Numerical Solution of Nonlinear Mixed Integral Equation with a Generalized Cauchy Kernel

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Abstract

In this article, we present approximate solution of the two-dimensional singular nonlinear mixed Volterra-Fredholm integral equations (V-FIE), which is deduced by using new strategy (combined Laplace homotopy perturbation method (LHPM)). Here we consider the V-FIE with Cauchy kernel. Solved examples illustrate that the proposed strategy is powerful, effective and very simple.

Keywords

Singular Integral Equation, Linear and Nonlinear V-FIE, Homotopy Perturbation Method (HPM), Cauchy Kernel

1. Introduction

The V-FIE arises from parabolic boundary value problems. Many authors have interested in solving the linear and nonlinear integral equation. The time collocation method was introduced by Pachpatta [1] and the projection method by Hacia [2]. Brunner [3] extended the Pachpatta's [2] results to nonlinear Volterra-Hammerstein integral equations. In [4] treated Maleknejad and Hadizadeh V-FIE by using the Adomian decomposition method (ADM) presented in [5]. Wazwaz [6] introduced the modified ADM for solving the V-FIE.

We consider the nonlinear mixed V-FIE with a generalized singular kernel

$$\varphi(\aleph,t) = g(\aleph,t) + \lambda \int_0^t \int_\Omega F(t,\zeta) k(|\aleph-\eta|) \gamma(\eta,\zeta,\varphi(\eta,\zeta)) \mathrm{d}\eta \mathrm{d}\zeta \qquad (1)$$

The functions $k(|\aleph-\eta|)$, $F(t,\zeta)$ and $g(\aleph,t)$ are given and called the kernel of Fredholm integral term, Volterra integral term and the free term respectively and $\lambda \neq 0$ denotes a (real or complex) parameter. Also, Ω is the domain of integration with respect to position, and the time $t,\zeta \in [0,\tilde{T}], \tilde{T} < \infty$.

While $\varphi(\aleph,t)$ is the unknown function to be determined in the space $L_p(\Omega) \times C[0,\tilde{T}]$. The existence and uniqueness results for Equation (1) were found in [7] [8].

Many authors have studied solutions of linear and nonlinear integral equations by utilizing different techniques, for example Abdou *et al.* in [7] [8] considered the integral equation with singular kernel and used Toeplitz matrix method (TMM) and product Nystrom method (PNM) to obtain the solution. Abdou *et al.* in [9] discussed the solution of linear and nonlinear Hammerstein integral equations with continuous kernel and used two different methods (Adomian decomposition method and homotopy analysis method). In [10], El-Kalla and Al-Bugami used ADM and degenerate kernel method for solving nonlinear V-FIE with continuous kernel.

With the quick advancement of nonlinear sciences, many analytical and numerical techniques have been produced and developed by various scientists, for example, the HPM introduced by He [11] [12]. Many research works have been conducted recently in applying this method to a class of linear and nonlinear equations [13] [14]. We extend the techniques to solve nonlinear mixed V-FIE with a generalized singular kernel.

In this article, we present new strategy which is the combined LHPM to obtain approximate solutions with high degree of accuracy for the nonlinear mixed V-FIE with a generalized Cauchy kernel.

2. The Homotopy Perturbation Method (HPM)

In this section, we will present the HPM. We consider a general integral equation

$$L\varphi = 0 \tag{2}$$

where L is an integral operator. Define a convex homotopy $\breve{H}(\vartheta,\wp)$ by

$$\breve{H}(\vartheta,\wp) = (1-\wp)F(\vartheta) + \wp L(\vartheta) = 0, \quad \wp \in [0,1], \tag{3}$$

where $F(\vartheta)$ is a functional operator with solution ϑ_0. Then

$$\breve{H}(\vartheta,0) = F(\vartheta) = 0, \quad \breve{H}(\vartheta,1) = L(\vartheta) = 0, \tag{4}$$

and the process of changing $\wp$ from 0 to 1 is just that of changing ϑ from ϑ_0 to φ. In topology, this is called deformation. $F(\vartheta)$ and $L(\vartheta)$ are called homotopies.

