Reciprocal Complementary Wiener Numbers of Non-Caterpillars

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Abstract
The reciprocal complementary Wiener number of a connected graph \( G \) is defined as

\[
RCW(G) = \sum_{u,v \in V(G)} \frac{1}{d + 1 - d(u,v,G)}
\]

where \( V(G) \) is the vertex set, \( d(u,v,G) \) is the distance between vertices \( u \) and \( v \), and \( d \) is the diameter of \( G \). A tree is known as a caterpillar if the removal of all pendant vertices makes it as a path. Otherwise, it is called a non-caterpillar. Among all \( n \)-vertex non-caterpillars with given diameter \( d \), we obtain the unique tree with minimum reciprocal complementary Wiener number, where \( 4 \leq d \leq n - 3 \). We also determine the \( n \)-vertex non-caterpillars with the smallest, the second smallest and the third smallest reciprocal complementary Wiener numbers.

Keywords
Reciprocal Complementary Wiener Number, Wiener Number, Caterpillar

1. Introduction
The Wiener number was one of the oldest topological indices, which was introduced by Harry Wiener in 1947. About the recent reviews on matrices and topological indices related to Wiener number, refer to [1]-[4]. The RCW number is one of the hottest additions in the family of such descriptors. The notion of RCW number was first put forward by Ivanciuc and its applications were discussed in [5]-[8].

Let $G$ be a simple connected graph with vertex set $V(G)$. For two vertices $u, v \in V(G)$, let $d(u, v|G)$ denote the distance between $u$ and $v$ in $G$. Then, the RCW number of $G$ is defined by

$$RCW(G) = \frac{1}{d + 1 - d(u, v|G)} \sum_{[u,v] \subseteq V(G)}$$

where $d$ is the diameter and the summation goes over all unordered pairs of distinct vertices of $G$. Some properties of the RCW number have been obtained in [9] [10].

A tree is called a caterpillar if the removal of all pendant vertices makes it as a path. Otherwise, it is called a non-caterpillar.

For integers $n$ and $d$ satisfying $4 \leq d \leq n - 3$, let $N_{n,d}$ be the tree obtained from the path $P_{d+1}$ labelled as $v_0, v_1, \cdots, v_d$ by attaching the path $P_2$ and $n - d - 3$ pendant vertices to vertex $v_i$ for $2 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$ (see Figure 1). Let $N_{n,d} = N_{nd[d]}$.

In this paper, we show that among all $n$-vertex non-caterpillars with given diameter $d$, $N_{n,d}$ is the unique tree with minimum RCW number where $4 \leq d \leq n - 3$. Furthermore, we determine the non-caterpillars with the smallest, the second smallest and the third smallest RCW numbers.

2. RCW Numbers of Non-Caterpillars

All $n$-vertex trees with diameter 2, 3, $n - 2$ and $n - 1$ are caterpillars. Let $n$ and $d$ be integers with $n \geq 7$ and $4 \leq d \leq n - 3$. Let $\text{NC}(n,d)$ be the class of non-caterpillars with $n$ vertices and diameter $d$. Let $\bar{\text{NC}}(n,d)$ be the class of non-caterpillars obtained by attaching the stars $s_{n_1}, \cdots, s_{n_t}$ at their centers and $s_{i_1}$ pendant vertices to one center (fixed if it is bicentral) of the path $P_{d+1}$, where $t \geq 1$, $s \geq 0$ and $n_i \geq 2$ for $i = 1, 2, \cdots, t$ (see Figure 2). Recall that $N_{n,d} = N_{n,d[d]}$. Obviously, $N_{n,d} \in \bar{\text{NC}}(n,d) \subseteq \text{NC}(n,d)$ and

$$\text{NC}(n,n-3) = \{N_{n,d-3}\}.$$

Let $T$ be a tree. For $u \in V(T)$ and $A \subseteq V(T)$, let $\delta_T(u)$ be the degree of $u$ in $T$ and $d_T(u, A)$ be the sum of all distances from $u$ to the vertices in $A$. i.e., $d_T(u, A) = \sum_{v \in A} d_T(u, v)$. Here and in the following $d_T(u, v)$ denotes the distance between vertices $u$ and $v$ in $T$.

