On a Class of Supereulerian Digraphs

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Abstract

The 2-sum of two digraphs \( D_1 \) and \( D_2 \), denoted \( D_1 \oplus D_2 \), is the digraph obtained from the disjunct union of \( D_1 \) and \( D_2 \) by identifying an arc in \( D_1 \) with an arc in \( D_2 \). A digraph \( D \) is supereulerian if \( D \) contains a spanning eulerian subdigraph. It has been noted that the 2-sum of two supereulerian (or even hamiltonian) digraphs may not be supereulerian. We obtain several sufficient conditions on \( D_1 \) and \( D_2 \) for \( D_1 \oplus D_2 \) to be supereulerian. In particular, we show that if \( D_1 \) and \( D_2 \) are symmetrically connected or partially symmetric, then \( D_1 \oplus D_2 \) is supereulerian.

Keywords

Supereulerian, Digraph 2-Sums, Arc-Strong-Connectivity, Hamiltonian-Connected Digraphs

1. Introduction

We consider finite graphs and digraphs, and undefined terms and notations will follow [1] for graphs and [2] for digraphs. Throughout this paper, the notation \((u, v)\) denotes an arc oriented from \( u \) to \( v \). A digraph \( D \) is strict if it contains no parallel arcs or loops; and is symmetric if for any vertices \( u, v \in V(D) \), if \((u, v) \in A(D)\), then \((v, u) \in A(D)\). If two arcs of \( D \) have a common vertex, we say that these two arcs are adjacent in \( D \). A directed path in a digraph \( D \) from a vertex \( u \) to a vertex \( v \) is called a \((u, v)\)-dipath. To emphasize the distinction between graphs and digraphs, a directed cycle or path in a digraph is often referred as a dicycle or dipath. A dipath \( P \) is a hamiltonian dipath if \( V(P) = V(D) \). A digraph \( D \) is hamiltonian if \( D \) contains a hamiltonian dicycle. An \((x, y)\)-hamiltonian dipath is a hamiltonian dipath from \( x \) to \( y \). A digraph \( D \) is hamiltonian-connected if \( D \) has an \((x, y)\)-hamiltonian dipath for every choice of distinct vertices \( x, y \in V(D) \).

As in [2], \( \lambda(D) \) denotes the arc-strong-connectivity of \( D \). A digraph \( D \) is strong if and only if \( \lambda(D) \geq 1 \). For \( X, Y \subseteq V(D) \), we define

\( (X,Y)_D = \{(x,y) \in A(D) : x \in X \text{ and } y \in Y\}; \) and \( \partial_D^+(X) = (X, V(D) - X)_D \).

For a subset \( A' \subseteq A(D) \), the subdigraph is arc-induced by \( A' \) is the digraph \( D[A'] = (V', A') \), where \( V' \) is the set of vertices in \( V \) which are incident with at least one arc in \( A' \).

Let \( d_D^+(X) = \left| \partial_D^+(X) \right| \) and \( d_D^-(X) = \left| \partial_D^-(X) \right| \).

When \( X = \{v\} \), we write \( d_D^+(v) = \left| \partial_D^+(v) \right| \) and \( d_D^-(v) = \left| \partial_D^-(v) \right| \). Let \( N_D^+(v) = \{u \in V(D) - v : (v,u) \in A(D)\} \) and \( N_D^-(v) = \{u \in V(D) - v : (u,v) \in A(D)\} \) denote the out-neighbourhood and in-neighbourhood of \( v \) in \( D \), respectively. Vertices in \( N_D^+(v), N_D^-(v) \) are called the out-neighbours, in-neighbours of \( v \). Thus for a digraph \( D \) and an integer \( k \geq 0 \),

\[
\lambda(D) \geq k \text{ if and only if for any } W \text{ with } \emptyset \neq W \subset V(D), \left| \partial_D^+(W) \right| \geq k. \tag{1}
\]

Boesch, Suffel, and Tindell [3] in 1977 proposed the supereulerian problem, which seeks to characterize graphs that have spanning eulerian subgraphs. They indicated that this problem would be very difficult. Pulleyblank [4] later in 1979 proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. Catlin [5] in 1992 presented the first survey on supereulerian graphs. Chen et al. [6] surveyed the reduction method associated with the supereulerian problem and their applications. An updated survey presenting the more recent developments can be found in [7].

