The Matching Equivalence Graphs with the Maximum Matching Root Less than or Equal to 2

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Abstract

In the paper, we give a necessary and sufficient condition of matching equivalence of two graphs with the maximum matching root less than or equal to 2.

Keywords
Matching Polynomial, Matching-Equivalent, Matching Unique

1. Introduction

Let $G$ be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. A spanning subgraph $H$ is called a matching of $G$, if every connected component of $H$ is isolated edge or isolated vertex. A $k$-matching of $G$ is a matching with $k$ edges. A matching polynomial of $G$ is defined as

$$
\mu(G,x) = \sum_{k \geq 0} (-1)^k p(G,k) x^{n-2k}
$$

where $p(G,k)$ is the number of $k$-matchings of $G$.

Two graphs $G$ and $H$ are called matching-equivalent if $\mu(G,x) = \mu(H,x)$, and denoted by $G \sim H$. A graph $G$ is called matching unique if $G \sim H$ implies that $G \cong H$. The union of two graphs $G$ and $H$, denoted by $G + H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. $kG$ denotes the union of $k$ graphs $G$.

More than 30 years ago E. J. Farrell in [1] introduced the concept of matching polynomials. Latterly, Godsil and Gutman in [2] gave another definition. Here we use the definition given by Godsil. Form then on, the research on the properties of matching polynomials has largely been done (see [3]-[13]). But the research on
The matching-equivalent of graphs is few. In this paper, we give a necessary and sufficient condition of matching equivalence of two graphs with the maximum matching root less than or equal to 2.

Throughout the paper, by $P_n$ ($n \geq 1$) and $C_n$ ($n \geq 3$), respectively, denote the path and the cycle with $n$ vertices. $\Delta(G)$ denotes the maximum degree of graph $G$. By $K_{i,k}$ denote the star graph with 5 vertices. By $I_{n}$ and $T_{i,j,k}$ denote the tree which has one 3-degree vertex and three 1-degree vertices respectively. $D_{m,n}$ is defined as the graph obtained by identifying one end of the path $P_{n-1}$ with a vertex of the cycle $C_m$. Let $T_{i,j,k}$ be the path obtained by adding pendant edges at vertices 2 and 3 of $P_{n-2}$, respectively. The graphs $D_{m,n}, I_n, T_{i,j,k}$ are shown in Figure 1.

2. Graphs with the Maximum Matching Root Less than or Equal to 2

Let $G$ be a graph with order $n$. Since the roots of $\mu(G,x)$ are real numbers (see [7]), the maximum root of $\mu(G,x)$ denoted by $M(G)$, the characteristic polynomial of graph $G$ denoted by $\phi(G,x)$ and the maximum root of $\phi(G,x)$ denoted by $\rho(G)$ ($\rho(G)$ is also called spectral radius of graph $G$), respectively. In this section, we determine graphs with the maximum matching root less than or equal to 2, we firstly give some useful lemmas as follows:

**Lemma 2.1.** [7] Let $G$ be a graph with $k$ components $G_1, G_2, \ldots, G_k$. Then

$$\mu(G,x) = \prod_{i=1}^{k} \mu(G_i,x).$$

**Lemma 2.2.** [7] Let $G$ be a forest. Then $\phi(G,x) = \mu(G,x)$.

**Lemma 2.3.** [7] Let $G$ be a connected graph and $u \in V(G)$. Then

1) $M(G)$ is a single root of $\mu(G,x)$ and $M(G) > M(G \setminus u)$.

2) $\rho(G)$ is a single root of $\phi(G,x)$ and $\rho(G) > \rho(G \setminus u)$.

**Definition 2.1.** Let $G$ be a connected graph with a vertex $u$. The path-tree $T(G,u)$ is a tree with the paths in $G$ which start at $u$ as its vertices, and where two such paths are joined by an edge if one is a maximal subpath of the other.

Clearly, if $G$ is a tree, then the path tree $T(G,u) = G$.

**Lemma 2.4.** [7] Let $u$ be a vertex in the graph $G$ and $T(T(G,u))$ be the path tree of $G$ with respect to $u$. Then

$$\mu(G \setminus u,x) = \mu(T \setminus u,x) \mu(T,x)$$

and $\mu(G,x)$ divides $\mu(T,x)$.

**Lemma 2.5.** Let $G$ be a connected graph and $u \in V(G)$. Then $M(G)$ is spectral radius of path-tree $T = T(G,u)$. i.e., $M(G) = \rho(T)$.