According to the HPM, we can use the embedding parameter $\wp$ as a "small parameter", and assume that the solution of Equation (3) can be written as a power series in $\wp$:

$$\vartheta = \varphi_0 + \wp\,\varphi_1 + \wp^2\varphi_2 + \cdots \tag{5}$$

when $\wp \to 1$, the approximate solution of Equation (2) is obtained with

$$\varphi = \lim_{\wp \to 1} \vartheta = \varphi_0 + \varphi_1 + \varphi_2 + \cdots \tag{6}$$

The series Equation (6) is convergent for most cases; however, the rate of convergence depends upon the nonlinear operator L [11].

3. The HPM Applied to Nonlinear Mixed V-FIE

To illustrate the HPM, for nonlinear mixed V-FIE let us consider the Equation (1)

$$\breve{H}(\vartheta,\wp) = \vartheta(\aleph,t) - g(\aleph,t)$$
$$- \wp\,\lambda \int_0^t \int_\Omega F(t,\zeta) k(|\aleph-\eta|) \gamma(\eta,\zeta,\vartheta(\eta,\zeta)) \mathrm{d}\eta\mathrm{d}\zeta \qquad (7)$$
$$= 0$$

By the HPM, we can expand $\vartheta(\aleph,t)$ into the form

$$\vartheta(\aleph,t) = \varphi_0(\aleph,t) + \wp\,\varphi_1(\aleph,t) + \wp^2\varphi_2(\aleph,t) + \cdots \qquad (8)$$

and the approximate solution is

$$\varphi(\aleph,t) = \lim_{\wp \to 1} \vartheta(\aleph,t) = \varphi_0(\aleph,t) + \varphi_1(\aleph,t) + \varphi_2(\aleph,t) + \cdots \qquad (9)$$

and in sum, according to [15], He's HPM considers the nonlinear term $\gamma(\varphi)$ as

$$\gamma(\varphi) = \sum_{i=0}^{\infty} \wp^i \breve{H}_i = \breve{H}_0 + \wp\breve{H}_1 + \wp^2\breve{H}_2 + \cdots, \qquad (10)$$

where $H'_n s$ are the so-called He's polynomials [15], which can be calculated by using the formula

$$\breve{H}_n = \frac{1}{n} \frac{\partial^n}{\partial \wp^n} \left[\gamma\left(\eta,\zeta,\sum_{i=0}^{\infty} \wp^i \varphi_i\right)\right]_{\wp=0}, \quad n = 0,1,2,\cdots \qquad (11)$$

Substituting (8) and (10) into (7) and equating the terms with identical powers of $\wp$, we have

$$\wp^0 : \varphi_0(\aleph,t) = g(\aleph,t),$$
$$\wp^{i+1} : \varphi_{i+1}(\aleph,t) = \lambda \int_0^t \int_\Omega F(t,\zeta) k(|\aleph-\eta|) \breve{H}_i \mathrm{d}\eta\mathrm{d}\zeta, \ i \geq 0 \qquad (12)$$

The components $\varphi_i(\aleph,t), i \geq 0$ can be computing by using the recursive relations (12).

4. The Combined LHPM Applied to Nonlinear Mixed V-FIE [16]

We assume that the kernel $k(|\aleph-\eta|)$ of Equation (7) takes the form

$$k(|\aleph-\eta|) = \frac{1}{(\aleph^2 - \eta^2)}$$

Applying the Laplace transform to both sides of Equation (7), we represent the linear term $\vartheta(\aleph,t)$ from Equation (8) and the nonlinear term $\gamma(\eta,\zeta,\vartheta(\eta,\zeta))$ will be represented by the He's polynomials from Equation (10) and equating the terms with identical powers of $\wp$, we have:

$$\wp^0 : \ell\{\varphi_0(\aleph,t)\} = \ell\{g(\aleph,t)\},$$
$$\wp^{i+1} : \ell\{\varphi_{i+1}(\aleph,t)\} = \lambda\ell\left[\{F(t,\zeta)\}\{k(|\aleph-\eta|)\}\{\breve{H}_i\}\right], \ i \geq 0 \qquad (13)$$

Applying the inverse Laplace transform to the first part of Equation (13) gives $\varphi_0(\aleph,t)$, that will define $\breve{H}_0$. Utilizing $\breve{H}_0$ will enable us to evaluate $\varphi_1(\aleph,t)$.