It is natural to consider the supereulerian problem in digraphs. A digraph \( D \) is eulerian if it contains a closed ditrail \( W \) such that \( A(W) = A(D) \), or, equivalently, if \( D \) is strong and for any \( v \in V(D) \), \( d_D^+(v) = d_D^-(v) \). A digraph \( D \) is supereulerian if \( D \) contains a closed ditrail \( W \) such that \( V(W) = V(D) \), or, equivalently, if \( D \) contains a spanning eulerian subdigraph. Some recent developments on supereulerian digraphs are given in [8]-[12].

A central problem is to determine or characterize supereulerian digraphs. In Section 2, the \textbf{2-sum} \( D_1 \oplus D_2 \) of two digraphs \( D_1 \) and \( D_2 \) is defined, and some basic properties of 2-sums are discussed. We will observe that a 2-sum of two supereulerian (or even hamiltonian) digraphs may not be supereulerian. Thus it is natural to seek sufficient conditions on \( D_1 \) and \( D_2 \) for the 2-sum of \( D_1 \) and \( D_2 \) to be supereulerian. In the last section, we will present several sufficient conditions for supereulerian 2-sums of digraphs. In particular, we show that if \( D_1 \) and \( D_2 \) are either symmetrically connected or partially symmetric (to be defined in Section 3), then \( D_1 \oplus D_2 \) is supereulerian.

\section{The 2-Sums of Digraphs}

The definition and some elementary properties of the 2-sums of digraphs are presented in this section. A digraph is nontrivial if it contains at least one arc. Throughout this section, all digraphs are assumed to be nontrivial.

\textbf{Definition 2.1} Let \( D_1 \) and \( D_2 \) be two vertex disjoint digraphs, and let \( a_1 = (v_{11}, v_{12}) \in A(D_1) \) and \( a_2 = (v_{21}, v_{22}) \in A(D_2) \) be two distinguished arcs. The \textbf{2-sum} \( D_1 \oplus_{a_1,a_2} D_2 \) of \( D_1 \) and \( D_2 \) with base arcs \( a_1 \) and \( a_2 \) is obtained from the union of \( D_1 \) and \( D_2 - a_2 \) by identifying \( v_{11} \) with \( v_{21} \) and \( v_{12} \) with \( v_{22} \), respectively. When the arcs \( a_1 \) and \( a_2 \) are not emphasized or is understood from the context, we often use \( D_1 \oplus D_2 \) for \( D_1 \oplus_{a_1,a_2} D_2 \).

\textbf{Lemma 1} Let \( D_1 \) and \( D_2 \) be two vertex disjoint strong digraphs. Then

\[ \lambda(D_1 \oplus D_2) \geq \min \{ \lambda(D_1), \lambda(D_2) \}. \]

\textbf{Proof.} Let \( k \geq 0 \) be an integer such that \( \min \{ \lambda(D_1), \lambda(D_2) \} = k \), and let \( \lambda(D_1 \oplus D_2) = k' \). We shall show that \( k' \geq k \). By (1), there exists a proper nonempty vertex subset \( X \subset V(D_1 \oplus D_2) \) such that \( \partial_{D_1 \oplus D_2}(X) = k' \). Let \( S = \partial_{D_1 \oplus D_2}(X) \). We argue by contradiction and assume that \( k' < k \).

By Definition 2.1, we have \( v_{11} = v_{21} \in V(D_1) \) and \( v_{12} = v_{22} \in V(D_2) \) in \( D_1 \oplus D_2 \). If \( X \cap V(D_i) \neq \emptyset \) and \( X \cap V(D_i) \neq \emptyset \), we obtain that \( v_{11} = v_{21} \notin X \) and \( v_{12} = v_{22} \notin X \), then \( X \cap V(D_i) \neq \emptyset \) and \( S = \partial_{D_i}(X) \). It follows by (1) that \( k' = |S| \geq \lambda(D_i) \geq k \), contrary to the assumption that \( k' < k \). Similarly, if \( X \cap V(D_i) = \emptyset \) and \( X \cap V(D_i) = \emptyset \), then \( X \subset V(D_1) \) and \( S = \partial_{D_2}(X) \), hence a contradiction to the assumption that \( k' < k \) is obtained from \( k' = |S| \geq \lambda(D_2) \geq k \).
Thus, we may assume that $X \cap V(D_1) \neq \emptyset$ and $X \cap V(D_2) \neq \emptyset$. Let $X' = X \cap V(D_1)$. Then $X'$ is a proper nonempty subset of $V(D_1)$, and