**Proof.** By Lemmas 2.2 and 2.4, we have $\mu(G,x) \phi(T \setminus u,x) = \mu(G \setminus u,x) \phi(T,x)$, by Lemma 2.3, we have $M(G) > M(G \setminus u)$ and $\rho(T) > \rho(T \setminus u)$, comparing with the maximum root of $\mu(G,x) \phi(T \setminus u,x)$ and $\mu(G \setminus u,x) \phi(T,x)$, we can obtain $M(G) = \rho(T)$.

**Lemma 2.6.** [14] Let $T$ be a tree. Then

1) $\rho(T) < 2$ if and only if $T \in \Gamma_1 = \{P_n, T_{1,1,1}, T_{1,2,2}, T_{1,3,3}, T_{2,2,2}\}$.

2) $\rho(T) = 2$ if and only if $T \in \Gamma_2 = \{I_n, K_{1,4,4}, T_{2,2,2,2}, T_{3,3,3,3}, T_{2,2,2,2,2}\}$.

**Theorem 2.1.** Let $G$ be a connected graph. Then

![Figure 1. The graphs $D_{m,n}, I_n$ and $T_{i,j,k}$.](image)
1) $M(G) < 2$ if and only if $G \in \Omega_1 = \{P_2, T_{1,2,2}, T_{1,2,2,3}, T_{1,2,4}, C_4, D_{3,1}\}$.

2) $M(G) = 2$ if and only if $G \in \Omega_2 = \{I_n, K_{1,4}, T_{2,2,2}, T_{1,3,3}, T_{1,2,5}, D_{4,1}, D_{4,2}\}$.

**Proof.** (1) Since the path-tree of $C_n$ respect to an arbitrary vertex and $D_{3,1}$ respect to the 3 degree vertex are $T_{2,2,2}$ and $T_{2,2,3}$, respectively. By Lemmas 2.5 and 2.6 the sufficiency is obvious.

Necessity:

- **Case 1.** If $G$ is a tree.
  - Clearly, $\rho(G) = M(G) < 2$. By Lemma 2.6, $G \in \Gamma_1 \subseteq \Omega_1$.

- **Case 2.** If $G$ isn’t a tree.
  - By Lemma 2.5 and 2.6, the path-tree respect to an arbitrary vertex $u$ of $G$ is $T(G, u) \in \Gamma$. Then we get that the maximum degree $\Delta(G) \leq 3$ and the number of 3-degree vertex of $G$ is at most 1 (otherwise, $T(G, u) \notin \Gamma_1$).

**Subcase 2.1.** If $\Delta(G) = 3$. It is clear that $G$ has only one 3 degree vertex, thus $G = D_{n,m}$ (otherwise, $T(G, u) \notin \Gamma_1$ or $G$ is a tree). Clearly, the path-tree of $G$ respect to the 3 degree vertex $u$ is $T(G, u) = T(D_{n,m}, u) = T_{m-1,m-1,n}$. Since $T(G, u) \in \Gamma_1$, thus we have $m = 3, n = 1, i.e., G = D_{3,1}$.

**Subcase 2.2.** If $\Delta(G) < 3$.
  - Since $G$ is connected and isn’t a tree, then $G$ is $C_n$. Thus $G \in \Omega_1$.

(2) Since the path-tree of $D_{2,2}$ and $D_{4,1}$ respect to the 3 degree vertex are $T_{2,2,2}$ and $T_{1,3,3}$, respectively. By Lemmas 2.5 and 2.6 the sufficiency is clear.

Necessity:

- **Case 1.** If $G$ is a tree.
  - Clearly, $\rho(G) = M(G) = 2$. By Lemma 2.6, $G \in \Gamma_2 \subseteq \Omega_2$.

- **Case 2.** If $G$ isn’t a tree.
  - By Lemma 2.5, the path-tree respect to an arbitrary vertex $u$ of $G$ is $T(G, u) \in \Gamma_2$, thus $3 \leq \Delta(G) \leq 4$.

Denote $V' = \{v | d_G(v) = \Delta\}$. In order to complete the proof, we will divide four subcases as follows:

**Subcase 2.1.** If $\Delta(G) = 4$.
  - Let $u$ is a 4 degree vertex of $G$. Since $T(G, u) \in \Gamma_2$, then $T(G, u) = K_{1,4}$, and thus $G = K_{1,4}$.

