γ and β Approximations via General Ordered Topological Spaces

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Abstract
In this paper, we introduce the concepts of γ and β approximations via general ordered topological
approximation spaces. Also, increasing (decreasing) γ, β boundary, positive and negative regions
are given in general ordered topological approximation spaces (GOTAS, for short). Some impor-
tant properties of them were investigated. From this study, we can say that studying any proper-
ties of rough set concepts via GOTAS is a generalization of Pawlak approximation spaces and gen-
eral approximation spaces.

Keywords
Rough Sets, Approximations, Ordered Topological Spaces

1. Introduction
Rough set theory was first proposed by Pawlak for dealing with vagueness and granularity in information sys-
tems. Various generalizations of Pawlak’s rough set have been made by replacing equivalence relations with
kinds of binary relations and many results about generalized rough set with the universe being finite were ob-
tained [1]-[7]. An interesting and natural research topic in rough set theory is studying it via topology [8] [9].
Neighborhood systems were first applied in generalizing rough sets in 1998 by T. Y. Lin as a generalization of
topological connections with rough sets. Lin also introduced the concept of granular computing as a form of to-
poalogical generalizations [10]-[13]. In this paper, we give the concept of γ, β via topological ordered spaces and
studied their properties which may be viewed as a generalization of previous studies in general approximation
spaces, as if we take the partially ordered relation as an equal relation, we obtain the concepts in general ap-
proximation spaces [14].

2. Preliminaries
In this section, we give an account of the basic definitions and preliminaries to be used in the paper.
Definition 2.1 [15]. A subset \( A \) of \( U \), where \((U, \rho)\) is a partially ordered set is said to be increasing (resp. decreasing) if for all \( a \in A \) and \( x \in U \) such that \( \rho x a \) (resp. \( \rho ax \)) imply \( x \in A \).

Definition 2.2 [15]. A triple \((U, \tau, \rho)\) is said to be a topological ordered space, where \((U, \tau)\) is a topological space and \( \rho \) is a partial order relation on \( U \).

Definition 2.3 [16]. Information system is a pair \((U, A)\), where \( U \) is a non-empty finite set of objects and \( A \) is a non-empty finite set of attributes.

Definition 2.4 [17]. A non-empty set \( U \) equipped with a general relation \( \tau \) which generates a topology \( \tau_R \) on \( U \) and partially order relation \( \rho \) written as \((U, \tau_R, \rho)\) is said to be general ordered topological approximation space (for short, GOTAS).

Definition 2.5 [18]. Let \((U, \tau_R, \rho)\) be a GOTAS and \( A \subseteq U \). We define:

1. \( R_{inc}(A) = A^{\tau_R'} \), \( A^{inc} \) is the greatest increasing open subset of \( A \).
2. \( R_{dec}(A) = A^{\tau_R'}, A^{dec} \) is the greatest decreasing open subset of \( A \).
3. \( R^{inc}(A) = \overline{A}^{\tau_R'}, \overline{A}^{inc} \) is the smallest increasing closed superset of \( A \).
4. \( R^{dec}(A) = \overline{A}^{\tau_R'}, \overline{A}^{dec} \) is the smallest decreasing closed superset of \( A \).
5. \( \alpha^{inc} = \frac{\text{card}(R_{inc}(A))}{\text{card}(R^{inc}(A))} \) (resp. \( \alpha^{dec} = \frac{\text{card}(R_{dec}(A))}{\text{card}(R^{dec}(A))} \)) and \( \alpha^{inc} \) (resp. \( \alpha^{dec} \)) is \( R \)-increasing (resp. decreasing) accuracy.

Definition 2.6 [17]. Let \((U, \tau_R, \rho)\) be a GOTAS and \( A \subseteq U \). We define:

1. \( S_{inc}(A) = A \cap R^{inc}(R_{inc}(A)), S^{inc}(A) \) is called \( R \)-increasing semi lower.
2. \( S^{inc}(A) = A \cup R^{inc}(R_{inc}(A)), S^{inc}(A) \) is called \( R \)-increasing semi upper.
3. \( S_{dec}(A) = A \cap R^{dec}(R_{dec}(A)), S^{dec}(A) \) is called \( R \)-decreasing semi lower.
4. \( S^{dec}(A) = A \cup R^{dec}(R_{dec}(A)), S^{dec}(A) \) is called \( R \)-decreasing semi upper.

\( A \) is \( R \)-increasing (resp. \( R \)-decreasing) semi exact if \( S_{inc}(A) = S^{inc}(A) \) (resp. \( S_{dec}(A) = S^{dec}(A) \)), otherwise \( A \) is \( R \)-increasing (resp. \( R \)-decreasing) semi rough.

Proposition 2.7 [18]. Let \((U, \tau_R, \rho)\) be a GOTAS and \( A \subseteq U \). Then

1. \( R_{inc}(A) \subseteq \alpha^{inc}(A) \subseteq S_{inc}(A) \) \( R_{dec}(A) \subseteq \alpha^{dec}(A) \subseteq S^{dec}(A) \).
2. \( S^{inc}(A) \subseteq \alpha^{inc}(A) \subseteq R^{inc}(A) \) \( S^{dec}(A) \subseteq \alpha^{dec}(A) \subseteq R^{dec}(A) \).

3. New Approximations and Their Properties

In this section, we introduce some definitions and propositions about near approximations, near boundary regions via GOTAS which is essential for a present study.