$\partial D_1(X') \subseteq S$. It follows by (1) that $k' = |S| \geq \lambda D_1(X') \geq \lambda(D_1) \geq k$

contrary to the assumption that $k' < k$.

Example 2.1 The converse of Lemma 1 may not always stand, as indicated by the example below, depicted in Figure 1. Let $V(D_1) = \{v_{11}, v_{12}, v_{13}, v_{14}\}$ and $V(D_2) = \{v_{21}, v_{22}, v_{23}, v_{24}\}$. Let $A(D_1) = \{(v_{11}, v_{12}), (v_{13}, v_{12}), (v_{14}, v_{13}), (v_{14}, v_{12}), (v_{11}, v_{13})\}$ and

$A(D_2) = \{(v_{21}, v_{22}), (v_{23}, v_{22}), (v_{24}, v_{23}), (v_{24}, v_{22}), (v_{21}, v_{22})\}$. Let $a_1 = (v_{11}, v_{12})$ and $a_2 = (v_{21}, v_{22})$.

Then, it is routine to verify that $\lambda D_1(D_2) \geq 1$. While $D_2$ is strong, the digraph $D_1$ contains a vertex $v_{11}$ with $d_{D_1}^+(v_{11}) = 0$, and so $\lambda(D_1) = 0$.

Lemma 2 A digraph $D$ is not supereulerian if for some integer $m > 0$, $V(D)$ has vertex disjoint subsets $\{B_i\}_{i=1}^m$ satisfying both of the following:

i) $|N_D(B_i)| \leq m - 1$.

ii) $N_D(B_i) \subseteq B_i$, $\forall i \in \{1, 2, \cdots, m\}$.

Proof. By contradiction, we assume that both i) and ii) hold and $D$ is supereulerian. Let $S$ be a spanning eulerian subdigraph of $D$, then $B \subseteq V(S) = V(D)$ and $A(S) \subset A(D)$. Since $S$ is eulerian, for any subset $X \subseteq V(S)$, it follows that $\partial D_1(X') = \partial D_2(X')$. Thus, by ii), we conclude that

$\partial D_1(B_j) \cap A(S) = \partial D_2(B_j) \cap A(S) \leq m - 1$. (2)

By i) and by (2), there must be a $B_j$ with $j \in \{1, 2, \cdots, m\}$ such that $\partial D_1(B_j) \cap A(S) = \emptyset$, contrary to the assumption that $V(S) = V(D)$.

Lemma 2 can be applied to find examples of hamiltonian digraphs whose 2-sum is not supereulerian, as shown in Example 2.2 below.

Example 2.2 Let $n_1, n_2 \geq 3$ be integers and $C_{n_1}$ and $C_{n_2}$ be two vertex disjoint dicycles with length $n_1$ and $n_2$, respectively. We claim that $C_{n_1} \oplus C_{n_2}$ is not supereulerian. To justify this claim, we denote $V(C_{n_1}) = \{v_{11}, v_{12}, \cdots, v_{n_1}\}$, and $V(C_{n_2}) = \{v_{12}, v_{13}, \cdots, v_{2n_2}\}$. Without loss of generality, we assume that $a_1 = (v_{11}, v_{12})$ and $a_2 = (v_{21}, v_{22})$, and $C_{n_1} \oplus C_{n_2} = C_{n_1} \oplus C_{n_2}$. Let $B_1$ and $B_2$ be subdigraphs of $C_{n_1} \oplus C_{n_2}$ with $V(B) = \{v_{12}, v_{13}\}$, $V(B_1) = \{v_{12}, v_{13}\}$ and $V(B_2) = \{v_{21}, v_{22}\}$, respectively. By Lemma 2, we conclude that $C_{n_1} \oplus C_{n_2}$ is not supereulerian (see Figure 2).