**Lemma 1** Let $T$ be a tree with minimum RCW number in $\text{NC}(n,d)$, where $4 \leq d \leq n - 3$. Then, $T \in \bar{\text{NC}}(n,d)$.

**Proof.** Suppose that $T \notin \text{NC}(n,d) \setminus \bar{\text{NC}}(n,d)$. Let $P(T) = v_0v_1 \cdots v_d$ be a diametral path of $T$. If $d$ is odd, we require that

$$\delta_T(v_{d/2}) \geq \delta_T(v_{d/2}).$$

Then at least one of $v_{d/2}, \cdots, v_{d-2}$ has degree at least three. There are two cases.

**Case 1.** One of $v_{d/2}, \cdots, v_{d-2}$ different from $v_{d/2}$ has degree at least three. Let $w_1, w_2, \cdots, w_k$ be all the neighbors outside $P(T)$ except those of $v_{d/2}$, where $w_i$ is a neighbor of $v_i \in V(P(T))$. Let $T_1$ be the subtree of $T - v_j$ containing $w_j$. $T^*$ be the tree formed from $T$ by deleting edges $w_jv_j$ and adding edges $w_jv_{d/2}$ for

![Figure 1. The tree $N_{n,d}$.](image)
all \(i = 1, 2, \cdots, k\). Obviously, \(T' \in \mathcal{NC}(n,d)\). Let \(A = V(P(T))\) and \(B = V(T) \setminus A\). It is easily seen that

\[
RCW(T) - RCW(T') = \sum_{u \in B, v \in A} \left[ \frac{1}{d + 1 - d_T(u,v)} - \frac{1}{d + 1 - d_{T'}(u,v)} \right] + \sum_{\{u,v\} \in B} \left[ \frac{1}{d + 1 - d_T(u,v)} - \frac{1}{d + 1 - d_{T'}(u,v)} \right]
\]

with equality if and only if \(\delta_T\left(\frac{d}{2}\right) = 2\). Since \(i' \neq \left\lfloor \frac{d}{2} \right\rfloor\), \(d_T(u,v_i) = d_{T'}(u,\frac{d}{2})\) for \(u \in V(T_i)\) and \(v_i \in A\) with \(1 \leq i \leq k\). We get

\[
\sum_{u \in A} \frac{1}{d + 1 - d_T(u,v)} = \sum_{v \in A} \frac{1}{d + 1 - d_T(u,v) - d_T(v,v)} = \sum_{v \in A} \frac{1}{d + 1 - d_T(u,v) - \left\lfloor \frac{d}{2} \right\rfloor - k} + \sum_{v \in A} \frac{1}{d + 1 - d_T(u,v) - \left\lceil \frac{d}{2} \right\rceil - k} = \sum_{v \in A} \frac{1}{d + 1 - d_T\left(\frac{d}{2}\right)} - d_T\left(\frac{d}{2}\right) = \sum_{v \in A} \frac{1}{d + 1 - d_T(u,v)}.
\]

Then

\[
\sum_{u \in A} \sum_{i \in \mathcal{NC}(T,d)} \left[ \frac{1}{d + 1 - d_T(u,v)} - \frac{1}{d + 1 - d_{T'}(u,v)} \right] = \sum_{u \in A} \left[ \sum_{i \in \mathcal{NC}(T,d)} \frac{1}{d + 1 - d_T(u,v)} - \sum_{v \in A} \frac{1}{d + 1 - d_{T'}(u,v)} \right] \geq 0
\]

