int_0^h f(s) I(t-s)\mathrm{d}s \\ &\quad - MI(\alpha+\gamma+\mu) - \gamma I^2 - \gamma RI \\ &= -\mu(K-S-I-R)^2 + \left(C\beta\int_0^h f(s) I(t-s)\mathrm{d}s - \gamma I\right)S \\ &\quad + \left[K\gamma - C(\alpha+\gamma+\mu)\right] I - \gamma I^2 - \gamma RI \\ &\le -\mu(K-S-I-R)^2 - \gamma I^2 - \gamma RI, \end{aligned} \tag{11}$$

then $\mathcal{L}V(S,I,R)$ becomes negative definite on $\mathbb{D}$, and hence it completes the proof of Theorem 4.1 by applying Lemma 2.2. □

Remark 4.1. *As we known, the basic reproduction number* R_0 *is one of the most important parameters in epidemiology, which reflects the expected number of secondary infections produced when one infected individual entered a fully susceptible population. If* $R_0<1$ *then the outbreak will disappear, on the other hand, if* $R_0>1$ *then the epidemic will spread a population. In this context, the basic reproduction number of the SIR model is* $R_0=\dfrac{\beta K}{\alpha+\gamma+\mu}$.

5. Stochastic Asymptotic Stability of Endemic Equilibrium

If $R_0>1$ and $F_i(S_2,I_2,R_2)=0\,(i=1,2)$, then there exists a unique endemic equilibrium solution (S_2,I_2,R_2) for the model (5), where

$$(S_2,I_2,R_2)=\left(\frac{\alpha+\gamma+\mu}{\beta},\frac{\mu}{\beta}\left(\frac{\beta K}{\alpha+\gamma+\mu}-1\right),\frac{\alpha}{\beta}\left(\frac{\beta K}{\alpha+\gamma+\mu}-1\right)\right)$$
$$=\left(\frac{K}{R_0},\frac{\mu}{\beta}(R_0-1),\frac{\alpha}{\beta}(R_0-1)\right).$$

Theorem 5.1. *The endemic equilibrium solution,* (S_2,I_2,R_2) *of the* Equation (5) *is stochastically asymptotically stable on* $\mathbb{D}=\{(S,I,R):S>0,I>0,R>0,S+I+R\le K\}$ *under the assumption of* $R_0>1$ *for some* $F_i(S,I,R)$ *such that* $F_i(S_2,I_2,R_2)=0$ *and satisfies* $G(S,I,R)\le 0$, *where*

$$\begin{aligned}G(S,I,R)=&-\frac{\mu}{S+I+R}(S+I+R-S_2-I_2-R_2)^2-\frac{\gamma}{S+I+R}(I-I_2)^2\\&-a\beta(S-S_2)^2\int_0^h f(s)I(t-s)\mathrm{d}s-b\mu(S-S_2)^2-b\mu(R-R_2)^2\\&+\frac{a}{2}S^2\left(\int_0^h f(s)I(t-s)\mathrm{d}s\right)^2F_1^2(S,I,R)\\&+\frac{b}{2}\left(\int_0^h f(s)I(t-s)\mathrm{d}s\right)^2F_2^2(S,I,R),\end{aligned}\tag{12}$$

and

$$a=\begin{cases}s>0, & \text{if } (S-S_2)(I-I_2)>0,(S-S_2)\left(\int_0^h f(s)I(t-s)\mathrm{d}s-I_2\right)>0;\\ 0, & \text{if } (S-S_2)(I-I_2)<0,(S-S_2)\left(\int_0^h f(s)I(t-s)\mathrm{d}s-I_2\right)<0;\\ a>\dfrac{K^2\gamma}{\beta S_2}, & \text{if } (S-S_2)(I-I_2)<0,(S-S_2)\left(\int_0^h f(s)I(t-s)\mathrm{d}s-I_2\right)>0;\\ =0, & \text{if } (S-S_2)(I-I_2)>0,(S-S_2)\left(\int_0^h f(s)I(t-s)\mathrm{d}s-I_2\right)<0,\end{cases}\tag{13}$$

$$b=\begin{cases}\inf\limits_{(S,I,R)\in\mathbb{D}}\dfrac{\gamma}{\alpha(S+I+R)}, & \text{if } (S-S_2)(R-R_2)>0;\\ \dfrac{\gamma}{\alpha K}, & \text{if } (S-S_2)(R-R_2)<0.\end{cases}\tag{14}$$