Definition 3.1. Let \((U, \tau_R, \rho)\) be a GOTAS and \( A \subseteq U \). We define:

1. \( \gamma^{inc}(A) = A \cap \overline{R^{inc}(R_{inc}(A))} \cup \overline{R^{inc}(R_{dec}(A))} \), \( \gamma^{inc}(A) \) is called \( R \)-increasing \( \gamma \) lower.
2. \( \gamma^{dec}(A) = A \cup \overline{R^{inc}(R_{dec}(A))} \cup \overline{R^{inc}(R_{dec}(A))} \), \( \gamma^{dec}(A) \) is called \( R \)-increasing \( \gamma \) upper.
3. \( \gamma^{dec}(A) = A \cap \overline{R^{dec}(R_{dec}(A))} \cup \overline{R^{inc}(R_{dec}(A))} \), \( \gamma^{dec}(A) \) is called \( R \)-decreasing \( \gamma \) lower.
4. \( \gamma^{dec}(A) = A \cup \overline{R^{dec}(R_{dec}(A))} \cup \overline{R^{inc}(R_{dec}(A))} \), \( \gamma^{dec}(A) \) is called \( R \)-decreasing \( \gamma \) upper.

\( A \) is \( R \)-increasing (resp. \( R \)-decreasing) \( \gamma \) exact if \( \gamma^{inc}(A) = \gamma^{dec}(A) \) (resp. \( \gamma^{dec}(A) = \gamma^{dec}(A) \)) otherwise \( A \) is \( R \)-increasing (resp. \( R \)-decreasing) \( \gamma \) rough.

Proposition 3.2. Let \((U, \tau_R, \rho)\) be a GOTAS and \( A, B \subseteq U \). Then

1. \( A \subseteq B \rightarrow \gamma^{inc}(A) \subseteq \gamma^{inc}(B) \) \( (A \supseteq B \rightarrow \gamma^{dec}(A) \subseteq \gamma^{dec}(B)) \).
2. \( \gamma^{inc}(A \cap B) \subseteq \gamma^{inc}(A) \cap \gamma^{inc}(B) \) \( (\gamma^{dec}(A \cap B) \subseteq \gamma^{dec}(A) \cap \gamma^{dec}(B)) \).
3. \( \gamma^{inc}(A \cup B) \supseteq \gamma^{inc}(A) \cup \gamma^{inc}(B) \) \( (\gamma^{dec}(A \cup B) \supseteq \gamma^{dec}(A) \cup \gamma^{dec}(B)) \).

Proof.
(1) Omitted.
One can prove the case between parentheses.

**Proposition 3.3.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A, B \subseteq U\). Then

1. \((A \subseteq B) \Rightarrow \overline{\Psi}^{\text{inc}}(A) \subseteq \overline{\Psi}^{\text{inc}}(B)\).
2. \(\Psi^{\text{inc}}(A \cap B) \subseteq \overline{\Psi}^{\text{inc}}(A) \cap \overline{\Psi}^{\text{inc}}(B)\).
3. \(\Psi^{\text{inc}}(A \cup B) \supseteq \overline{\Psi}^{\text{inc}}(A) \cup \overline{\Psi}^{\text{inc}}(B)\).

**Proof.**

1. Easy.

2. \(\Psi^{\text{inc}}(A \cap B) = (A \cap B) \cap \left[\overline{\Psi}^{\text{inc}} \left( \overline{\Psi}^{\text{inc}}(A \cap B) \cup \overline{\Psi}^{\text{inc}}(A) \right) \right]\)

   \(\supseteq (A \cap B) \cap \left[\overline{\Psi}^{\text{inc}} \left( \overline{\Psi}^{\text{inc}}(A) \cap \overline{\Psi}^{\text{inc}}(B) \right) \right]\)

   \(\supseteq (A \cap B) \cap \left[\overline{\Psi}^{\text{inc}} \left( \overline{\Psi}^{\text{inc}}(A) \cap \overline{\Psi}^{\text{inc}}(B) \right) \right]\)

   \(\supseteq A \cap \left[\overline{\Psi}^{\text{inc}} \left( \overline{\Psi}^{\text{inc}}(A) \cap \overline{\Psi}^{\text{inc}}(B) \right) \right]\)

   \(\subseteq \overline{\Psi}^{\text{inc}}(A) \cap \overline{\Psi}^{\text{inc}}(B)\).

3. \(\Psi^{\text{inc}}(A \cup B) = (A \cup B) \cap \left[\overline{\Psi}^{\text{inc}} \left( \overline{\Psi}^{\text{inc}}(A \cup B) \cup \overline{\Psi}^{\text{inc}}(A) \right) \right]\)

   \(\supseteq (A \cup B) \cap \left[\overline{\Psi}^{\text{inc}} \left( \overline{\Psi}^{\text{inc}}(A) \cup \overline{\Psi}^{\text{inc}}(B) \right) \right]\)

   \(\supseteq (A \cup B) \cap \left[\overline{\Psi}^{\text{inc}} \left( \overline{\Psi}^{\text{inc}}(A) \cup \overline{\Psi}^{\text{inc}}(B) \right) \right]\)

   \(\supseteq \overline{\Psi}^{\text{inc}}(A) \cup \overline{\Psi}^{\text{inc}}(B)\).

One can prove the case between parentheses.

**Proposition 3.4.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A, B \subseteq U\). If \(A\) is \(R\)-increasing (resp. decreasing) exact then \(A\) is \(R\)-increasing (resp. decreasing) \(\gamma\) exact.

**Proof.**

Let \(A\) be \(R\)-increasing exact. Then \(\overline{\Psi}^{\text{inc}}(A) = \overline{\Psi}^{\text{inc}}(A)\), thus \(\overline{\Psi}^{\text{inc}}(A) = \overline{\Psi}^{\text{inc}}(A)\) and \(\overline{\Psi}^{\text{inc}}(A) = \overline{\Psi}^{\text{inc}}(A)\).

