New Result for Strongly Starlike Functions

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Abstract

In this paper, using Salagean differential operator, we define and investigate a new subclass of univalent functions \( S_\alpha^\beta \). We also establish a characterization property for functions belonging to the class \( S_\alpha^\beta \).

Keywords

Strongly Starlike Functions, Strongly Convex Functions, Salagean Differential Operator

1. Introduction

Let \( A \) be the class of functions of the form

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k
\]

which are analytic in the unit disk \( U = \{ z \in C : |z| < 1 \} \). A function \( f(z) \in A \) is said to be starlike of order \( \alpha \) if and only if

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad 0 \leq \alpha < 1 \quad (z \in U)
\]

We denote by \( S^{\alpha} \) the subclass of \( A \) consisting of functions which are starlike of order \( \alpha \) in \( U \).

Also, a function \( f(z) \in A \) is said to be convex of order \( \alpha \) if and only if

\[
\Re \left( 1 + \frac{zf^*(z)}{f(z)} \right) > \alpha, \quad 0 \leq \alpha < 1 \quad (z \in U)
\]

We denote by \( C^{\alpha} \) the subclass of \( A \) consisting of functions which are convex of order \( \alpha \) in \( U \).

If \( f(z) \in A \) satisfies

\[
\left| \arg \left( \frac{zf'(z)}{f(z)} - \alpha \right) \right| < \frac{\pi \beta}{2}, \quad 0 \leq \alpha < 1, \quad 0 < \beta \leq 1 \quad (z \in U)
\]
then $f(z)$ is said to be strongly starlike of order $\beta$ and type $\alpha$ in $U$, denoted by $[1]$. 

If $f(z) \in A$ satisfies 
\[
\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} - \alpha \right) \right| < \frac{\pi \beta}{2}, \quad 0 \leq \alpha < 1, \quad 0 < \beta \leq 1, \quad (z \in U) \tag{5}
\]
then $f(z)$ is said to be strongly convex of order $\beta$ and type $\alpha$ in $U$, denoted by $C_{\alpha \beta}[1]$. 

The following lemma is needed to derive our result for class $S^\alpha_\beta$. 

**Lemma (1)** [2] [3] [4] [5]. Let a function $p(z)$ be analytic in $U$, $p(0) = 1$ and $p(z) \neq 0 \ (z \in U)$, if there exists a point $z_0 \in U$ such that 
\[
\arg(p(z)) < \frac{\pi \beta}{2} \quad (|z| < |z_0|) \quad \text{and} \quad \arg(p(z_0)) = \frac{\pi \beta}{2} \quad \text{with} \quad 0 < \beta \leq 1,
\]
then 
\[
\frac{z_0 p'(z_0)}{p(z_0)} = i k \beta
\]
\[
(6)
\]
where 
\[
k > \frac{1}{2} \left( a + \frac{1}{a} \right) \quad (\text{when} \ \arg(p(z_i))) = \frac{\pi \beta}{2}
\]
\[
k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad (\text{when} \ \arg(p(z_i))) = -\frac{\pi \beta}{2}
\]
And $p(z_0) = z_0^{1/2} \ (a > 0)$. 

**Definition 1.** A function $f(z) \in A$ is said to be in the class $S^\alpha_\beta$ if 
\[
\left| \arg \left( \frac{D^{n+1}f(z)}{D^n f(z)} - \alpha \right) \right| < \frac{\pi \beta}{2}, \quad (z \in U) \tag{7}
\]
For some $\alpha$, $0 \leq \alpha < 1, \ n \in N_0 = N \cup \{0\}, \ 0 < \beta \leq 1$. 

**Remark** 
When $n = 0$ then $S^\alpha_\beta$ is the class studied by [1]. 

**Definition 2.** For functions $f(z) \in A$ the Salagean differential operator [6] is 
$D^\alpha : A \rightarrow A$. 
\[
D^\alpha f(z) = f(z), \quad D^1 f(z) = zf''(z), \cdots D^n f(z) = D^n [D^{n-1} f(z)], \quad n = 0,1,2,3,\cdots
\]

The main focus of this work is to provide a characterization property for the class of functions belonging to the class $S^\alpha_\beta$. 

**2. Main Result** 

**Theorem 1.** If $f(z) \in A$ satisfies 
\[
(i) \quad \frac{D^{n+1}f(z)}{D^n f(z)} \neq \frac{1}{2} \\
(ii) \quad \left| \frac{D^{n+2}f(z)/D^{n+1}f(z)}{D^{n+1}f(z)/D^n f(z)} - 1 \right| < \frac{\beta}{2}, \quad (z \in U)
\]
\[
\tag{3}
\]
for some $\beta$, $0 < \beta \leq 1, \ n \in N_0 = N \cup \{0\}$, then $f(z) \in S^\alpha_\beta$. 