with equality if and only if \(i' = \left\lfloor \frac{d}{2} \right\rfloor\) (which is only possible for odd number \(d\)). But \(\delta_T\left(\frac{d}{2}\right) \geq \delta_{T'}\left(\frac{d}{2}\right)\), and
thus if $\delta_T\left(\frac{d_j}{d}\right) = 2$ then $\delta_T\left(\frac{v_i}{d}\right) = 2$. So $i' = \left\lfloor \frac{d}{2} \right\rfloor$ for $i = 1, 2, \cdots, k$. Thus

$$RCW(T) - RCW(T^*)$$

$$> \sum_{u \in V(T) \setminus V(P_T)} \left[ \frac{1}{d + 1 - d_x(u,v)} - \frac{1}{d + 1 - d_{x'}(u,v)} \right] \geq 0,$$

since $d_x(u,v) \geq d_{x'}(u,v)$ for $u \in V(T), v \in V(T)$ \hspace{1cm} ($1 \leq i < j \leq k$). It follows that $RCW(T) > RCW(T^*)$. This is a contradiction.

**Case 2.** Any vertex $v_i$ with $i = 2, 3, \cdots, d - 2$ and $i' \neq \left\lfloor \frac{d}{2} \right\rfloor$ has degree two. Obviously, $\delta_T\left(\frac{v_i}{d}\right) \geq 3$. Let $xyz\cdots v_i$ be the (unique) path from $x$ to $v_i$ in $T$ such that $d_x\left(\frac{v_i}{d}\right) = \max_{u \in V(T) \setminus V(P_T)} d_x\left(\frac{u}{d}\right)$. Since $T \notin NC(n,d)$, we have $d_x\left(\frac{v_i}{d}\right) \geq 3$. Let $x, y, z, v_i$ be the neighbors of $v_i$ in $T$, where $x_i = x$ and $r \geq 1$.

Let $T^*$ be the tree obtained from $T$ by deleting edges $xy$ and adding edges $xz$ for all $i = 1, 2, \cdots, r$. Then $T^* \notin NC(n,d)$. Let $N_y = \{x_1, \cdots, x_r\}$, $C = V(T) \setminus N_y$. Since $d_x(u,v) = d_{x'}(u,v) + 1$ for $u \in N_y, v \in C \setminus \{v, z\}$, we get

$$RCW(T) - RCW(T^*)$$

$$= \sum_{u \in N_y, v \in C} \left[ \frac{1}{d + 1 - d_x(u,v)} - \frac{1}{d + 1 - d_{x'}(u,v)} \right]$$

$$= \frac{r - 1}{d + 1 - 1} + \frac{r - 1}{d + 1 - 2} + \sum_{u \in N_y, v \in C \setminus \{v, z\}} \frac{1}{d + 1 - d_x(u,v)} - \frac{r - 1}{d + 1 - 1} - \frac{r - 1}{d + 1 - 2} - \sum_{u \in N_y, v \in C \setminus \{v, z\}} \frac{1}{d + 1 - d_{x'}(u,v)}$$

$$> 0.$$  

This is a contradiction.

By combining Cases 1 and 2, we find that $T \in NC(n,d) \setminus NC(n,d)$ is impossible. The result follows.

**Lemma 2** Let $T \in NC(n,d)$ with $4 \leq d \leq n - 3$. Then

$$RCW(T) \geq RCW(N_{n,d}),$$

with equality if and only if $T = N_{n,d}$.

**Proof.** Let $T$ be a tree with the minimum RCW number in $NC(n,d)$. Let $P(T) = v_0v_1\cdots v_d$ be a diametral path of $T$.

Suppose that there is a vertex $u \in V(T) \setminus V(P_T)$ with $\delta_T(u) \geq 3$. Let $u_1, u_2, \cdots, u_s$ be the neighbors of $u$ different from $v_i$ in $T$, where $s \geq 2$. Clearly, $u_i$ are pendant vertices for $i = 1, 2, \cdots, s$. Let $T'$ be the tree obtained from $T$ by deleting edges $uu_1$ and adding edges $v_iu_i$ for $i = 2, \cdots, s$. Obviously, $T' \in NC(n,d)$.