Therefore \(\overline{\Psi}^{\text{inc}}(A) = \overline{\Psi}^{\text{inc}}(A)\).

One can prove the case between parentheses.

\(R\)-increasing (resp. decreasing) exact \(\rightarrow\) \(R\)-increasing (resp. decreasing) \(\gamma\) exact.

**Proposition 3.5.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A \subseteq U\). Then \(\overline{\Psi}^{\text{inc}}(A) \subseteq \overline{\Psi}^{\text{inc}}(A) \cap \overline{\Psi}^{\text{inc}}(A)\).
Proof.
Since \( R_{\text{inc}}(A) \subseteq A \) and \( R_{\text{inc}}(A) \subseteq R_{\text{inc}}(R_{\text{inc}}(A)) \), then \( R_{\text{inc}}(A) \subseteq R_{\text{inc}}(R_{\text{inc}}(A)) \cup R_{\text{inc}}(R_{\text{inc}}(A)) \). Therefore, \( R_{\text{inc}}(A) \subseteq A \cap R_{\text{inc}}(R_{\text{inc}}(A)) \cup R_{\text{inc}}(R_{\text{inc}}(A)) \). Thus \( R_{\text{inc}}(A) \subseteq \gamma_{\text{inc}}(A) \).

One can prove the case between parentheses.

Proposition 3.6. Let \((U, \tau, \rho)\) be a GOTAS and \(A \subseteq U\). Then \( \gamma_{\text{inc}}(A) \subseteq R_{\text{inc}}(A) \) \( (\gamma_{\text{dec}}(A) \subseteq R_{\text{dec}}(A)) \).

Proof. Since \( A \subseteq R_{\text{inc}}(A) \) and \( R_{\text{inc}}(A) \subseteq A \subseteq R_{\text{inc}}(A) \), then \( R_{\text{inc}}(R_{\text{inc}}(A)) \subseteq A \subseteq R_{\text{inc}}(A) \). Thus \( R_{\text{inc}}(R_{\text{inc}}(A)) \subseteq \gamma_{\text{inc}}(A) \).

One can prove the case between parentheses.

Proposition 3.7. Let \((U, \tau, \rho)\) be a GOTAS and \(A \subseteq U\). Then \( \gamma_{\text{dec}}(A) \subseteq R_{\text{dec}}(A) \) \( (\gamma_{\text{inc}}(A) \subseteq R_{\text{inc}}(A)) \).

Proof. Let \( x \in P_{\text{inc}}(A) = A \cap R_{\text{inc}}(R_{\text{inc}}(A)) \). Then \( x \in A \) and \( x \in R_{\text{inc}}(R_{\text{inc}}(A)) \). Therefore \( x \in A \) and \( x \in R_{\text{inc}}(R_{\text{inc}}(A)) \).

Thus \( x \in A \cap R_{\text{inc}}(R_{\text{inc}}(A)) = \gamma_{\text{inc}}(A) \). Hence \( P_{\text{inc}}(A) \subseteq \gamma_{\text{inc}}(A) \).

One can prove the case between parentheses.

Proposition 3.8. Let \((U, \tau, \rho)\) be a GOTAS and \(A \subseteq U\). Then \( \gamma_{\text{dec}}(A) \subseteq R_{\text{dec}}(A) \) \( (\gamma_{\text{inc}}(A) \subseteq R_{\text{inc}}(A)) \).

Proof. Let \( x \in S_{\text{inc}}(A) = A \cap R_{\text{inc}}(R_{\text{inc}}(A)) \). Then \( x \in A \) and \( x \in R_{\text{inc}}(R_{\text{inc}}(A)) \). Therefore \( x \in A \) and \( x \in R_{\text{inc}}(R_{\text{inc}}(A)) \).

Thus \( x \in A \cap R_{\text{inc}}(R_{\text{inc}}(A)) = \gamma_{\text{inc}}(A) \). Hence \( S_{\text{inc}}(A) \subseteq \gamma_{\text{inc}}(A) \).

One can prove the case between parentheses.

Proposition 3.9. Let \((U, \tau, \rho)\) be a GOTAS and \(A \subseteq U\). Then \( \gamma_{\text{dec}}(A) \subseteq R_{\text{dec}}(A) \) \( (\gamma_{\text{inc}}(A) \subseteq R_{\text{inc}}(A)) \).

Proof. Let \( x \in \overline{\gamma}_{\text{inc}}(A) = A \cup R_{\text{inc}}(R_{\text{inc}}(A)) \). Then \( x \in A \) and \( x \in R_{\text{inc}}(R_{\text{inc}}(A)) \). Therefore \( x \in A \) and \( x \in R_{\text{inc}}(R_{\text{inc}}(A)) \).

Thus \( \overline{\gamma}_{\text{inc}}(A) \subseteq \overline{\gamma}_{\text{inc}}(A) \).

Proposition 3.10. Let \((U, \tau, \rho)\) be a GOTAS and \(A \subseteq U\). Then \( \overline{\gamma}_{\text{inc}}(A) \subseteq \overline{\gamma}_{\text{inc}}(A) \) \( (\overline{\gamma}_{\text{dec}}(A) \subseteq \overline{\gamma}_{\text{dec}}(A)) \).

Proof. Omitted.