3. Sufficient Conditions for Supereulerian 2-Sums of Digraphs

In this section, we will show several sufficient conditions on $D_1$ and $D_2$ to assure that the 2-sum $D_1 \oplus D_2$...
is supereulerian.

**Proposition 1** Let \(D_1\) and \(D_2\) be two vertex disjoint supereulerian digraphs with \(a_1 = (v_{11}, v_{12}) \in A(D_1)\) and \(a_2 = (v_{21}, v_{22}) \in A(D_2)\), and let \(D_1 \oplus D_2\) denote \(D_1 \oplus a_{12} D_2\). Each of the following holds.

i) For some \(i \in \{1, 2\}\), if \(D_i\) has a spanning eulerian subdigraph \(S_i\) such that \(a_i \notin A(S_i)\), then \(D_1 \oplus D_2\) is supereulerian.

ii) If for some \(i \in \{1, 2\}\), \(D_i\) is hamiltonian-connected, then \(D_1 \oplus D_2\) is supereulerian.

**Proof.** i) Since \(D_1\) and \(D_2\) are supereulerian digraphs, \(D_1\) and \(D_2\) are strongly connected, and so by Lemma 1, \(D_1 \oplus D_2\) is also strongly connected. Without loss of generality, we assume that \(i = 1\) and \(D_1\) has a spanning eulerian subdigraph \(S_1\) such that \(a_1 \notin A(S_1)\). Since \(D_2\) is supereulerian, we can pick a spanning eulerian subdigraph \(S_2\) in \(D_2\). Then \(A(S_1) \cap A(S_2) = \emptyset\) and \(V(S_1) \cap V(S_2) \neq \emptyset\). It follows that \(D \left[ A(S_1) \cup A(S_2) \right] \) is a spanning eulerian subdigraph in \(D_1 \oplus D_2\).

ii) Without loss of generality, we assume that \(i = 1\) and \(D_1\) is hamiltonian-connected, and so \(D_1\) has a \((v_{11}, v_{12})\)-hamiltonian dipath \(T_1\) and a \((v_{21}, v_{22})\)-hamiltonian dipath \(T_2\). Since \(D_2\) is supereulerian, \(D_2\) contains a spanning eulerian subdigraph \(S_2\). Define

\[
S = \left\{ D \left[ A(T_1) \cup A(S_2 - \{(v_{21}, v_{22})\}) \right] \right\} \text{ if } (v_{21}, v_{22}) \in A(S_2) \],

\[
D \left[ \left( A(T_2) \cup \{(v_{11}, v_{12})\} \right) \cup A(S_2) \right] \right\} \text{ if } (v_{11}, v_{12}) \notin A(S_2) .
\]

As in any case, \(S\) is strongly connected and every vertex \(v \in V(S)\) satisfies \(d^+_S(v) = d^-_S(v)\), and so \(S\) is eulerian. Since \(V(S) = V(T_1) \cup V(S_2) = V(D_1) \cup V(D_2)\), for \(i \in \{1, 2\}\), we conclude that \(S\) is a spanning eulerian subdigraph of \(D_1 \oplus D_2\), and so \(D_1 \oplus D_2\) is supereulerian.

**Theorem 2** [13] If a strict digraph on \(n \geq 3\) vertices has \((n-1)^2 + 1\) or more arcs, then it is hamiltonian-connected.

**Corollary 1** Let \(D_1\) be a strict digraph on \(n_1 \geq 3\) vertices and with \(\left| A(D_1) \right| \geq (n_1 - 1)^2 + 1\). If \(D_2\) is a supereulerian digraph, then \(D_1 \oplus D_2\) is supereulerian.

**Proof.** By Theorem 2, \(D_1\) is hamiltonian-connected. Then by Proposition 1 (ii), \(D_1 \oplus D_2\) is supereulerian.

Two classes of supereulerian digraphs seem to be of particular interests in studying supereulerian digraph 2-sums. We first present their definitions.