Proof. It is a fact that the endemic equilibrium solution of system (5) exists if $R_0>1$ and $F_i(S_2,I_2,R_2)=0$. Introduce a Lyapunov function

$$V(S,I,R)=S+I+R-(S_2+I_2+R_2)-(S_2+I_2+R_2)\ln\frac{S+I+R}{S_2+I_2+R_2}$$
$$+\frac{a}{2}(S-S_2)^2+\frac{b}{2}(R-R_2)^2+c,$$

on $\mathbb{D}$, where a and b are defined as Equations ((13) and (14)), c is an arbitrary positive constant. An elementary computation leads to $V>0$ for any point $(S,I,R)\in\mathbb{D}$, and we have

$$\mathcal{L}V(S,I,R)=\left[-\beta S\int_0^h f(s)I(t-s)\mathrm{d}s+\mu(K-S)\right]\left[1-\frac{S_2+I_2+R_2}{S+I+R}+a(S-S_2)\right]$$
$$+\frac{a}{2}S^2\times\left[\int_0^h f(s)I(t-s)\mathrm{d}s\right]^2F_1^2(S,I,R)+\frac{b}{2}\left[\int_0^h f(s)I(t-s)\mathrm{d}s\right]^2F_2^2$$
$$+\left[\beta S\int_0^h f(s)I(t-s)\mathrm{d}s-(\alpha+\gamma+\mu)I\right]\left[1-\frac{S_2+I_2+R_2}{S+I+R}\right]$$
$$+(\alpha I-\mu R)\left[1-\frac{S_2+I_2+R_2}{S+I+R}+b(R-R_2)\right].$$

From the following formulas and the definitions of a, b can help to simplify $\mathcal{L}(S,I,R)$

i) $\mu(K-I-S-R)-\gamma I=-\mu(S+I+R-S_2-I_2-R_2)-\gamma(I-I_2)$;

ii) $$-\beta S\int_0^h f(s)I(t-s)\mathrm{d}s+\mu(K-S)$$
$$=-\beta(S-S_2)\int_0^h f(s)I(t-s)\mathrm{d}s-\beta S_2\left(\int_0^h f(s)I(t-s)\mathrm{d}s-I_2\right)-\mu(S-S_2);$$

iii) $\alpha+\beta+\mu=\beta S_2$;

iv) $\alpha I-\mu R=\alpha(I-I_2)-\mu(R-R_2)$.

Then

$$\mathcal{L}V=\left[\mu(K-S-I-R)-\gamma I\right]\left[1-\frac{S_2+I_2+R_2}{S+I+R}\right]+a(S-S_2)\Big[-\beta(S-S_2)$$
$$\times\int_0^h f(s)I(t-s)\mathrm{d}s-\beta S_2\left(\int_0^h f(s)I(t-s)\mathrm{d}s-I_2\right)-\mu(S-S_2)\Big]$$
$$+\frac{a}{2}S^2\left[\int_0^h f(s)I(t-s)\mathrm{d}s\right]^2F_1^2(S,I,R)+\frac{b}{2}\left[\int_0^h f(s)I(t-s)\mathrm{d}s\right]^2F_2^2$$
$$=-\frac{\mu}{S+I+R}\left[S+I+R-S_2-I_2-R_2\right]^2-\frac{\gamma}{S+I+R}(I-I_2)^2$$
$$-\frac{\gamma}{S+I+R}(S-S_2)(I-I_2)-\frac{\gamma}{S+I+R}(I-I_2)(R-R_2)$$
$$-a\beta(S-S_2)^2\int_0^h f(s)I(t-s)\mathrm{d}s-a\beta S_2(S-S_2)\left(\int_0^h f(s)I(t-s)\mathrm{d}s-I_2\right)$$
$$+\frac{a}{2}S^2\left[\int_0^h f(s)I(t-s)\mathrm{d}s\right]^2F_1^2(S,I,R)+\frac{b}{2}\left[\int_0^h f(s)I(t-s)\mathrm{d}s\right]^2F_2^2$$
$$\le-\frac{\mu}{S+I+R}\left[S+I+R-S_2-I_2-R_2\right]^2-\frac{\gamma}{S+I+R}(I-I_2)^2$$
$$-a\mu(S-S_2)^2+b(R-R_2)\left[\alpha(I-I_2)-\mu(R-R_2)\right]$$
$$+b\alpha(I-I_2)(R-R_2)-b\mu(R-R_2)^2-a\mu(S-S_2)^2$$
$$-a\beta(S-S_2)^2\int_0^h f(s)I(t-s)\mathrm{d}s-b\mu(R-R_2)^2$$
$$+\frac{a}{2}S^2\left[\int_0^h f(s)I(t-s)\mathrm{d}s\right]^2\times F_1^2(S,I,R)+\frac{b}{2}\left[\int_0^h f(s)I(t-s)\mathrm{d}s\right]^2F_2^2.$$