Definition 3.11. Let \((U, \tau, \rho)\) be a GOTAS and \(A \subseteq U\). We define:
(1) \( \overline{\gamma}_{\text{inc}}(A) = A \cap R_{\text{inc}}(R_{\text{inc}}(A))) \), \( \overline{\gamma}_{\text{inc}}(A) \) is called \( R \)-increasing \( \beta \) lower.
(2) \( \overline{\gamma}_{\text{dec}}(A) = A \cup R_{\text{inc}}(R_{\text{inc}}(A))) \), \( \overline{\gamma}_{\text{dec}}(A) \) is called \( R \)-increasing \( \beta \) upper.
(3) \( \overline{\gamma}_{\text{dec}}(A) = A \cap R_{\text{dec}}(R_{\text{dec}}(A))) \), \( \overline{\gamma}_{\text{dec}}(A) \) is called \( R \)-decreasing \( \beta \) lower.
(4) \( \overline{\gamma}_{\text{dec}}(A) = A \cup R_{\text{dec}}(R_{\text{dec}}(A))) \), \( \overline{\gamma}_{\text{dec}}(A) \) is called \( R \)-decreasing \( \beta \) upper.

A is \( R \)-increasing (decreasing) \( \beta \) exact if \( \overline{\gamma}_{\text{inc}}(A) = \overline{\gamma}_{\text{inc}}(A) \) (resp. \( \overline{\gamma}_{\text{dec}}(A) = \overline{\gamma}_{\text{dec}}(A) \)), otherwise \( A \) is \( R \)-increasing (decreasing) \( \beta \) rough.

Proposition 3.12. Let \((U, \tau, \rho)\) be a GOTAS and \(A, B \subseteq U\). Then
(1) \( A \subseteq B \rightarrow \overline{\gamma}_{\text{inc}}(A) \subseteq \overline{\gamma}_{\text{inc}}(B) \) \( (A \subseteq B \rightarrow \overline{\gamma}_{\text{dec}}(A) \subseteq \overline{\gamma}_{\text{dec}}(B)) \).
(2) \( \overline{\gamma}_{\text{inc}}(A \cap B) \subseteq \overline{\gamma}_{\text{inc}}(A) \cap \overline{\gamma}_{\text{inc}}(B) \) \( (\overline{\gamma}_{\text{dec}}(A \cap B) \subseteq \overline{\gamma}_{\text{dec}}(A) \cap \overline{\gamma}_{\text{dec}}(B)) \).
(3) \( \overline{\gamma}_{\text{inc}}(A \cup B) \subseteq \overline{\gamma}_{\text{inc}}(A) \cup \overline{\gamma}_{\text{inc}}(B) \) \( (\overline{\gamma}_{\text{dec}}(A \cup B) \subseteq \overline{\gamma}_{\text{dec}}(A) \cup \overline{\gamma}_{\text{dec}}(B)) \).

Proof.
(1) Omitted.
(2) \( \overline{B}^{\text{Inc}} (A \cap B) = (A \cap B) \cup R_{\text{Inc}} \left( R_{\text{Inc}} (A \cap B) \right) \)
\( = (A \cap B) \cup R_{\text{Inc}} \left( R_{\text{Inc}} (A) \cap R_{\text{Inc}} (B) \right) \)
\( \subseteq (A \cap B) \cup R_{\text{Inc}} \left( R_{\text{Inc}} (A) \cap \overline{R}_{\text{Inc}} (R_{\text{Inc}} (B)) \right) \)
\( \subseteq (A \cap B) \cup R_{\text{Inc}} \left( R_{\text{Inc}} (A) \cap \overline{R}_{\text{Inc}} (R_{\text{Inc}} (B)) \right) \)
\( \subseteq A \cup R_{\text{Inc}} \left( R_{\text{Inc}} (A) \right) \cap B \cup R_{\text{Inc}} \left( R_{\text{Inc}} (B) \right) \)
\( \subseteq \overline{R}_{\text{Inc}} (A) \cap \overline{R}_{\text{Inc}} (B) \).

(3) \( \overline{B}^{\text{Inc}} (A \cup B) = (A \cup B) \cup R_{\text{Inc}} \left( R_{\text{Inc}} (A \cup B) \right) \)
\( = (A \cup B) \cup R_{\text{Inc}} \left( R_{\text{Inc}} (A) \cup R_{\text{Inc}} (B) \right) \)
\( \supseteq (A \cup B) \cup R_{\text{Inc}} \left( R_{\text{Inc}} (A) \cup \overline{R}_{\text{Inc}} (R_{\text{Inc}} (B)) \right) \)
\( \supseteq (A \cup B) \cup R_{\text{Inc}} \left( R_{\text{Inc}} (A) \cup \overline{R}_{\text{Inc}} (R_{\text{Inc}} (B)) \right) \)
\( \supseteq A \cup R_{\text{Inc}} \left( R_{\text{Inc}} (A) \right) \cup B \cup R_{\text{Inc}} \left( R_{\text{Inc}} (B) \right) \)
\( \supseteq \overline{R}_{\text{Inc}} (A) \cup \overline{R}_{\text{Inc}} (B) \).

One can prove the case between parentheses.

**Proposition 3.13.** Let \( (U, \tau, \rho) \) be a GOTAS and \( A, B \subseteq U \). Then
(1) \( A \subseteq B \Rightarrow \beta_{\text{Inc}} (A) \subseteq \beta_{\text{Inc}} (B) \) (\( A \subseteq B \Rightarrow \beta_{\text{Dec}} (A) \subseteq \beta_{\text{Dec}} (B) \)).
(2) \( \beta_{\text{Inc}} (A \cap B) \supseteq \beta_{\text{Inc}} (A) \cap \beta_{\text{Inc}} (B) \) (\( \beta_{\text{Inc}} (A \cap B) \supseteq \beta_{\text{Dec}} (A) \cap \beta_{\text{Dec}} (B) \)).
(3) \( \beta_{\text{Inc}} (A \cup B) \supseteq \beta_{\text{Inc}} (A) \cup \beta_{\text{Inc}} (B) \) (\( \beta_{\text{Inc}} (A \cup B) \supseteq \beta_{\text{Dec}} (A) \cup \beta_{\text{Dec}} (B) \)).

