A Characterization of Graphs with Rank No More Than 5

Haicheng Ma, Xiaohua Liu

Department of Mathematics, Qinghai Nationalities University, Xining, China
Email: qhmymhc@163.com

Abstract

The rank of a graph is defined to be the rank of its adjacency matrix. In this paper, the Matlab was used to explore the graphs with rank no more than 5; the performance of the proposed method was compared with former methods, which is simpler and clearer; and the results show that all graphs with rank no more than 5 are characterized.

Keywords
Graph, Matrix, Rank, Nullity

1. Introduction

In this paper only consider simple graph of finite and unordered. $G = (V(G), E(G))$ is a graph, $V(G) = \{v_1, v_2, \ldots, v_n\}$ is vertices set of a graph $G$, the adjacency matrix $A(G)$ of a graph $G$ is the $n \times n$ symmetric matrix $[a_{ij}]$ such that $a_{ij} = 1$ if $v_i$ is adjacent to $v_j$, and $a_{ij} = 0$ otherwise. Obviously, $A(G)$ is a real symmetric matrix, and all eigenvalues are real number, denoted by eigenvalues of a graph $G$. The rank of a graph $G$, written as $r(G)$, is defined to be the number of the rank of matrix $A(G)$. The nullity of a graph $G$ is the multiplicity of the zero eigenvalues of matrix $A(G)$ and denoted by $\eta(G)$. Clearly, $\eta(G) + r(G) = |V(G)|$. In chemistry, the nullity is correlated with the stability of hydrocarbon that a graph $G$ represented (see [1]-[6]). Collatz and Sinogowitz [1] posed the problem of characterizing all non- singular graphs, which is required to describe the issue of all nullity greater than zero; although this problem is very hard, still a lot of literature research it (see [5] [7] [8]). It is known that the rank $r(G)$ of a graph $G$ is equal to 0 if and only if $G$ is a null graph (i.e. a graph without edges), and there is no graph with rank 1. The graph $G$ with the rank $r(G)$ is equal to 2 or 3, which is completely characterized in [8]. The graph $G$ with the rank $r(G)$ is equal to 4, which is
completely characterized in [9]. Although in [10], the graphs with rank 5 are characterized by using forbidden subgraph. In the paper, we completely characterize the graphs with rank no more than 5 by using Matlab. Compared to the method in [10], the method of this paper is simpler and clearer.

For a vertex \( x \) in \( G \), the set of all vertices in \( G \) that are adjacent to \( x \) is denoted by \( N_G(x) \). The distance between \( u \) and \( v \), denoted by \( \text{dist}_G(u,v) \), is the length of a shortest \( u, v \)-path in graph \( G \). The distance between a vertex \( u \) and a subgraph \( H \) of \( G \), denoted by \( \text{dist}_G(u,H) \), is defined to be the value \( \min \{ \text{dist}_G(u,v) : v \in V(H) \} \). Given a subset \( S \subseteq V(G) \), the subgraph of \( G \) induced by \( S \), is written as \( G[S] \). The \( n \)-path, the \( n \)-cycle and the \( n \)-complete graph are denoted by \( P_n \), \( C_n \) and \( K_n \), respectively.

A subset \( I \subseteq V(G) \) is called an independent set of \( G \) if the subgraph \( G[I] \) is a null graph. Next we define a graph operation (see page 53 of [6]). Give a graph \( G \) with \( V(G) = \{v_1, v_2, \ldots, v_n\} \). Let \( m = (m_1, m_2, \ldots, m_n) \) be a vector of positive integers. Denoted by \( G \odot m \) the graph is obtained from \( G \) by replacing each vertex \( v_i \) of \( G \) with an independent set of \( m_i \) vertices \( v'_1, v'_2, \ldots, v'_{m_i} \) and joining \( v'_i \) with \( v'_j \) if and only if \( v_i \) and \( v_j \) are adjacent in \( G \). The resulting graph \( G \odot m \) is said to be obtained from \( G \) by multiplication of vertices. For graphs \( G_1, G_2, \ldots, G_k \), we denote by \( \mathcal{M}(G_1, G_2, \ldots, G_k) \) the class of all graphs that can be obtained from one of the graphs in \( \{G_1, G_2, \ldots, G_k\} \) by multiplication of vertices.

2. Preliminaries

**Lemma 2.1.** [9] Suppose that \( G \) and \( H \) are two graphs. If \( G \in \mathcal{M}(H) \), then \( r(G) = r(H) \).