**Subcase 2.2.** If $\Delta(G) = 3$ and $|V'| > 2$.
  - It is clear that the number of 3 degree vertex of path-tree $T(G, u)$ respect to an arbitrary vertex $u$ of $G$ is also greater than 2. Hence $T(G, u) \notin \Gamma_2$.

**Subcase 2.3.** If $\Delta(G) = 3$ and $|V'| = 2$, then $G$ is one of the graphs $G_1, G_2, G_3$ and $G_4$ (see Figure 2) Clearly, $T(G, u) \in \Gamma_2$.

**Subcase 2.4.** If $\Delta(G) = 3$ and $|V'| = 1$.
  - It is clear that $G = D_{n,m}$ and the path-tree respect to the 3 degree vertex $u$ is $T(G, u) = T(D_{n,m}, u) = T_{m-1,m-1,n}$. Since $T(G, u) \in \Gamma_2$, thus $m = 3, n = 2$ or $m = 4, n = 1$, i.e., $G = D_{3,2}$ or $D_{4,1}$.

By Theorem 2.1 and Lemma 2.1, we can easily obtain the following Theorem 2.2:

**Theorem 2.2.** Let $G$ be a graph. Then

1) $M(G) < 2$ if and only if every connected component of $G$ belongs to $\Omega_1$.

2) $M(G) = 2$ and $2$ is multiple root of $\mu(G, x)$ if and only if $m$ connected components of $G$ belong to $\Omega_2$ and others belong to $\Omega_1$.

![Figure 2. Four connected graphs with $\Delta(G) = 3$ and $|V'| = 2$.](image-url)
3. Sufficient and Necessary Condition for Matching Equivalence of Graphs

In this section, the sufficient and necessary condition for matching equivalence of graphs with the maximum matching root less than or equal to 2 is determined. Firstly, we give some lemmas as follows:

Lemma 3.1. [7] Let \( G \) be a connected graph and \( u \in V(G) \). Then

\[
\mu(G, x) = x \mu(G \setminus u, x) - \sum_{i \in N_G(u)} \mu(G \setminus \{u, i\}, x)
\]

where \( N_G(u) \) is neighbor vertex set of \( u \) in graph \( G \).

Lemma 3.2. 1) \( \mu(P_{2m+1}, x) = \mu(P_m, x) \mu(C_{m+1}, x) \),

2) \( \mu(T_{1,1,n}, x) = \mu(P_1, x) \mu(C_{n+2}, x) \),

3) \( \mu(T_{1,2,2}, x) = \mu(P_2, x) \mu(D_{3,1}, x) \),

4) \( \mu(P_1, x) \mu(C_{n}, x) = \mu(P_1, x) \mu(D_{3,1}, x) \),

5) \( \mu(P_1, x) \mu(C_{n}, x) = \mu(C_{n}, x) \mu(T_{1,2,3}, x) \),

6) \( \mu(P_1, x) \mu(C_{n}, x) = \mu(C_{n}, x) \mu(T_{1,2,4}, x) \),

7) \( \mu(D_{1,1}, x) = \mu(D_{3,2}, x) \),

8) \( \mu(I_n, x) = \mu(P_1, x) \mu(D_{3,2}, x) \),

9) \( \mu(T_{2,2,2}, x) = \mu(P_1, x) \mu(D_{3,2}, x) \),

10) \( \mu(T_{2,3,3}, x) = \mu(P_2, x) \mu(D_{3,2}, x) \),

11) \( \mu(T_{1,2,5}, x) = \mu(P_1, x) \mu(D_{3,2}, x) \),

12) \( \mu(P_2, x) \mu(K_{1,4}, x) = [\mu(P_1, x)]^2 \mu(D_{3,2}, x) \),

13) \( \mu(P_2, x) \mu(I_m, x) = \mu(P_1, x) \mu(P_{m-4}, x) \mu(D_{3,2}, x) \).

Proof: (1) Let the vertices sequence of path \( P_{2m+1} \) is \( u_1, u_2, \ldots, u_{2m+1} \), by Lemma 3.1, consider \( P_{2m+1} \) with \( u = u_0 \) and \( C_{m+1} \) with any one vertex, thus (1) holds.

(2) Let \( v \) be the 3 degree vertex and \( u \) be a such pendant vertex of \( T_{1,1,n} \) that the distance between \( u \) and \( v \) is 1. By Lemma 3.1, consider \( T_{1,1,n} \) with \( u \) and \( C_{n+2} \) with any one vertex, thus (2) holds.