**Definition 3.2** Let \(D\) be a digraph such that either \(D = K_1\) or \(A(D) \neq \emptyset\). If for any \(u, v \in V(D)\), \(D\) contains a symmetric dipath from \(u\) to \(v\), then \(D\) is called a **symmetrically connected** digraph.

Given a digraph \(D\), define a relation \(\sim \) on \(V(D)\) such that \(u \sim v\) if and only if \(u = v\) or \(D\) has a symmetrically connected subdigraph \(H\) with \(u, v \in V(H)\). By definition, one can routinely verify that \(\sim\) is an equivalence relation. Each equivalence class induces a symmetrically connected component of \(D\). Hence \(D\) is symmetrically connected if and only if \(D\) has only one symmetrically connected component. A symmetrically connected component of \(D\) is also called a maximal symmetrically connected subdigraph of \(D\). When \(D\) has more than one symmetrically connected components, we have the following definition.

**Definition 3.3** Let \(D\) be a weakly connected digraph and \(\{H_1, H_2, \cdots, H_c\}\) be the set of maximal symmetrically connected subdigraphs of \(D\) with \(c \geq 2\). If for any proper nonempty subset \(J \subset \{H_1, H_2, \cdots, H_c\}\),

there exist an \(H_j \in J\), a vertex \(v \in V(H_j)\), and an \(H_j \notin J\) such that

\[
N^+_D(v) \cap V(H_j) \neq \emptyset \text{ and } N^-_D(v) \cap V(H_j) \neq \emptyset,
\]

then \(D\) is **partially symmetric**.

It is known that both symmetrically connected digraphs and partially symmetric digraphs are supereulerian.

**Theorem 3** ([14] and [15]) Each of the following holds.

i) Every symmetrically connected digraph is supereulerian.

ii) Every partially symmetric digraph is supereulerian.

A main result of this section is to show that the digraph 2-sums of symmetrically connected or partially symmetric digraphs are supereulerian.

**Lemma 3** Let \(D_1\) and \(D_2\) be two vertex disjoint digraphs with \(a_1 = (v_{11}, v_{12}) \in A(D_1)\) and \(a_2 = (v_{21}, v_{22}) \in A(D_2)\), and let \(D_1 \oplus D_2\) denote \(D_1 \oplus a_{12} D_2\). Each of the following holds.

i) If \(D_1\) and \(D_2\) are symmetrically connected, then \(D_1 \oplus D_2\) is symmetrically connected.
ii) If $D_1$ and $D_2$ are partially symmetric, then $D_1 \oplus D_2$ is partially symmetric.

iii) If $D_1$ is symmetric and $D_2$ is partially symmetric, then $D_1 \oplus D_2$ is partially symmetric.

Proof: i) For any vertices $x, y \in V(D_1 \oplus D_2)$, we shall show that $D_1 \oplus D_2$ always has a symmetric $(x, y)$-path. If for some $i \in \{1, 2\}$, we have $x, y \in V(D_i)$, then as $D_i$ is symmetrically connected, $D_i$ contains a symmetric $(x, y)$-path $P$. Since $D_1$ is a subdigraph of $D_1 \oplus D_2$, $P$ is also a symmetric $(x, y)$-path of $D_1 \oplus D_2$. Hence we may assume that $x \in V(D_1)$ and $y \in V(D_2)$. Since $D_1$ and $D_2$ are symmetrically connected, $D_i$ contains a symmetric $(x, v_{i1})$-path $P_i$ and $D_2$ contains a symmetric $(v_{i2}, y)$-path $P_2$.

By Definition 2.1, $v_{i1}$ and $v_{i2}$ represent the same vertex in $D_1 \oplus D_2$, and so $D_1 \oplus D_2 [A(P_i) \cup A(P_2)]$ is a symmetric $(x, y)$-path in $D_1 \oplus D_2$.