By Lemma 2.1, we know that the rank of a graph doesn’t change by multiplication of vertices. Let \( G \) be a graph, if exists a graph \( H (\not\cong G) \) such that \( G \in \mathcal{M}(H) \), we call \( G \) is a non-basic graph. Otherwise, \( G \) is called a basic graph. The following we need find all basic graphs with rank no more than 5.

**Lemma 2.2.** [3] (1) Let \( G = H_1 \cup H_2 \), where \( H_1 \) and \( H_2 \) be two graphs. Then \( r(G) = r(H_1) + r(H_2) \).

(2) Let \( H \) be an induced subgraph of \( G \). Then \( r(H) \leq r(G) \).

**Lemma 2.3.** Let \( G \) be a connected graph with rank \( k (\geq 2) \). Then there exists an induced subgraph \( H \) (of \( G \)) on \( k \) vertices such that \( r(H) = k \), and \( \text{dist}_G(u,H) \leq 1 \) for each vertex \( u \) of \( G \).

**Proof.** Without loss of generality, suppose the previous \( k \) row vectors of \( A(G) \) are linear independence, and the rest of the row vectors of \( A(G) \) are linear combination of the previous \( k \) row vectors. Since \( A(G) \) is a symmetrical matrix, we know that the rest of the column vectors of \( A(G) \) are linear combination of the previous \( k \) column vectors. Therefore we can obtain the following matrix by using elementary transformation for \( A(G) \),

\[
\begin{bmatrix}
A(H) & 0 \\
0 & 0
\end{bmatrix}
\]
where $H$ is the induced subgraph (of $G$) with the $k$ vertices which is correspondent to the previous $k$ vectors, and $r(H) = r(G) = k$.

Suppose $v \in V(G)$ satisfying $\text{dist}_G(v, H) = 2$. Then there exists an induced subgraph $F$ of $G$ such that

\[
A(F) = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
1 & 0 & x_1 & x_2 & \cdots & x_k \\
0 & x_1 & 0 & \cdots & \cdots & A(H) \\
0 & x_2 & \cdots & \cdots & \cdots & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \\
0 & x_k & \cdots & \cdots & \cdots & 0
\end{bmatrix},
\]

where $x_i \in \{0, 1\}, i = 1, 2, \ldots, k$. Obviously, $r(F) = r(H) + 2 = k + 2$, this contradicts to $r(G) = k$. \qed

Let $H$ be an induced subgraph of $G$. For a vertex subset $U$ of $V(H)$, denote by $S^H_U$ (Abbreviated as $S_U$) the set

\[
\{x \in V(G) \setminus V(H) \mid N_G(x) \cap V(H) = U\}.
\]

### 3. Main Result

Let $G$ be a graph with $n$ vertices, $v_1, v_2, \ldots, v_n$ be ordered vertices of $G$. $u \in V(G)$, $n$-dimensional column vector $\alpha_u = (x_1, x_2, \ldots, x_n)^T$ is called adjacency vector of $u$, where $x_i = 1$ if $u$ is adjacent to $v_i$, and $x_i = 0$ otherwise.

For obtaining all connected basic graphs with rank $r$, we have two steps.

**Step 1.** Find out all graphs with rank $r$ which have exactly $r$ vertices. Denote them by $G_1, G_2, \ldots, G_s$.

**Step 2.** Find out all connected graphs with rank $r$ which have more than $r$ vertices. Let $G$ be a graph with rank $r$. By Lemma 2.3, we know that $G$ contains an induced subgraph $G_i (i \in \{1, 2, \ldots, s\})$ with rank $r$ and $\text{dist}_G(u, G_i) \leq 1$ for each vertices $u$ of $G$. Therefore, we consider the adjacent relation between $u$ and the vertices of $G_i$. Let

\[
B = \begin{bmatrix}
A(G_i) & \alpha \\
\alpha^T & 0
\end{bmatrix},
\]

satisfying

\[
r(B) = r(A(G_i)) = r, \quad (*)
\]

where $A(G_i)$ is adjacency matrix of $G_i$, $\alpha = (x_1, x_2, \ldots, x_r)$, $x_i \in \{0, 1\}$ $(i = 1, 2, \ldots, r)$ is a $|V(G)|$-dimensional column vector. We calculate all vectors $\alpha$ satisfying condition $(*)$ by MATLAB.