**Proof.**
(1) Easy.
(2) \( \beta_{\text{Inc}} (A \cap B) = (A \cap B) \cap \overline{R}_{\text{Inc}} \left( R_{\text{Inc}} (A \cap B) \right) \)
\( \subseteq (A \cap B) \cap \overline{R}_{\text{Inc}} \left( R_{\text{Inc}} (A) \cap \overline{R}_{\text{Inc}} (B) \right) \)
\( \subseteq (A \cap B) \cap \overline{R}_{\text{Inc}} \left( R_{\text{Inc}} (A) \cap \overline{R}_{\text{Inc}} (B) \right) \)
\( \subseteq A \cap R_{\text{Inc}} \left( R_{\text{Inc}} (A) \right) \cap B \cap \overline{R}_{\text{Inc}} \left( R_{\text{Inc}} (B) \right) \)
\( \subseteq \beta_{\text{Inc}} (A) \cap \beta_{\text{Inc}} (B) \).

(3) \( \beta_{\text{Inc}} (A \cup B) = (A \cup B) \cap \overline{R}_{\text{Inc}} \left( R_{\text{Inc}} (A \cup B) \right) \)
\( \subseteq (A \cup B) \cap \overline{R}_{\text{Inc}} \left( R_{\text{Inc}} (A) \cup \overline{R}_{\text{Inc}} (B) \right) \)
\( \subseteq (A \cup B) \cap \overline{R}_{\text{Inc}} \left( R_{\text{Inc}} (A) \cup \overline{R}_{\text{Inc}} (B) \right) \)
\( \subseteq A \cap R_{\text{Inc}} \left( R_{\text{Inc}} (A) \right) \cup B \cap \overline{R}_{\text{Inc}} \left( R_{\text{Inc}} (B) \right) \)
\( \subseteq \beta_{\text{Inc}} (A) \cup \beta_{\text{Inc}} (B) \).

One can prove the case between parentheses.

**Proposition 3.14.** Let \( (U, \tau, \rho) \) be a GOTAS and \( A, B \subseteq U \). If \( A \) is \( R \)-increasing (resp. decreasing) exact then \( A \) is \( \beta \)-increasing (resp. decreasing) exact.

**Proof.**
Let \( A \) be \( R \)-increasing exact. Then \( \overline{R}_{\text{Inc}} (A) = R_{\text{Inc}} (A) \). Therefore \( \overline{B}^{\text{Inc}} (A) = R_{\text{Inc}} (A) \). \( \beta_{\text{Inc}} (A) = R_{\text{Inc}} (A) \).
Thus \( \overline{B}^{\text{Inc}} (A) = \beta_{\text{Inc}} (A) \). Hence \( A \) is \( R \)-increasing \( \beta \) exact.
One can prove the case between parentheses.

**Proposition 3.15.** Let \( (U, \tau_r, \rho) \) be a GOTAS and \( A \subseteq U \). Then
\[
\mathcal{R}_{\text{inc}}(A) \subseteq \mathcal{R}_{\text{inc}}(A) \left( \mathcal{R}_{\text{dec}}(A) \subseteq \mathcal{R}_{\text{dec}}(A) \right).
\]

**Proof.**
Since \( \mathcal{R}_{\text{inc}}(A) \subseteq A \subseteq \mathcal{R}_{\text{inc}}(A) \) and \( \mathcal{R}_{\text{inc}}(A) \subseteq \mathcal{R}_{\text{dec}}(A) \). Then
\[
\mathcal{R}_{\text{inc}}(A) \subseteq \mathcal{R}_{\text{inc}}(A) \left( \mathcal{R}_{\text{dec}}(A) \subseteq \mathcal{R}_{\text{dec}}(A) \right).
\]
Therefore \( \mathcal{R}_{\text{inc}}(A) \subseteq A \cap \mathcal{R}_{\text{inc}}(A) \). Thus \( \mathcal{R}_{\text{inc}}(A) \subseteq \mathcal{R}_{\text{inc}}(A) \) . One can prove the case between parentheses.

**Proposition 3.16.** Let \( (U, \tau_r, \rho) \) be a GOTAS and \( A \subseteq U \). Then
\[
\beta \subseteq \mathcal{R}_{\text{inc}}(A) \left( \mathcal{R}_{\text{dec}}(A) \subseteq \mathcal{R}_{\text{dec}}(A) \right).
\]

**Proof.**
Since \( \mathcal{R}_{\text{inc}}(A) \subseteq \mathcal{R}_{\text{inc}}(A) \) and \( \mathcal{R}_{\text{inc}}(A) \subseteq \mathcal{R}_{\text{dec}}(A) \). Then
\[
\mathcal{R}_{\text{inc}}(A) \subseteq \mathcal{R}_{\text{inc}}(A) \left( \mathcal{R}_{\text{dec}}(A) \subseteq \mathcal{R}_{\text{dec}}(A) \right).
\]
Therefore \( A \cup \mathcal{R}_{\text{inc}}(A) \subseteq \mathcal{R}_{\text{inc}}(A) \) . Hence \( \beta \subseteq \mathcal{R}_{\text{inc}}(A) \).