ii) Fix $i \in \{1, 2\}$. Since $D_i$ is partially symmetric, for some integer $c_i > 1$, let $\{H'_{i1}, H'_{i2}, \ldots, H'_{ic_i}\}$ be the set of all maximal symmetrically connected subdigraphs of $D_i$. Without loss of generality, we assume that $v_{i1} \in V(H'_{i1})$ and $v_{i2} \in V(H'_{i2})$; and for some $s, t$ with $1 \leq s < c_i$ and $1 \leq t < c_i$, $v_{i1} \in V(H'_{is})$ and $v_{i2} \in V(H'_{it})$. (We allow the possibility that $s = 1$ and/or $t = 1$). Define, for $1 \leq h \leq c_i$ and $1 \leq j \leq c_i$,

$$H_{ih} = \begin{cases} H'_{ih} & \text{if } h \notin \{1, s\} \\ H'_{is} \cup H'_{it} & \text{if } h = 1 \text{ and } j = 1 \\ H'_{ih} \cup H'_{it} & \text{if } h = s \text{ and } j = t \\ H'_{ih} \cup H'_{it} & \text{if } j \notin \{1, t\} \end{cases}$$

Then, $\mathcal{H} = \{H_{i1}, H_{i2}, \ldots, H_{ic_i}, H_{i1}, H_{i2}, \ldots, H_{ic_i}\}$ is the set of all maximal symmetrically connected subdigraphs of $D_i \oplus D_2$. Note that $H_{i1} = H_{i2}$ and $H_{ic_i} = H_{ic_i}$. We shall show by definition that $D_1 \oplus D_2$ is partially symmetric. To do that, let $\mathcal{J}$ be a nonempty proper subset of $\mathcal{H}$. We shall show that (3) holds.

Since $\mathcal{H} = \{H_{i1}, H_{i2}, \ldots, H_{ic_i}, H_{i1}, H_{i2}, \ldots, H_{ic_i}\}$, we either have $\mathcal{J} \cap \{H_{i1}, H_{i2}, \ldots, H_{ic_i}\} \neq \emptyset$ or $\mathcal{J} \cap \{H_{i1}, H_{i2}, \ldots, H_{ic_i}\} = \emptyset$. By symmetry, we may assume that $\mathcal{J} \cap \{H_{i1}, H_{i2}, \ldots, H_{ic_i}\} \neq \emptyset$.

Suppose first that $\{H_{i1}, H_{i2}, \ldots, H_{ic_i}\} - \mathcal{J} \neq \emptyset$. Let $\mathcal{J}' = \{H'_{ih} | H_{ih} \in \mathcal{J}\}$. Then $\{H'_{i1}, H'_{i2}, \ldots, H'_{ic_i}\} - \mathcal{J}' \neq \emptyset$. Since $D_i$ is partially symmetric, there exists an $H'_{ih} \in \mathcal{J}'$, a vertex $v \in V(H'_{ih})$, and an $H'_{ih} \in \{H'_{i1}, H'_{i2}, \ldots, H'_{ic_i}\} - \mathcal{J}'$ such that

$$N_{D_i}(v) \cap V(H'_{ih}) \neq \emptyset \quad \text{and} \quad N_{D_i}(v) \cap V(H'_{ih}) \neq \emptyset.$$

This implies that the vertex $v \in V(H_{ih})$, $H_{ih} \in \mathcal{J}$, and $H_{ih} \not\in \mathcal{J}$ such that

$$N_{D_1 \oplus D_2}(v) \cap V(H_{ih}) \neq \emptyset \quad \text{and} \quad N_{D_1 \oplus D_2}(v) \cap V(H_{ih}) \neq \emptyset.$$

Thus (3) holds in this case.

Hence we may assume that $\{H_{i1}, H_{i2}, \ldots, H_{ic_i}\} \subset \mathcal{J}$. Since $\mathcal{J}$ is a proper subset, we must have $\{H_{i1}, H_{i2}, \ldots, H_{ic_i}\} - \mathcal{J} \neq \emptyset$. Since $H_{i1} = H_{i1} \in \mathcal{J}$, we also have $\{H_{i2}, H_{i2}, \ldots, H_{ic_i}\} \cap \mathcal{J} \neq \emptyset$. With a similar argument, we conclude that (3) must also hold in this case.

iii) Let $H_0 = D_i$ and let $\{H'_1, H'_2, \ldots, H'_{ic_i}\}$ be the set of all maximal symmetrically connected subdigraphs of $D_i$ with $v_{i1} \in V(H'_i)$ and for some $j \in \{1, 2, \ldots, c_i\}$, $v_{i2} \in V(H'_j)$. (We allow the possibility that $j = 1$). Define

$$H_{ij} = \begin{cases} H'_{i} \cup H'_0 & \text{if } i = 1 \text{ or } i = j \\ H'_{i} & \text{if } i \notin \{1, j\} \end{cases}$$

Then $\mathcal{H} = \{H_i, H_2, \ldots, H_{ic_i}\}$ is the set of all maximal symmetrically connected subdigraphs of $D_i \oplus D_2$. Note that $H_i = H_j$ with this notation. Let $\mathcal{J}$ be a nonempty proper subset of $\mathcal{H}$. We shall show that (3) holds.