Obviously, $\alpha = (0, 0, \ldots, 0)$ and adjacency vectors of any vertex $v$ in $G_i$ satisfy $(*)$; this implies that $u$ is not adjacent to $G_i$ or $u \in S^G_{n_G(v(i))}$. This is not the connected basic graphs that we need to find. Therefore, these $r + 1$ vectors are called trivial vectors and the rest of the vectors (if it exist) are non-trivial vectors. If there exist non-trivial vectors $\alpha_1, \alpha_2, \cdots, \alpha_t$ such that $(*)$ holds, then for any vector $\alpha_j$ $(j = 1, 2, \ldots, t)$, we can obtain a basic graph $G_{ij}$ on $r + 1$ vertices; its adjacency matrix is
$$B = \begin{bmatrix} A(G_i) & \alpha_j \\ \alpha_j^T & 0 \end{bmatrix},$$

$$r(G_j) = r(G_i) = r$$

(In fact, suppose $G_j$ is not a basic graph. Then it is obtained from some graph $H \not\cong G_j$ by multiplication of vertices. Thus there are two vertices $v_i$ and $v_j$ which are not adjacent in $G_j$; the adjacent relation between $v_i$ and any vertex of $G_j$ and the adjacent relation between $v_j$ and any vertex of $G_j$ are the same. Since $\alpha_j$ is non-trivial vector, we have $u \not\in \{v_i, v_j\}$. Hence the adjacent relation between $v_i$ and any vertex of $G_j \setminus u$ and the adjacent relation between $v_j$ and any vertex of $G_j \setminus u$ are the same. (where $G_j \setminus u = G_i$ is the graph obtained from $G_j$ by removing the vertex $u$ and all edges associated with $u$. Note $G_j \setminus u = G_i$, we have $r(G_i) < r$, a contradiction.)

Repeat the above process for $G_j$, it will obtain a family of basic graphs. Continue to repeat the above process for these basic graphs until every basic graph does not produce non-trivial vectors. We can find out all basic graphs with rank $r$. Now we give two examples.

**Example 3.1.** Let $G$ be a connected graph and $r(G) = 2$, then $G \in \mathcal{M}(K_2)$.

In fact, $K_2$ is a unique graph [7] with rank 2 which have exactly two vertices. Calculating by MATLAB, have three and only three vectors

$$\alpha = (x_1, x_1)^T = (0, 0)^T, (1, 0)^T \text{ and } (1, 0)^T$$

satisfying that the rank of matrix $B$ is 2, and they are trivial.

$$B = \begin{bmatrix} 0 & 1 & x_1 \\ 1 & 0 & x_2 \\ x_1 & x_2 & 0 \end{bmatrix}$$

Hence, $K_2$ is unique basic graph with rank 2, thus $G \in \mathcal{M}(K_2)$.

**Example 3.2.** Let $G$ be a connected graph and $r(G) = 3$, then $G \in \mathcal{M}(K_3)$.

In fact, $K_3$ is a unique graph [7] with rank 3 which have exactly three vertices. Calculating by MATLAB, have four and only four vectors

$$\alpha = (x_1, x_2, x_3)^T = (0, 0, 0)^T, (1, 1, 0)^T, (1, 0, 1)^T \text{ and } (1, 1, 0)^T$$

satisfying that the rank of matrix $B$ is 3, and they are trivial.

$$B = \begin{bmatrix} 0 & 1 & 1 & x_1 \\ 1 & 0 & 1 & x_2 \\ 1 & 1 & 0 & x_3 \\ x_1 & x_2 & x_3 & 0 \end{bmatrix}$$

Hence, $K_3$ is unique basic graph with rank 3, thus $G \in \mathcal{M}(K_3)$.

The paper [9] has given all basic graphs with rank 4 (see Figure 1). It is easy to obtain these graphs with our method. We write the following theorem without proof.

**Theorem 3.1.** [9] Let $G$ be a graph. Then $r(G) = 4$ if and only if $G$ can be obtained from one of the graphs shown in Figure 1 by multiplication of vertices.

**Theorem 3.2.** [7] Suppose that $G$ is a graph on 5 vertices. Then $r(G) = 5$ if and only if $G$ is one of the graphs shown in Figure 2.
Theorem 3.3. Let $G$ be a graph without isolated vertices. Then $r(G) = 5$ if and only if $G$ can be obtained from one of the graphs shown in Figure 2 and Figure 3 by multiplication of vertices.