(3)-(12) The results (3)-(12) can easily obtained by the following equalities.

\[
\mu(P_1, x) = x^3 - 1, \mu(P_2, x) = x^3 - 2x, \mu(P_3, x) = x^3 - 3x^2 + 1,
\]

\[
\mu(C_3, x) = x^3 - 3x, \mu(C_2, x) = x^3 - 5x^2 + 5x, \mu(C_1, x) = x^3 - 6x^2 + 9x^2 - 2,
\]

\[
\mu(C_{n}, x) = x^{n} - 9x^{n-2} + 27x^{n-3} - 30x^n + 9x,
\]

\[
\mu(C_{1,n}, x) = x^{3n} - 15x^{3n-3} + 90x^{3n-5} - 275x^{3n-7} + 450x^{3n-9} - 378x^{3n-11} + 140x^{3n-13} - 15x^n,
\]

\[
\mu(D_{1,1}, x) = x^4 - 4x^3 + 1, \mu(D_{2,2}, x) = x^3 - 5x^2 + 4x, \mu(D_{3,1}, x) = x^3 - 5x^2 + 4x,
\]

\[
\mu(T_{1,2,2}, x) = x^6 - 5x^4 + 5x^2 - 1, \mu(T_{1,2,3}, x) = x^7 - 6x^5 + 9x^2 - 3x,
\]

\[
\mu(T_{1,2,4}, x) = x^8 - 7x^6 + 14x^4 - 8x^2 + 1, \mu(T_{2,2,2}, x) = x^7 - 6x^5 + 9x^2 - 4x,
\]

\[
\mu(T_{1,3,3}, x) = x^8 - 7x^6 + 14x^4 - 8x^2 + 20x^2 - 17x^3 + 4x,
\]

\[
\mu(I_n, x) = x^{n} - 5x^4 + 4x^2, \mu(K_{14}, x) = x^3 - 4x^3.
\]

(13) By Lemma 3.1, \( \mu(I_m, x) = x \mu(T_{1,m-4}, x) - x \mu(T_{1,m-6}, x) \) and

\[
\mu(T_{1,1,n}, x) = x \mu(T_{1,1,n-1}, x) - x \mu(T_{1,1,n-2}, x). \]

Then \( \mu(I_m, x) = x \mu(I_{m-4}, x) - \mu(I_{m-2}, x) \).

Now, by using mathematical induction to prove (13). Firstly, By (8) and \( \mu(I_7, x) = x^7 - 6x^5 + 8x^3 \),

(13) holds for \( m = 6, 7 \). If \( m \geq 8 \),

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\[ \mu(P_{1},x) \mu(I_{m},x) = \mu(P_{2},x) \mu(I_{m-1},x) - \mu(P_{3},x) \mu(I_{m-2},x) \]
\[ = \mu(P_{1},x) \mu(P_{5},x) \mu(D_{3},x) - \mu(P_{3},x) \mu(P_{6},x) \mu(D_{3},x) \]
\[ = \mu(P_{1},x) \mu(D_{3},x) [\mu(P_{5},x) - \mu(P_{6},x)] \]
\[ = \mu(P_{1},x) \mu(P_{4},x) \mu(D_{3},x). \]

Hence (13) holds for \( m \geq 6 \). □

**Lemma 3.3.**
1) \( M(P_{m}) > M(P_{n}), (m > n), \)
2) \( M(C_{m}) = M(P_{2m-1}) > M(P_{2m-3}) = M(C_{m}), (m > n \geq 3). \)
3) \( M(T_{1,1,n}) = M(C_{n-2}) = M(P_{2m-1}), \)
4) \( M(T_{1,2,2}) = M(D_{3}) = M(C_{n}) = M(P_{1}), \)
5) \( M(T_{1,2,2}) = M(C_{n}) = M(P_{2}), \)
6) \( M(T_{1,2,2}) = M(C_{n}) = M(P_{29}). \)

**Proof.** Clearly, by Lemma 2.3, we obtain Lemma 3.3(1) immediately. And comparing with the maximum root of two sides of equalities in Lemma 3.2, other results in Lemma 3.3 is also obvious. □