Let $N_u = \{u_2, \cdots, u_s\}$, $D = V(T) \setminus N_u$, and $K = \{u_i, v_i \}$. Since $d_x(u,v) = d_{x'}(u,v) + 1$ for $u \in N_u, v \in D \setminus K$, we get
and then $RCW(T) > RCW(T')$, this is a contradiction. Thus any vertex of $T$ outside $P(T)$ has degree at most two.

Suppose that there are at least two vertices of $T$ outside $P(T)$ with degree two. Let $y \in V(T) \setminus V(P(T))$ with $\delta_T(y) = 2$ and let $x$ be the neighbor of $y$ which is different from $v_{d,T}$ in $T$. Let $T''$ be the tree formed from $T$ by deleting edge $yx$ and adding edge $v_{d,T}$. Obviously, $T'' \notin NC(n,d)$. Let $F = \{x,y,v_{d,T}\}$. Since $RCW(T) > RCW(T'')$ and $d_f(x,v) = d_f(x,y) + 1$ for $v \in V(T) \setminus F$, we get

$$ RCW(T) - RCW(T'') $$

$$ = \frac{1}{d+1-1} + \frac{1}{d+1-2} + \frac{1}{d+1-3} - \sum_{v \in F} \frac{1}{d+1-d_f(x,v)} $$

$$ > 0, $$

This is a contradiction. Thus there is exactly one vertex outside $P(T)$ with degree two and all other vertices of $T$ outside $P(T)$ are pendant vertices. Then, $T = N_{n,d}$.

By a direct calculation, we get

$$ RCW \left( N_{n,d, \left\lfloor \frac{d}{2} \right\rfloor} \right) $$

$$ = d + \frac{n-d+1}{d-2} + \frac{(n-d-2)(n-d-3)-2}{2(d-1)} $$

$$ + (n-d-1) \left\{ \sum_{k=1}^{d} \frac{2}{d-k} + \frac{1}{d} \right\} \text{ where } d \text{ is even; } $$

$$ RCW \left( N_{n,d, \left\lceil \frac{d}{2} \right\rceil} \right) $$

$$ = d + \frac{n-d-3}{d-2} + \frac{2}{d-3} + \frac{(n-d-2)(n-d-3)+4(n-d)-2}{2(d-1)} $$

$$ + (n-d-1) \left\{ \sum_{k=1}^{d-1} \frac{2}{d-k} + \frac{1}{d} \right\} \text{ where } d \text{ is odd. } $$

Combining Lemmas 1 and 2, we get

**Theorem 1** Let $T \in NC(n,d)$, and $4 \leq d \leq n-3$. Then

$$ RCW(T) \geq RCW\left( N_{n,d} \right) $$

with equality if and only if $T = N_{n,d}$. 
Lemma 3 For $4 \leq d \leq n - 3$, there is $RCW\left( \frac{N_{d}}{[\frac{d}{2}]} \right) > RCW\left( \frac{N_{d+1}}{[\frac{d+1}{2}]} \right)$.

**Proof.** If $d$ is even, then

$$RCW\left( \frac{N_{d}}{[\frac{d}{2}]} \right) - RCW\left( \frac{N_{d+1}}{[\frac{d+1}{2}]} \right) = \left( -1 + \sum_{k=1}^{\frac{d}{2}} \frac{2}{d-k} \right) + \frac{5(n-d-1)}{d} - \frac{n-d-2}{d+1} - \frac{4(n-d-2)+1}{d} + \frac{n-d-1}{d-2} - \frac{n-d-3}{d-1} + \frac{(n-d-2)(n-d-3)}{2(d-1)} - \frac{(n-d-3)(n-d-4)}{2d} > \left( -1 + \sum_{k=1}^{\frac{d}{2}} \frac{2}{d-k} \right) > 0.$$  