Let $\mathcal{J}' = \{H'_i | H_i \in \mathcal{J}\}$. Since $\mathcal{J}$ is proper, $\mathcal{J}'$ is a nonempty proper subset of $\{H'_1, H'_2, \ldots, H'_{ic_i}\}$. Since $D_i$ is partially symmetric, by Definition 3.2, there exist an $H'_{ih} \in \mathcal{J}'$, a vertex $v \in V(H'_{ih})$, and an $H'_{ih} \in \{H'_1, H'_2, \ldots, H'_{ic_i}\} - \mathcal{J}'$ such that

$$N_{D_i}(v) \cap V(H'_{ih}) \neq \emptyset \quad \text{and} \quad N_{D_i}(v) \cap V(H'_{ih}) \neq \emptyset.$$
This implies that vertex \( v \in V(H_k) \), \( H_k \in \mathcal{F} \) and \( H_k \notin \mathcal{F} \) such that

\[
N_{\delta_D \oplus \delta_2}(v) \cap V(H_k) \neq \emptyset \quad \text{and} \quad N_{\delta_D \oplus \delta_2}^-(v) \cap V(H_k) \neq \emptyset.
\]

Thus (3) holds, and so by definition, \( D \oplus_2 D_2 \) is partially symmetric.

**Theorem 4** Let \( D_1 \) and \( D_2 \) be two digraphs. Each of the following holds.

i) If \( D_1 \) and \( D_2 \) are symmetrically connected, then \( D_1 \oplus_2 D_2 \) is supereulerian.

ii) If \( D_1 \) and \( D_2 \) are partially symmetric, then \( D_1 \oplus_2 D_2 \) is supereulerian.

iii) If \( D_1 \) is symmetric and \( D_2 \) is partially symmetric, then \( D_1 \oplus_2 D_2 \) is supereulerian.

**Proof.** This follows from Theorem 3 and Lemma 3.

It is also natural to consider sufficient conditions on \( D_1 \) and \( D_2 \) for \( D_1 \oplus_2 D_2 \) to be hamiltonian.

**Theorem 5** If \( D_1 \) is hamiltonian and \( D_2 \) is hamiltonian-connected digraphs, then \( D_1 \oplus_2 D_2 \) is hamiltonian.

**Proof.** Let \( V(D_1) = \{ v_1, v_2, \ldots, v_{n_1} \} \) with \( C = v_1 v_2 \cdots v_{n_1}v_1 \) be a hamiltonian dicycle of \( D_1 \) and \( V(D_2) = \{ v_{21}, v_{22}, \ldots, v_{2m_2} \} \). Let \( \alpha_1 = (v_{21}, v_{22}) \in A(D_2) \) and \( \alpha_2 = (v_{21}, v_{22}) \in A(D_2) \), and \( D_1 \oplus_2 D_2 = D_1 \oplus_0 D_2 \).

Since \( D_1 \) is hamiltonian-connected, \( D_2 \) contains a \((v_{21}, v_{22})\)-hamiltonian dipath \( P \). Thus \((C - \{ \alpha_1 \}) \cup \bar{P} \) is a hamiltonian dicycle in \( D_1 \oplus_2 D_2 \).

**Theorem 6** (Thomassen [16]) If a semicomplete digraph \( D \) is 4-strong, then \( D \) is hamiltonian-connected.

By Theorem 5 and 6, we have the following corollary.

**Corollary 2** Let \( D_1 \) and \( D_2 \) be two 4-strong semicomplete digraphs, then \( D_1 \oplus_2 D_2 \) is hamiltonian.

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### References