**Definition 3.17.** Let \( (U, \tau_r, \rho) \) be a GOTAS and \( A \subseteq U \). Then
(1) \( j_{\text{inc}}(A) = \mathcal{R}_{\text{inc}}(A) \) is increasing (resp. \( j_{\text{dec}}(A) = \mathcal{R}_{\text{dec}}(A) \) ), is increasing (resp. decreasing) \( j \) positive region.
(2) \( j_{\text{inc}}(A) = \mathcal{R}_{\text{inc}}(A) \) is increasing (resp. \( j_{\text{dec}}(A) = \mathcal{R}_{\text{dec}}(A) \) ), is increasing (resp. decreasing) \( j \) negative region. Where \( j_{\text{inc}}(A) \) the near lower approximations s.t. \( \gamma \in \beta \).

**Proposition 3.18.** Let \( (U, \tau_r, \rho) \) be a GOTAS and \( A, B \subseteq U \). Then
(1) \( \gamma_{\text{inc}}(A \cup B) \subseteq \gamma_{\text{inc}}(A) \cup \gamma_{\text{inc}}(B) \) ( resp. \( \gamma_{\text{dec}}(A \cup B) \subseteq \gamma_{\text{dec}}(A) \cup \gamma_{\text{dec}}(B) \) ).
(2) \( \gamma_{\text{inc}}(A \cap B) \supseteq \gamma_{\text{inc}}(A) \cap \gamma_{\text{inc}}(B) \) ( resp. \( \gamma_{\text{dec}}(A \cap B) \supseteq \gamma_{\text{dec}}(A) \cap \gamma_{\text{dec}}(B) \) ).

**Proof.**
(1) \( \gamma_{\text{inc}}(A \cup B) = U - \left[ \mathcal{R}_{\text{dec}}(A \cup B) \right] \)
\[
\subseteq U - \left[ \mathcal{R}_{\text{dec}}(A) \cup \mathcal{R}_{\text{dec}}(B) \right] \]
\[
\subseteq U - \left[ \mathcal{R}_{\text{dec}}(A) \cup \mathcal{R}_{\text{dec}}(B) \right] \]
\[
\subseteq U - \left[ \mathcal{R}_{\text{dec}}(A) \cup \mathcal{R}_{\text{dec}}(B) \right] \]
\[
\subseteq U - \left[ \mathcal{R}_{\text{dec}}(A) \cup \mathcal{R}_{\text{dec}}(B) \right] \]
\[
\subseteq U - \left[ \mathcal{R}_{\text{dec}}(A) \cup \mathcal{R}_{\text{dec}}(B) \right] \]
\[
\subseteq U - \left[ \mathcal{R}_{\text{dec}}(A) \cup \mathcal{R}_{\text{dec}}(B) \right] \]
\[
\subseteq U - \left[ \mathcal{R}_{\text{dec}}(A) \cup \mathcal{R}_{\text{dec}}(B) \right] \]
(2) \( \gamma_{\text{inc}}(A \cap B) = U - \left[ \mathcal{R}_{\text{dec}}(A) \cup \mathcal{R}_{\text{dec}}(B) \right] \)
\[
\subseteq U - \left[ \mathcal{R}_{\text{dec}}(A) \cup \mathcal{R}_{\text{dec}}(B) \right] \]
\[
\subseteq U - \left[ \mathcal{R}_{\text{dec}}(A) \cup \mathcal{R}_{\text{dec}}(B) \right] \]
\[
\subseteq U - \left[ \mathcal{R}_{\text{dec}}(A) \cup \mathcal{R}_{\text{dec}}(B) \right] \]
\[
\subseteq U - \left[ \mathcal{R}_{\text{dec}}(A) \cup \mathcal{R}_{\text{dec}}(B) \right] \]
\[
\subseteq U - \left[ \mathcal{R}_{\text{dec}}(A) \cup \mathcal{R}_{\text{dec}}(B) \right] \]
\[
\subseteq U - \left[ \mathcal{R}_{\text{dec}}(A) \cup \mathcal{R}_{\text{dec}}(B) \right] \]
\[
\subseteq U - \left[ \mathcal{R}_{\text{dec}}(A) \cup \mathcal{R}_{\text{dec}}(B) \right] \]
One can prove the case between parentheses.

**Proposition 3.19.** Let \( (U, \tau_r, \rho) \) be a GOTAS and \( A, B \subseteq U \). Then
(1) \( \text{Neg}_{\text{Inc}}(A \cup B) \subseteq \text{Neg}_{\text{Inc}}(A) \cup \text{Neg}_{\text{Inc}}(B) \) (Neg_{\text{Inc}}(A \cup B) \subseteq \text{Neg}_{\text{Inc}}(A) \cup \text{Neg}_{\text{Dec}}(B)).

\( \text{Neg}_{\text{Dec}}(A \cap B) \supseteq \text{Neg}_{\text{Dec}}(A) \cap \text{Neg}_{\text{Dec}}(B) \) (Neg_{\text{Dec}}(A \cap B) \supseteq \text{Neg}_{\text{Dec}}(A) \cap \text{Neg}_{\text{Dec}}(B)).

Proof.