If $d$ is odd, then

$$RCW\left( \frac{N_{d+1}}{[\frac{d+1}{2}]} \right) - RCW\left( \frac{N_{d}}{[\frac{d}{2}]} \right) = \left( -1 + \sum_{k=1}^{\frac{d+1}{2}} \frac{2}{d-k} \right) + \frac{n-d-3}{d-2} - \frac{n-d-4}{d} + \frac{(n-d-2)(n-d-3)}{2(d-1)} - \frac{(n-d-3)(n-d-4)}{2d} > \left( -1 + \sum_{k=1}^{\frac{d+1}{2}} \frac{2}{d-k} \right) > 0.$$

The result follows.

**Theorem 2** For $n \geq 9$, there is

$$RCW\left( \frac{N_{n-3}}{[\frac{n-3}{2}]} \right) < RCW\left( \frac{N_{n-5}}{[\frac{n-5}{2}]} \right) < RCW\left( \frac{N_{n-4}}{[\frac{n-4}{2}]} \right).$$

And $RCW(T) > RCW\left( \frac{N_{n,n-4}}{[\frac{n-4}{2}]} \right)$ for any $n$-vertex non-caterpillar $T$ different from $N_{n,n-4}\left[\frac{n-3}{2}\right]$, $N_{n,n-4}\left[\frac{n-5}{2}\right]$, $N_{n,n-4}\left[\frac{n-4}{2}\right]$.

**Proof.** Let $T \in NC(n, d)$, where $4 \leq d \leq n - 3$. If $d = n - 3$, then $T$ is a non-caterpillar $N_{n,n-3}$, where $1 \leq i \leq \left[\frac{n-3}{2}\right]$. It follows that

$$RCW\left( N_{n,n-3} \right) = n - 3 + \frac{2}{n-3} + \frac{1}{n-4} + \frac{\sum_{k=3}^{i} \frac{2}{n-k}}{i} + \frac{\sum_{k=3}^{i-1} \frac{2}{n-k}}{i-1} + \frac{1}{i-4} + \frac{1}{i-1},$$

and hence $RCW\left( N_{n,n-3} \right)$ is monotonically decreasing for $1 \leq i \leq \left[\frac{n-3}{2}\right]$. This implies
Now suppose that $d \leq n - 4$. By Theorem 1 and Lemma 3, there is

$$RCW(T) \geq RCW\left(N_{n,n-4,\frac{n-4}{2}}\right)$$

where equality holds if and only if $T \cong N_{n,n-4,\frac{n-4}{2}}$. We need only to show

$$RCW\left(N_{n,n-3,\frac{n-5}{2}}\right) < RCW\left(N_{n,n-4,\frac{n-4}{2}}\right).$$

**Case 1.** $n$ is odd. Let $i = \left\lfloor \frac{n-5}{2} \right\rfloor = \frac{n-5}{2}$ and $n \geq 9$. Then there is

$$RCW\left(N_{n,n-4,\frac{n-5}{2}}\right) - RCW\left(N_{n,n-3,\frac{n-5}{2}}\right) = \left(-1 + \sum_{k=0}^{\frac{n-5}{2}} \frac{2}{n-k}\right) + \frac{2}{n-5} + \frac{1}{n-6} > 0.$$  

**Case 2.** $n$ is even. Let $i = \left\lfloor \frac{n-5}{2} \right\rfloor = \frac{n-6}{2}$. Then there is

$$RCW\left(N_{n,n-4,\frac{n-6}{2}}\right) - RCW\left(N_{n,n-3,\frac{n-6}{2}}\right) = \left(-1 + \sum_{k=0}^{\frac{n-6}{2}} \frac{2}{n-k}\right) + \frac{2}{n-5} + \frac{1}{n-6} + \frac{4}{n-4} - \frac{2}{n-3} - \frac{2}{n-8} \geq 0.$$

Thus, the proof is finished.

**References**