Proof. We now prove the necessary part. Assume that $G$ is not connected, then $G = H_1 \cup H_2$ and $r(H_1) = 2$, $r(H_2) = 3$, where $H_1$ and $H_2$ are two graphs. By the example 1 and example 2, we have $G \in \mathcal{M}(K_2 \cup K_1)$. Now assume that $G$ is connected. By Lemma 2.3, there exist induced subgraphs $H = G_i (i = 1, 2, \cdots, 9)$ of $G$ (see Figure 2) such that $\text{dist}_G(u, H) \leq 1$ for each vertex $u$ of $G$. According to the differences of induced subgraphs that $G$ contains, we consider the following Case 1-Case 5.
Figure 3. The basic graphs $G_i (i = 10, 11, \cdots, 25)$ have more than five vertices with rank 5.

**Case 1.** $G$ contains an induced subgraph $G = K_5$, $V(G) = \{1, 2, 3, 4, 5\}$.

$$A(G) = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{bmatrix}.$$  

The following we first determine basic graph contain $G_i$. Let

$$B = \begin{bmatrix}
A(G_i) & \alpha^T \\
\alpha & 0
\end{bmatrix}$$

where $\alpha = (x_1, x_2, x_3, x_4, x_5)$, $x_i \in \{0, 1\}$ ($i = 1, 2, \cdots, 5$). For $\alpha$ satisfying $r(B) = r(A(G_i)) = 5$ (or $\det(B) = 0$), calculating by MATLAB, we obtain

$$\alpha = (0, 0, 0, 0, 0), (0, 1, 1, 1, 1), (1, 0, 1, 1, 1), (1, 1, 0, 1, 1), (1, 1, 1, 0, 1), \text{ or } (1, 1, 1, 0).$$

This implies that not exist non-trivial vectors such that $r(B) = 5$, hence $G_i$ is unique basic graph contain $G_i$ with rank 5, then $G \in \mathcal{M}(G_i)$.

**Case 2.** $G$ contains an induced subgraph $G = C_5$, $V(G) = \{1, 2, 3, 4, 5\}$. Similar with Case 1, we know $G \in \mathcal{M}(G_i)$. 
Case 3. $G$ contains an induced subgraph $G_3$, $V(G_3) = \{1,2,3,4,5\}$. Similar with Case 1, we know $G \in \mathcal{M}(G_3)$.

Case 4. $G$ contains an induced subgraph $G_4$, $V(G_4) = \{1,2,3,4,5\}$, 

\[
A(G_4) = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}.
\]

First considering basic graph contain $G_4$, let
\[
B = \begin{bmatrix}
A(G_4) & \alpha^- \\
\alpha & 0
\end{bmatrix}
\]

where $\alpha = (x_1, x_2, x_3, x_4, x_5)$, $x_i \in \{0,1\} \ (i=1,2,\cdots,5)$. For $\alpha$ satisfying $r(B) = 5$, calculating by MATLAB, we obtain

\[
\alpha = (0,0,0,0,0),(0,1,0,0,0),(1,0,1,0,0),(1,1,0,1,0),(0,0,1,0,1),(0,0,0,1,0),
(1,0,1,1,0),(0,1,1,0,0),(1,0,0,1,1),(0,1,0,1,1),(1,1,0,0,0).
\]

we know the first six vectors is trivial.

Case 4.1. For non-trivial vector $\alpha = (0,1,1,1,0)$, (or $(1,0,1,1,0)$), then there exists a graph $G_{10}$ (the adjacency matrix of $G_{10}$ is $B$), it is a basic graph contain $G_4$ with rank 5. $V(G_{10}) = \{1,2,3,4,5,6\}$, Same as above, calculating by MATLAB for $G_{10}$, we obtain 3 non-trivial vectors. $\alpha = (0,1,0,1,1,1),(1,0,1,1,0,1)$, (or $(1,1,0,0,1,0)$).

Case 4.1.1. For non-trivial vector $\alpha = (0,1,0,1,1,1)$, then there exists a graph $G_{11}$ is a basic graph contain $G_{10}$ with rank 5. Same as above, calculating by MATLAB for $G_{11}$, we obtain not exist non-trivial vectors. Hence $G_{11}$ is a unique basic graph contain $G_{11}$ with rank 5.

Case 4.1.2. For non-trivial vector $\alpha = (1,0,1,1,0,1)$, then there exists a graph $G_{12}$ is a basic graph contain $G_{10}$ with rank 5. Same as above, calculating by MATLAB for $G_{12}$, we obtain not exist non-trivial vectors $(1,1,0,0,0,1,1)$, the resulting produce a graph $G_{13}$ is a basic graph contain $G_{12}$ with rank 5. Calculating by MATLAB for $G_{13}$, we obtain not exist non-trivial vectors, Hence $G_{13}$ is a unique basic graph contain $G_{13}$ with rank 5.