Proof. Let

\[ p(z) = 2 \frac{D^{n+1}f(z)}{D^nf(z)} - 1, \quad n \in \mathbb{N}, n = 0, 1, 2, \ldots \]  

(8)

Taking the logarithmic differentiation in both sides of Equation (8), we have

\[
\frac{p'(z)}{p(z)} = \left[ \frac{D^nf(z)2(D^{n+1}f(z))'}{D^nf(z)} - \frac{2D^{n+1}f(z)[D^nf(z)']}{D^nf(z)} \right] \left[ \frac{D^nf(z)}{2D^{n+1}f(z) - D^nf(z)} \right] 
\]

(9)

Multiplying Equation (9) through by \( p(z) \), to get

\[
p'(z) = \frac{2(D^{n+1}f(z))'}{D^nf(z)} - \frac{2D^{n+1}f(z)[D^nf(z)']}{(D^nf(z))^2} 
\]

(10)

Multiply Equation (10) by \( z \) to obtain

\[
zp'(z) = \frac{2z(D^{n+1}f(z))'}{D^nf(z)} - \frac{2D^{n+1}f(z)z[D^nf(z)']}{(D^nf(z))^2} 
\]

(11)

Multiply Equation (11) through by 2 and divide through by \((1+p(z))^2\) to give

\[
\frac{2zp'(z)}{(1+p(z))^2} = \frac{4(D^{n+2}f(z))}{D^nf(z)(1+p(z))^2} - 1 
\]

(12)

Multiplying Equation (12) by \( \frac{D^{n+1}f(z)}{D^nf(z)} = \frac{1+p(z)}{2} \), and further simplification, we obtain

\[
\frac{D^{n+1}f(z)}{D^nf(z)} \left( 1 + \frac{2zp'(z)}{(1+p(z))^2} \right) = \frac{D^{n+2}f(z)}{D^{n+1}f(z)}, \quad z \in U, \quad n \in \mathbb{N} 
\]

(13)

therefore,

\[
\frac{D^{n+2}f(z)/D^{n+1}f(z)}{D^{n+1}f(z)/D^nf(z)} = 1 + \frac{2zp'(z)}{(1+p(z))^2} 
\]

(14)

If \( \exists \) a point \( z_0 \in U \) which satisfies \( |\arg p(z)| < \frac{\pi \beta}{2} \) \((|z| < |z_0|)\) and \( |\arg p(z_0)| = \frac{\pi \beta}{2} \)
then by lemma [2]

\[
\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta
\]

\[
k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{and} \quad p(z_0) = a^\beta e^{\frac{i\alpha\beta}{z}} \quad \text{or} \quad p(z_0) = a^\beta e^{-\frac{i\alpha\beta}{z}} \quad (a > 0)
\]

Now,

\[
\frac{D^{n+2}f(z_0)/D^{n+1}f(z_0) - 1}{D^{n+1}f(z_0)/D^n f(z_0) - 1} = 2k \beta \frac{|p(z_0)|}{(1 + p(z_0))^2}
\]

\[
\geq \frac{2\beta}{1 + 2a} \left( a + \frac{1}{a} \right) |p(z_0)|
\]

Since,

\[
\frac{1}{(1 + p(z_0))^2} \geq \frac{1}{1 + 2|p(z_0)| + |p(z_0)|^2}
\]

\[
\frac{D^{n+2}f(z_0)/D^{n+1}f(z_0) - 1}{D^{n+1}f(z_0)/D^n f(z_0) - 1} \geq \beta \left( a + \frac{1}{a} \right) \frac{|p(z_0)|}{1 + 2|p(z_0)| + |p(z_0)|^2}
\]

\[
p(z_0) = a^\beta e^{\frac{i\alpha\beta}{z}}, \quad a > 0 \Rightarrow |p(z_0)| = a^\beta
\]

But

\[
\beta \left( a + \frac{1}{a} \right) a^\beta = \frac{(a + 1)^2 a^\beta}{1 + 2a^\beta + a^{2\beta}}
\]

\[
= \left( a + \frac{1}{a} \right) \beta
\]

\[
= a^\beta + 2 + a^{2\beta}
\]

Let

\[
S(a) = \frac{(a + 1)^2 a^\beta}{a^\beta + 2 + a^{2\beta}}
\]

then

\[
S'(a) = \frac{2(a^2 - 1) + (1 - \beta)(a^2(1 + \beta) - 1) + (1 + \beta)(a^2(2 + a^\beta) - 1)}{a^2(a^\beta + 2 + a^{2\beta})^2}
\]

Hence, \( S'(a) = 0 \Rightarrow a = 1 \).

It implies that

\[
S'(a) < 0 \text{ when } 0 < a < 1 \text{ and } S'(a) > 0 \text{ when } a > 1, \text{ hence, } a = 1 \text{ is a minimum point of } S(a) \cdot S(1) = \frac{1}{2}.
\]

Therefore, we have that

\[
\frac{D^{n+2}f(z_0)/D^{n+1}f(z_0) - 1}{D^{n+1}f(z_0)/D^n f(z_0) - 1} \geq \frac{\beta}{2}, \quad n \in N, \quad z \in U
\]
which contradicts the condition of the theorem.

Hence, it is concluded from lemma [2] that

$$\left| \arg p(z) \right| = \left| \arg \left( \frac{D^n f(z)}{D^s f(z)} \right) - \frac{1}{2} \right| < \frac{\pi \beta}{2}, \quad z \in U, \quad n \in N_0$$

so that

$$f(z) \in S^\beta_i \left( \frac{\pi}{2} \right).$$

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References