The determination of $\varphi_0(\aleph,t)$ and $\varphi_1(\aleph,t)$ leads to the determination of $\breve{H}_1$ that will allows us to determine $\varphi_2(\aleph,t)$, and so on. This in turn will lead to the complete determination of the components of $\varphi_i, i \geq 0$, upon utilizing the second part of Equation (13). The series solution follows immediately after using Equation (9). The obtained series solution may converge to an exact solution if such a solution exists.

5. Numerical Examples

Example 5.1 [7]:

Consider the linear mixed V-FIE with a generalized Cauchy kernel

$$\varphi(\aleph,t) = g(\aleph,t) + \lambda \int_0^t \int_{-1}^1 \zeta^2 \frac{1}{\left(\aleph^2 - \eta^2\right)} \varphi(\eta,\zeta) \, d\eta \, d\zeta \tag{14}$$

$$\lambda = 1.5, N = 20, \quad \text{the exact solution} \quad \varphi(\aleph,t) = \aleph^5 t^6$$

we obtain Table 1.

Example 5.2 [8]:

Consider the nonlinear mixed V-FIE with a generalized Cauchy kernel

$$\varphi(\aleph,t) = g(\aleph,t) + \lambda \int_0^t \int_{-1}^1 \zeta^2 \frac{1}{\left(\aleph^2 - \eta^2\right)} \varphi^3(\eta,\zeta) \, d\eta \, d\zeta \tag{15}$$

$$\lambda = 1.5, N = 20, \quad \text{the exact solution} \quad \varphi(\aleph,t) = \aleph^5 t^6$$

we obtain Table 2.

The results for this examples using the LHPM obtained in Table 1 and Table 2 are best from the results in [7] [8] where the solution was obtained using TMM.

Table 1. Results obtained for example 1 and error.

$\aleph$	Exact	App. (φ)	Error
		$t = 0.03$	
−1.00E+00	−7.29000E−10	−7.28829E−10	1.71000E−13
−8.00E−01	−2.38879E−10	−2.38850E−10	2.90000E−14
−6.00E−01	−5.66870E−11	−5.66841E−11	2.90000E−15
6.00E−01	5.66870E−11	5.66899E−11	2.90000E−15
8.00E−01	2.38879E−10	2.38908E−10	2.90000E−14
1.00E+00	7.29000E−10	7.29171E−10	1.71000E−13
		$t = 0.7$	
−1.00E+00	−1.17649E−01	−1.04112E−01	1.35370E−02
−8.00E−01	−3.855120E−02	−3.61709E−02	2.38030E−03
−6.00E−01	−9.14839E−03	−8.90295E−03	2.45440E−04
6.00E−01	9.14839E−03	9.40544E−03	2.57050E−04
8.00E−01	3.85512E−02	4.12071E−02	2.65590E−03
1.00E+00	1.17649E−01	1.34400E−01	1.67510E−02

Table 2. Results obtained for example 2 and error.

ℵ	Exact	App. (φ)	Error
	$t = 0.03$		
−1.00E+00	−7.29000E−10	−7.29000E−10	0.0000E+00
−8.00E−01	−2.38879E−10	−2.38879E−10	0.0000E+00
−6.00E−01	−5.66870E−11	−5.66870E−11	0.0000E+00
6.00E−01	5.66870E−11	5.66870E−11	0.0000E+00
8.00E−01	2.38879E−10	2.38879E−10	0.0000E+00
1.00E+00	7.29000E−10	7.29000E−10	0.0000E+00
	$t = 0.7$		
−1.00E+00	−1.17649E−01	−1.17573E−01	7.60000E−05
−8.00E−01	−3.85512E−02	−3.85499E−02	1.30000E−06
−6.00E−01	−9.14839E−03	−9.14838E−03	1.10000E−08
6.00E−01	9.14839E−03	9.14840E−03	1.80000E−08
8.00E−01	3.85512E−02	3.85525E−02	1.30000E−06
1.00E+00	1.17649E−01	1.17725E−01	7.60000E−05

6. Conclusion

In this article, we proposed LHPM and used it for solving nonlinear mixed V-FIE with a generalized singular kernel. As examples show, the displayed technique diminishes the computational difficulties of other methods. An interesting feature of this method is that the error is too small and all the calculations can be done straightforward. It can be concluded that LHPM is a very simple, powerful and effective method.

Acknowledgements

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