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Existence and Stability Analysis of Fractional Order BAM Neural Networks with a Time Delay

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Abstract

Based on the theory of fractional calculus, the contraction mapping principle, Krasnoselskii fixed point theorem and the inequality technique, a class of Caputo fractional-order BAM neural networks with delays in the leakage terms is investigated in this paper. Some new sufficient conditions are established to guarantee the existence and uniqueness of the nontrivial solution. Moreover, uniform stability of such networks is proposed in fixed time intervals. Finally, an illustrative example is also given to demonstrate the effectiveness of the obtained results.

Keywords

BAM Neural Networks, Caputo Fractional-Order, Existence, Fixed Point Theorems, Uniform Stability

1. Introduction

Fractional order calculus was firstly introduced 300 years ago, but it did not attract much attention for a long time since it lack of application background and the complexity. In recent decades, the study of fractional-order calculus has re-attracted tremendous attention of much researchers because it can be applied to physics, applied mathematics and engineering [1]-[6]. We know that the fractional-order derivative is nonlocal and has weakly singular kernels. It provides an excellent instrument for the description of memory and hereditary properties of dynamical processes where such effects are neglected or difficult to describe to the integer order models.

We know that the next state of a system not only depends upon its current state but also upon its history information. Since a model described by fractional-order equations possesses memory, it is precise to describe the

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states of neurons. Moreover, the superiority of the Caputo's fractional derivative is that the initial conditions for fractional differential equations with Caputo derivatives take on the similar form as those for integer-order differentiation. Therefore, it is necessary and interesting to study fractional-order neural networks both in theory and in applications.

Recently, some important and interesting results on fractional-order neural networks have been obtained and various issues have been investigated [7]-[14] by many authors. In [11], the authors proposed a fractional-order Hopfield neural network and investigated its stability by using energy function. In [12], the authors investigated stability, multistability, bifurcations, and chaos for fractional-order Hopfield neural networks. In [13], Chen *et al*. obtained a sufficient condition for uniform stability of a class of fractional-order delayed neural networks. In [14], we investigated the finite-time stability for Caputo fractional-order BAM neural networks with distributed delay and established a delay-dependent stability criterion by using the theory of fractional calculus and generalized Gronwall-Bellman inequality approach. In [15], Song and Cao considered the existence, uniqueness of the nontrivial solution and also uniform stability for a class of neural networks with a fractional-order derivative, by using the contraction mapping principle, Krasnoselskii fixed point theorem and the inequality technique.

The integer-order bidirectional associative memory (BAM) neural networks models, first proposed and studied by Kosko [16]. This neural network has been widely studied due to its promising potential for applications in pattern recognition and automatic control. In recent years, integer-order BAM neural networks have been extensively studied [17]-[21]. Recently, some authors considered the uniform stability of delayed neural networks; for example, see [22]-[24] and references therein. However, to the best of our knowledge, there are few results on the uniform stability analysis of fractional-order BAM neural networks.

Motivated by the above-mentioned works, this paper considers the uniform stability of a class of fractional-order BAM neural networks with delays in the leakage terms described by

$$\begin{cases} {}^{C}D^{\alpha}x_i(t) = -c_i x_i(t-\sigma) + \sum_{j=1}^{m} a_{ij} f_j(y_j(t)) + I_i, & t \ge 0, i = 1,\cdots,n \\ {}^{C}D^{\beta}y_j(t) = -d_j y_j(t-\sigma) + \sum_{i=1}^{n} b_{ji} g_i(x_i(t)) + J_j, & t \ge 0,\ j = 1,\cdots,m, \end{cases} \tag{1}$$

where $0<\alpha,\beta<1$, ${}^{C}D^{\alpha}$ and ${}^{C}D^{\beta}$ denote the Caputo fractional derivative of order α,β, respectively; $x_i(t)$ $(i=1,\cdots,n)$ and $y_j(t)$ $(j=1,\cdots,m)$ are the activations of the *i*th neuron in the neural field F_x and the *j*th neuron in in the neural field F_y at time *t*, respectively; $f_j(y_j(t))$ denotes the activation function of the *j*th neuron from the neural field F_y at time *t* and $g_i(x_i(t))$ denotes the activation function of the *i*th neuron from the neural field F_x at time *t*; I_i and J_j are constants, which denote the external inputs on the *i*th neuron from F_x and the *j*th neuron from F_y, respectively; the positive constants c_i and d_j denote the rates with which the *i*th neuron from the neural field F_x and the *j*th neuron from the neural field F_y will reset their potential to the resting state in isolation when disconnected from the networks and external inputs, respectively; the constants a_{ij} and b_{ji} represent the connection strengths; the nonnegative constant σ denotes the leakage delay.