**Definition 3.1.** Let \( G \) and \( H_{i} (i = 1, 2, \cdots, n) \) be graphs, if
\[ \mu(G,x) = \mu(H_{1}, x)^{k_{1}} \mu(H_{2}, x)^{k_{2}} \cdots \mu(H_{n}, x)^{k_{n}}, \]
where \( k_{i} (i = 1, 2, \cdots, n) \) be integers. Then \( G \) is called a linear combination of \( H_{i} \), and denote
\[ G = k_{1}H_{1} + k_{2}H_{2} + \cdots + k_{n}H_{n}. \]

Note that some \( k_{i} \) is allowed to be negative. In fact, if all \( k_{i} \) are positive, then \( G = k_{1}H_{1} + k_{2}H_{2} + \cdots + k_{n}H_{n} \) doesn't stand for a graph. In any case, \( G = k_{1}H_{1} + k_{2}H_{2} + \cdots + k_{n}H_{n} \) implies that polynomials \( \mu(G,x) \) and \( \mu(H_{1}, x)^{k_{1}} \mu(H_{2}, x)^{k_{2}} \cdots \mu(H_{n}, x)^{k_{n}} \) are equal. For example, since \( \mu(P_{2m-1}, x) = \mu(P_{m}, x) \mu(C_{m+1}, x) \), we can denote \( C_{m+1} = P_{2m-1} - P_{m}. \)

By Lemma 3.2, the following representations are also obvious.

**Lemma 3.4.**
1) \( C_{m} = P_{2m-1} - P_{m}, \)
2) \( T_{1,2,n} = P_{2m-3} - P_{m+1} + P_{1}, \)
3) \( D_{3} = P_{1} - P_{2} - P_{3} + P_{1}, \)
4) \( T_{1,2,2} = P_{11} - P_{2} - P_{3} + P_{2} + P_{3}, \)
5) \( T_{1,2,3} = P_{11} - P_{2} - P_{3} + P_{2} + P_{3}, \)
6) \( T_{1,2,4} = P_{29} - P_{14} - P_{3} + P_{4} - P_{2} + P_{3} + P_{1}, \)
7) \( D_{3} = P_{2}, \)
8) \( T_{2,2,2} = D_{3} + P_{2}, \)
9) \( T_{1,3,3} = D_{3} + P_{2}, \)
10) \( T_{1,2,3} = D_{3} + P_{4}, \)
11) \( K_{1,4} = D_{3} + P_{2}, \)
12) \( I_{m} = D_{3} + P_{m-4} - P_{2} + P_{1}, \)

**Lemma 3.5.** If \( M(G) < 2 \). Then \( G \) can uniquely be represented as a linear combination of the form
\[ G = a_{1}P_{m_{1}} + a_{2}P_{m_{2}} + \cdots + a_{i}P_{m_{i}}, \]
and the non-vanishing coefficient \( a_{i} \), with the greatest \( m_{i} \), is positive. Furthermore, if \( P_{m_{i}} \) is the longest path with the non-vanishing coefficient \( a_{i} \), \( M(G) = M(P_{m_{i}}). \)

**Proof.** Since \( M(G) < 2 \), by Theorem 2.2, every connected component of \( G \) belongs to \( \Omega_{1} \). According to Lemma 3.4, we get that \( G \) can be represented as a linear combination of paths. Next, without loss of generality, assume that \( G \) can be represented as
\[ a_{1}P_{m_{1}} + a_{2}P_{m_{2}} + \cdots + a_{i}P_{m_{i}} = b_{1}P_{n_{1}} + b_{2}P_{n_{2}} + \cdots + b_{j}P_{n_{j}}, \]
(1)
where \( m_{1} < m_{2} < \cdots < m_{i} \) and \( n_{1} < n_{2} < \cdots < n_{j} \).

Now by transposition terms from side to side of Equations (1) to (2) such that the coefficients of \( P_{m_{i}} \) and \( P_{n_{j}} \) are positive, without loss of generality, Assumes that the Equation (2) as follows:
\[ a_{1}P_{m_{1}} + a_{2}P_{m_{2}} + \cdots + a_{i}P_{m_{i}} = b_{1}P_{n_{1}} + b_{2}P_{n_{2}} + \cdots + b_{j}P_{n_{j}}, \]
(2)
where \( m_{1} < m_{2} < \cdots < m_{i} \), \( n_{1} < n_{2} < \cdots < n_{j} \) and \( a_{1} > 0, b_{j} > 0 (i = 1, 2, \cdots, k, j = 1, 2, \cdots, s). \)

Compare with the maximum root and its multiplicity of graphs in two sides of (2), we shall get \( n_{s} = m_{k}, b_{j} = a_{k} \). Thus
Repeat this proceeding, we shall get \( k = s \) and \( \alpha_i = m_i, b_i = a_i \) for \( i = 1, 2, \ldots, k \). That is, \( G \) can uniquely be represented as a linear combination of paths.