(1) \( \text{Neg}_{\text{Inc}}(A \cup B) = U - \left[ (A \cup B) \cup \text{R}_{\text{Dec}} \left( \text{R}_{\text{Dec}}(A) \cup \text{R}_{\text{Dec}}(B) \right) \right] \)

\( \subseteq U - \left[ (A \cup B) \cup \text{R}_{\text{Dec}} \left( \text{R}_{\text{Dec}}(A) \cup \text{R}_{\text{Dec}}(B) \right) \right] \)

\( \subseteq U - \left[ (A \cup B) \cup \text{R}_{\text{Dec}} \left( \text{R}_{\text{Dec}}(A) \cup \text{R}_{\text{Dec}}(B) \right) \right] \)

\( \subseteq \text{Neg}_{\text{Dec}}(A) \cap \text{Neg}_{\text{Dec}}(B). \)

(2) \( \text{Neg}_{\text{Inc}}(A \cap B) = U - \left[ (A \cap B) \cup \text{R}_{\text{Dec}} \left( \text{R}_{\text{Dec}}(A) \cup \text{R}_{\text{Dec}}(B) \right) \right] \)

\( \supseteq U - \left[ (A \cap B) \cup \text{R}_{\text{Dec}} \left( \text{R}_{\text{Dec}}(A) \cup \text{R}_{\text{Dec}}(B) \right) \right] \)

\( \supseteq U - \left[ (A \cap B) \cup \text{R}_{\text{Dec}} \left( \text{R}_{\text{Dec}}(A) \cup \text{R}_{\text{Dec}}(B) \right) \right] \)

\( \supseteq \text{Neg}_{\text{Dec}}(A) \cap \text{Neg}_{\text{Dec}}(B). \)

One can prove the case between parentheses.

Proposition 3.20. Let \((U, \tau_{B}, \rho)\) be a GOTS and \(A \subseteq U\). Then

\( S_{\text{Inc}}(A) \subseteq \gamma_{\text{Dec}}(A) \subseteq \beta_{\text{Dec}}(A) \) (\( S_{\text{Dec}}(A) \subseteq \gamma_{\text{Dec}}(A) \subseteq \beta_{\text{Dec}}(A) \)).

Proof.

Let \( x \in S_{\text{Inc}}(A) \). Then \( x \in \text{R}_{\text{Inc}}(\text{R}_{\text{Dec}}(A)) \). Therefore \( x \in \text{R}_{\text{Inc}}(\text{R}_{\text{Dec}}(A)) \cup \text{R}_{\text{Inc}}(\text{R}_{\text{Dec}}(A)). \) Thus

\( x \in A \cap \left[ \text{R}_{\text{Inc}}(\text{R}_{\text{Dec}}(A)) \cup \text{R}_{\text{Inc}}(\text{R}_{\text{Dec}}(A)) \right] \)

and thus \( x \in \gamma_{\text{Dec}}(A) \).

Hence

\( S_{\text{Inc}}(A) \subseteq \gamma_{\text{Inc}}(A) \) (1).

Since \( x \in \text{R}_{\text{Inc}}(A) \), then \( x \in \text{R}_{\text{Inc}}(A) \). Therefore \( x \in \text{R}_{\text{Inc}}(\text{R}_{\text{Dec}}(A)). \)

Thus \( x \in \text{R}_{\text{Inc}}(\text{R}_{\text{Dec}}(A)), \) and thus \( x \in A \cap \text{R}_{\text{Dec}}(\text{R}_{\text{Dec}}(A)). \) Hence

\( x \in \beta_{\text{Inc}}(A) \) (2).

From (1) and (2) we have,

\( S_{\text{Inc}}(A) \subseteq \gamma_{\text{Dec}}(A) \subseteq \beta_{\text{Inc}}(A). \)

One can prove the case between parentheses.

Proposition 3.21. Let \((U, \tau_{B}, \rho)\) be a GOTS and \(A \subseteq U\). Then

\( \beta_{\text{Inc}}(A) \subseteq \gamma_{\text{Dec}}(A) \subseteq S_{\text{Inc}}(A) \) (\( \beta_{\text{Dec}}(A) \subseteq \gamma_{\text{Dec}}(A) \subseteq S_{\text{Dec}}(A) \)).

Proof.

Let \( x \in \beta_{\text{Inc}}(A) \). Then \( x \in A \cup \text{R}_{\text{Inc}}(\text{R}_{\text{Dec}}(A)). \) Therefore \( x \in A \) or \( x \in \text{R}_{\text{Inc}}(\text{R}_{\text{Dec}}(A)). \) Thus \( x \in A \) or \( x \in \text{R}_{\text{Inc}}(\text{R}_{\text{Dec}}(A)). \) So \( x \in A \cup \text{R}_{\text{Inc}}(\text{R}_{\text{Dec}}(A)) \), and so \( x \in A \cup \left[ \text{R}_{\text{Inc}}(\text{R}_{\text{Dec}}(A)) \cup \text{R}_{\text{Dec}}(\text{R}_{\text{Dec}}(A)) \right] \).
Thus \( x \in \overline{\tau}^\text{inc}(A) \). Hence
\[
\overline{\beta}^\text{inc}(A) \subseteq \overline{\tau}^\text{inc}(A). \tag{1}
\]
Since \( x \in \overline{\tau}^\text{inc}(A), \ x \in A \) or \( x \in R^\text{inc}_\text{Inc}(A), \) then \( x \in A \cup R^\text{inc}_\text{Inc}(A) \). Therefore
\[
x \in S^\text{inc}(A) \tag{2}
\]
From (1) and (2) we have, \( \overline{\beta}^\text{inc}(A) \subseteq \overline{\tau}^\text{inc}(A) \subseteq S^\text{inc}(A) \).
One can prove the case between parentheses.

**Definition 3.22.** Let \((U, \tau, \rho)\) be a GOTAS and \(A\) is a non-empty finite subset of \(U\). Then the increasing (decreasing) \(j\) accuracy of a finite non-empty subset \(A\) of \(U\) is given by:
\[
\eta^\text{inc}_j(A) = \left[ \frac{\overline{\tau}^\text{inc}(A)}{\overline{\tau}^\text{inc}(A)} \right], \ j \in \{\beta, \gamma\}.
\]

**Proposition 3.23.** Let \((U, \tau, \rho)\) be a GOTAS and \(A\) non-empty finite subset of \(U\). Then we have
\[
\eta(A) \leq \eta^\text{inc}_j(A) \big( \eta(A) \leq \eta^\text{dec}_j(A) \big), \ \text{for all} \ j \in \{\beta, \gamma\}, \ \text{where} \ \eta(A) = \left[ \frac{R^\text{inc}(A)}{R(A)} \right].
\]
**Proof.** Omitted.