Case 4.1.3. For non-trivial vector $\alpha = (1,1,0,0,1,0)$, then there exists a graph $G_{14}$ is a basic graph contain $G_{10}$ with rank 5. Calculating by MATLAB for $G_{14}$, we obtain exist a non-trivial vectors $\alpha = (1,0,1,1,0,1,1)$. The resulting produce a graph $G_{15}$ is a basic graph contain $G_{14}$ with rank 5. Similar with Case 4.1.2, $G_{15}$ is a unique basic graph contain $G_{15}$ with rank 5.

Case 4.2. For non-trivial vector $\alpha = (0,1,0,1,1)$, (or $(1,0,0,1,1)$), then there exists a graph $G_{16}$ is a basic graph contain $G_{4}$ with rank 5. $V(G_{16}) = \{1,2,3,4,5,6\}$, Same as above, calculating by MATLAB for $G_{16}$, we obtain 3 non-trivial vectors. $\alpha = (1,1,1,0,1,0),(1,0,0,1,1,1)$, (or $(0,1,1,1,0,1)$).

Case 4.2.1. For non-trivial vector $\alpha = (1,1,1,0,1,0)$, (or $(1,0,0,1,1,1)$), then there exists a graph $G_{17}$ is a basic graph contain $G_{16}$ with rank 5. Same as above, calculating by MATLAB for $G_{17}$ exist a non-trivial vectors $(1,0,0,1,0,1,1)$,
the resulting produce a graph $G_{17}$ is a basic graph contain $G_{16}$ with rank 5. Calculating with MATLAB for $G_{17}$, we obtain not exist non-trivial vectors. Hence $G_{17}$ is a unique basic graph contain $G_{16}$ with rank 5.

**Case 4.2.2.** For non-trivial vector $\alpha = (0,1,1,0,1)$, then there exists a graph $G_{11}$ is a basic graph contain $G_{15}$ with rank 5. Similar with Case 4.1.1, $G_{11}$ is a unique basic graph contain $G_{15}$ with rank 5.

**Case 4.3.** For non-trivial vector $\alpha = (1,1,0,0,0)$, there exists a graph $G_{18}$ is a basic graph contain $G_{4}$ with rank 5. Same as above, calculating by MATLAB for $G_{18}$, we obtain three non-trivial vectors $\alpha = (1,1,0,1,0),(0,1,1,1,0,1), (or (1,0,1,1,0,1))$.

**Case 4.3.1.** For non-trivial vector $\alpha = (1,1,1,0,1,0)$, there exists a graph $G_{19}$ is a basic graph contain $G_{18}$ with rank 5. Same as above, calculating by MATLAB for $G_{19}$, we obtain not exist non-trivial vectors. Hence $G_{19}$ is a unique basic graph contain $G_{18}$ with rank 5.

**Case 4.3.2.** For non-trivial vector $\alpha = (0,1,1,1,0,1), (or \alpha = (1,0,1,1,0,1))$, there exists a graph $G_{14}$ is a basic graph included $G_{18}$ with rank 5. Similar with Case 4.1.3, we obtain $G_{13}$ and $G_{14}$ it is only one basic graph contain $G_{14}$ with rank 5.

In a word, basic graph contain $G_{4}$ with rank 5 are $G_{4}, G_{i} (i = 10,11,\cdots,19)$. Let $G$ be a graph contain $G_{4}$ with rank 5, then it must be a multiplication of vertices graph of one of $G_{4}, G_{i} (i = 10,11,\cdots,19)$, thus $G \in M(G) (i = 4,10,11,\cdots,19)$.

**Case 5.** $G$ contains an induced subgraph which is $G_{i} (i = 5,6,7,8)$, similar with Case 4, we first find basic graphs contain $G_{i}$ with rank 5. The result and logic levels below in Figure 4 and process is omitted.

![Figure 4](image-url)

*Figure 4.* The level indicate figure of basic graphs contain $G_{i} (i = 4,5,\cdots,9)$ with rank 5.
Summarize the previous cases, we can obtain $G \in \mathcal{M}(G_i)(i = 1, 2, \cdots, 25)$. Sufficiency is obvious by the proof process of the necessity. The proof is completed.

By Examples 3.1, 3.2 and Theorems 3.1-3.3, we immediately get the following Theorem

**Theorem 3.4.** Let $G$ be a graph, then $r(G) \leq 5$ if and only if $G \in \mathcal{M}(H)$, where $H$ is an induced subgraph of $G_1, G_2, G_3, G_{11}, G_{13}, G_{17}, G_{19}$ and $G_{24}$ (see Figure 2 and Figure 3).

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