Furthermore, assume that \( G \) be represented as a linear combination

\[
G = \alpha_1 p_{m_1} + \alpha_2 p_{m_2} + \cdots + \alpha_k p_{m_k},
\]

and \( \alpha_k \) is the non-vanishing coefficient of the longest path in (3). Then \( \alpha_k > 0 \).

In fact, assume that \( \alpha_k < 0 \), then by transposition terms from side to side of Equation (3) such that the coefficients of \( p_{m_k} \) are positive, we can obtain Equation (4).

\[
G + (\alpha_k) p_{m_k} + \cdots + \beta_p p_{m_p} = \beta_1 p_{n_1} + \beta_2 p_{n_2} + \cdots + \beta_q p_{n_q},
\]

where \( \beta_i = \pm \alpha_i \) and \( \beta_i > 0 \). By comparing with the maximum root of \( G + (\alpha_k) p_{m_k} + \cdots + \beta_p p_{m_p} \) and \( \beta_1 p_{n_1} + \beta_2 p_{n_2} + \cdots + \beta_q p_{n_q} \), we can obtain

\[
M(G + (\alpha_k) p_{m_k} + \cdots + \beta_p p_{m_p}) \geq M(p_{m_k}),
\]

\[
M(\beta_1 p_{n_1} + \beta_2 p_{n_2} + \cdots + \beta_q p_{n_q}) < M(p_{m_k}),
\]

it is a contradiction. Thus \( \alpha_k > 0 \) and then modify (4) as

\[
G + \cdots + \beta_p p_{m_p} = \alpha_k p_{m_k} + \beta_1 p_{n_1} + \beta_2 p_{n_2} + \cdots + \beta_q p_{n_q},
\]

compare with the maximum root of \( G + \cdots + \beta_p p_{m_p} \) and \( \alpha_k p_{m_k} + \beta_1 p_{n_1} + \beta_2 p_{n_2} + \cdots + \beta_q p_{n_q} \) we can obtain

\[
M(G) = M(p_{m_k}).
\]

**Lemma 3.6.** If \( M(G) = 2 \), then \( G \) can uniquely be represented as a linear combination of the form

\[
a_0 D_{3,2} + a_1 n_{m_1} + a_2 n_{m_2} + \cdots + a_k n_{m_k} \quad \text{and} \quad a_0 \quad \text{equals to the multiplicity of root} \ 2 \ \text{of} \ \mu(G, x).
\]

**Proof.** Since \( M(G) = 2 \), by Theorem 2.2, every connected component of \( G \) belongs to \( \Omega_1 \cup \Omega_2 \). According to Lemma 3.4, we easily obtain that \( G \) can be represented as a linear combination of \( D_{3,2} \) and some paths. Next, without loss of generality, assume that \( G \) can be represented as

\[
a_0 D_{3,2} + a_1 p_{m_1} + a_2 p_{m_2} + \cdots + a_k p_{m_k} = b_0 D_{3,2} + b_1 p_{n_1} + b_2 p_{n_2} + \cdots + b_k p_{n_k}.
\]

By transposition terms and comparing with the multiplicity of root 2, we have \( a_0 = b_0 \) equal to the multiplicity of root 2 of \( \mu(G, x) \). Thus

\[
a_0 D_{3,2} + a_1 p_{m_1} + a_2 p_{m_2} + \cdots + a_k p_{m_k} = b_0 D_{3,2} + b_1 p_{n_1} + b_2 p_{n_2} + \cdots + b_k p_{n_k}.
\]

Furthermore, we can obtain \( s = k \) and \( n_i = m_i, a_i = b_i \) for \( i = 1, 2, \ldots, k \). By Lemmas 3.5, 3.6 and Definition 3.1 we immediately get.

**Theorem 3.1.** Let \( G, H \) be graphs. Then

1) If \( M(G) < 2, M(H) < 2 \) , then \( G \sim H \) if and only if \( G \) and \( H \) have the same linear combination representation of paths.

2) If \( M(G) = 2, M(H) = 2 \) , then \( G \sim H \) if and only if \( G \) and \( H \) have the same linear combination representation of \( D_{3,2} \) and some paths.

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