In the following example we illustrate most of the properties that have been proved in the previous propositions.

**Example 3.24.** Let \(U = \{a, b, c, d\}, \ U/R = \{\{a\}, \{b, c\}, \{c, d\}\}, \ \tau = \{U, \phi, \{a, b\}, \{c, d\}, \{a\}, \{a, d, c\}\}, \ \tau^\beta = \{U, \phi, \{a, c\}, \{b\}, \{c, d\}, \{b\}\} \) and \(\rho = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, d), (a, d), (a, c), (c, d)\}\).

For \(A = \{a, c\}\), we have:
\[
R^\text{dec}_\text{Dec}(A) = \{a\}, \ R^\text{inc}_\text{Inc}(A) = \{a, b\}, \ \overline{R^\text{dec}}_\text{Dec}(A) = U, \ \overline{R^\text{inc}}_\text{Inc}(A) = U.
\]
\[
S^\text{dec}_\text{Dec}(A) = U, \ \overline{S^\text{inc}}_\text{Inc}(A) = U, \ \overline{B}^\text{dec}_\text{Dec}(A) = \{b, c, d\}, \ \text{Neg}^\text{dec}_\text{Dec} = \phi.
\]
\[
\overline{\gamma}^\text{dec}_\text{Dec}(A) = A \cup A = A, \ \overline{\tau}^\text{inc}(A) = A \cup U = U, \ \overline{B}^\text{dec}_\text{Dec}(A) = \{b, d\}, \ \text{Neg}^\text{dec}_\text{Dec} = \phi.
\]
\[
\overline{\beta}^\text{inc}_\text{Inc}(A) = A \cap U = A, \ \overline{\tau}^\text{inc}(A) = \{a, b, c\}, \ \overline{B}^\text{dec}_\text{Dec}(A) = \{b\}, \ \text{Neg}^\text{dec}_\text{Dec} = \{d\}.
\]

**Proposition 3.25.** Let \((U, \tau, \rho)\) be a GOTAS and \(A \subseteq U\). Then we have
\[
B^\text{inc}_\text{Dec}(A) \subseteq B^\text{inc}_\text{Dec}(A) \subseteq B^\text{dec}_\text{Dec}(A) \subseteq B^\text{dec}_\text{Dec}(A) = B^\text{dec}_\text{Dec}(A).
\]
**Proof.** Omitted.

**Remark 3.26.** \(B^\text{inc}_\text{Dec}(A) \subseteq B^\text{dec}_\text{Dec}(A) \ (B^\text{dec}_\text{Dec}(A) \subseteq B^\text{dec}_\text{Dec}(A))\).

**Remark 3.27.** \(B^\text{inc}_\text{Dec}(A) \subseteq B^\text{inc}_\text{Dec}(A) \ (B^\text{dec}_\text{Dec}(A) \subseteq B^\text{dec}_\text{Dec}(A))\).

**Proposition 3.28.** Let \((U, \tau, \rho)\) be a GOTAS and \(A\) be a non-empty finite subset of \(U\). Then
\[
\eta^\text{inc}_j(A) \leq \eta^\text{dec}_j(A) \leq \eta^\text{dec}_j(A) \leq \eta^\text{dec}_j(A).
\]
**Proof.** Omitted.

**Proposition 3.28.** Let \((U, \tau, \rho)\) be a GOTAS and \(A \subseteq U\). Then \(\overline{\gamma}^\text{dec}_\text{Dec}(A) \subseteq \overline{\beta}^\text{inc}_\text{Dec}(A) \ (\overline{\beta}^\text{inc}_\text{Dec}(A) \subseteq \overline{\beta}^\text{inc}_\text{Dec}(A))\)

**Proof.** Let \(x \in \overline{\gamma}^\text{dec}_\text{Dec}(A) = A \cap \overline{\tau}^\text{inc}(A) \cup R^\text{inc}_\text{Inc}(A)\). Then \(x \in A\) and
\[
x \in \overline{\tau}^\text{inc}(R^\text{inc}_\text{Inc}(A)) \cup R^\text{inc}_\text{Inc}(\overline{\tau}^\text{inc}(A)).
\]
Therefore \(x \in A\) and \(x \in \overline{\tau}^\text{inc}(R^\text{inc}_\text{Inc}(A)) \) or \(x \in R^\text{inc}_\text{Inc}(\overline{\tau}^\text{inc}(A))\). Thus \(x \in A\) and \(x \in R^\text{inc}_\text{Inc}(\overline{\tau}^\text{inc}(A))\) and thus \(x \in A\) and \(x \in \overline{\tau}^\text{inc}(R^\text{inc}_\text{Inc}(A))\). Hence \(x \in A \cap \overline{\tau}^\text{inc}(R^\text{inc}_\text{Inc}(A))\). Therefore \(\overline{\gamma}^\text{dec}_\text{Dec}(A) \subseteq \overline{\beta}^\text{inc}_\text{Dec}(A)\).

One can prove the case between parentheses.

4. Conclusion

In this paper, we generalize rough set theory in the framework of topological spaces. Our results in this paper became the results about of \(\gamma, \beta\) approximation in [2] in the case of \(\rho\) is the equal relation. Also, the new
approximation which we give became as Pawlak’s approximation in the case of $\rho$ is the equal relation and $R$ is the equivalence relation. This theory brings in all these techniques to information analysis and knowledge processing.

References


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