$\mathcal{L}V(S,I,R)=0$ if and only if $(S,I,R)=(S_2,I_2,R_2)$ and by the given condition one can obtain $\mathcal{L}V<0$ on $\mathbb{D}\backslash(S_2,I_2,R_2)$. Therefore $\mathcal{L}V(S,I,R)$ is negative definite on $\mathbb{D}$ for some suitable $F_i(S,I,R)$. Then Lemma 2.2 (ii) leads to the stochastically asymptotical stability of the endemic equilibrium with $R_0>1$ and for some suitable functions $F_i(S,I,R)$ such that $F_i(S,I,R)$ satisfies Equation (12) and $F_i(S_2,I_2,R_2)=0$. □

6. Example

In this section, we visualize our results with some simulation to confirm them. Due to the difficulty of the research on the drawing of the disease equilibrium point, many scholars have not given the relevant examples. Along this clue, we only give the figures of the disease-free equilibrium point (**Figure 1**). We consider the special case $\int_0^1 f(s)I(t-s)\mathrm{d}s=I(t)$ which only satisfies the condition of Theorem 4.1 $\int_0^h f(s)I(t-s)\mathrm{d}s\le\frac{\alpha+\gamma+\mu}{K\beta}I(t)$, that is, $\frac{K\beta}{\alpha+\gamma+\mu}<1$, and hence we can obtain the disease free equilibrium solution $(S_1,I_1,R_1)=(K,0,0)$ of Equation (5) is globally stochastically stable on $\mathbb{D}$. In the simulation, the parameters are chosen as follows

$$K=1000,\quad \mu=1/75=0.013,\quad \alpha=52,\quad \gamma=52,\quad \beta=0.05.$$

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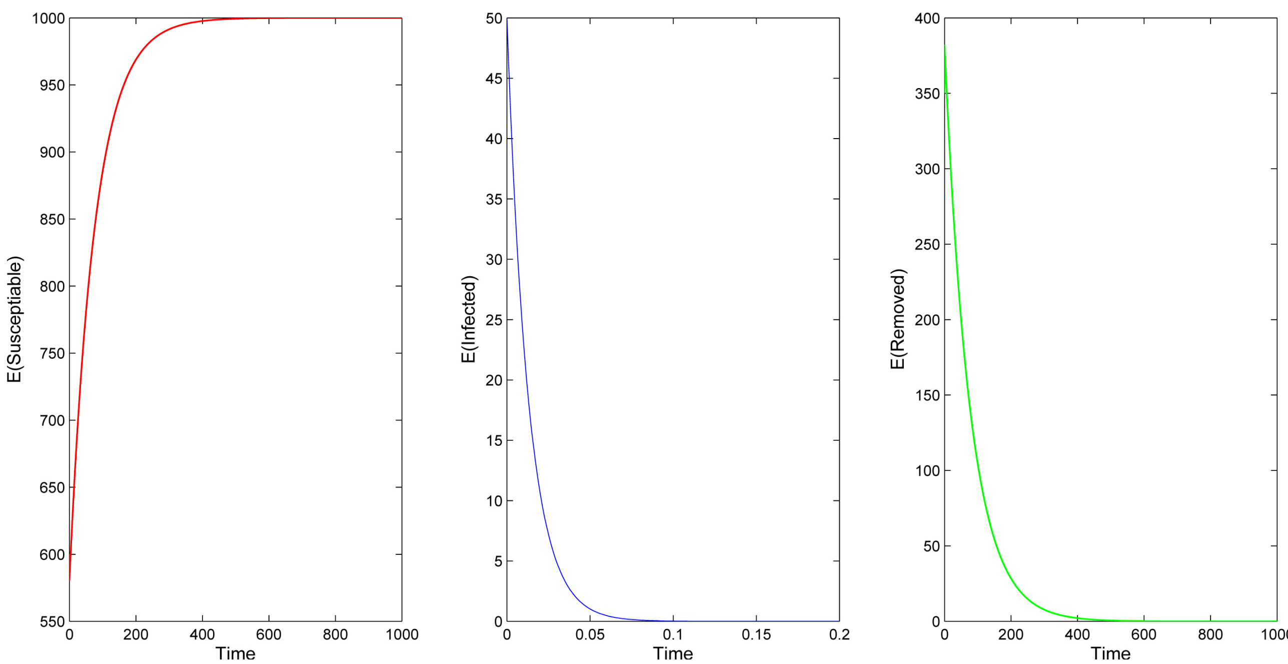

Figure 1. The disease free equilibrium $(S,I,R)=(1000,0,0)$ is globally stochastically asymptotically stable since $\frac{K\beta}{\alpha+\gamma+\mu}<\frac{K\beta}{\gamma}=0.97<1$ for $\alpha=52$, $\mu=0.013$, $K=1000$, $\beta=0.05$ and $\gamma=52$.

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Conflict of Interests

The authors declare that the study was realized in collaboration with the same responsibility.

Competing Interests

The authors declare that they have no competing interests regarding the publication of this paper.

Authors Contributions

All of the authors, XHX, LM and JFX contributed substantially to this paper, participated in drafting and checking the manuscript, and have approved the version to be published.

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