This paper is organized as follows. In Section 2, some definitions of fractional-order calculus and some necessary lemmas are given. In Section 3, some new sufficient conditions to ensure the existence, uniqueness of the nontrivial solution and also uniform stability of the fractional-order BAM neural networks 1 is obtained. Finally, an example is presented to manifest the effectiveness of our theoretical results in Section 4.

2. Preliminaries

For the convenience of the reader, we first briefly recall some definitions of fractional calculus, for more details, see [1] [2] [5], for example.

Definition 1. *The Riemann-Liouville fractional integral of order* $\alpha>0$ *of a function* $u:(0,\infty)\to R$ *is given by*

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\,\mathrm{d}s$$

provided the right side is pointwise defined on $(0,\infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2. *The Caputo fractional derivative of order* $\gamma > 0$ *of a function* $u:(0,\infty)\to R$ *can be written as*

$$ {}^{C}D^{\gamma}u(t)=\frac{1}{\Gamma(n-\gamma)}\int_0^t \frac{u^{(n)}(s)}{(t-s)^{\gamma+1-n}}\mathrm{d}s,\quad n-1<\gamma<n. $$

Let $X=\left\{x \mid x=(x_1,x_2,\cdots,x_n)^{\mathrm{T}}, x_i\in C([0,T],\mathbb{R})\right\}$, $Y=\left\{y \mid y=(y_1,y_2,\cdots,y_m)^{\mathrm{T}}, y_j\in C([0,T],\mathbb{R})\right\}$. For *p*, *q* > 1, we know that *X* is a Banach space with the norm $\|x\|_p=\sup_{0\le t\le T}\left(\sum_{i=1}^{n}|x_i(t)|^p\right)^{1/p}$, and *Y* is a Banach space with the norm $\|y\|_q=\sup_{0\le t\le T}\left(\sum_{j=1}^{m}|y_j(t)|^q\right)^{1/q}$. It is easy to see that $X\times Y$ is a Banach space with the norm $\|(x,y)\|=\|x\|_p+\|y\|_q$.

The initial conditions associated with system (1) are of the form

$$ x_i(\theta)=\varphi_i(\theta),\quad y_j(\theta)=\psi_j(\theta),\quad \theta\in[-\sigma,0],\quad i=1,\cdots,n,\ j=1,\cdots,m, \tag{2} $$

where $\varphi_i,\psi_j\in C([-\sigma,0],\mathbb{R})$.

To prove our results, the following lemmas are needed.

Lemma 1. ([25]). *Let* $\alpha>0$, *then the fractional differential equation*

$$ {}^{C}D^{\alpha}x(t)=h(t), $$

has solutions

$$ x(t)=I^{\alpha}h(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}, $$

where $c_i\in\mathbb{R}$, $n=[\alpha]+1$.

Lemma 2. ([26]). *Let D be a closed convex and nonempty subset of a Banach space X. Let* ϕ_1, ϕ_2 *be the operators such that*

1) $\phi_1x+\phi_2y\in D$ wherever $x,y\in D$;

2) ϕ_1 is compact and continuous;

3) ϕ_2 is a contraction mapping.

Then, there exists $x\in D$ such that $\phi_1x+\phi_2x=x$.

In order to obtain main result, we make the following assumptions.

($H1$) The neurons activation functions f_i and g_j are Lipschitz continuous, that is, there exist positive constants L_i,l_j ($i=1,\cdots,n$, $j=1,\cdots,m$) such that

$$ |f_i(u)-f_i(v)|\le L_i|u-v|,\quad |g_j(u)-g_j(v)|\le l_j|u-v|,\quad \forall u,v\in\mathbb{R}. $$

($H2$) For $i=1,\cdots,n$, $j=1,\cdots,m$, there exist $M_1,M_2>0$ such that for $u,v\in\mathbb{R}$, $|f_i(u)|\le M_1$ and $|g_j(v)|\le M_2$.

3. Main Results

For convenience, let

$$ c_0=\max_{1\le i\le n}c_i,\quad d_0=\max_{1\le j\le m}d_j,\quad a_{i0}=\sum_{j=1}^{m}|a_{ij}|,\quad b_{j0}=\sum_{i=1}^{n}|b_{ji}|, \tag{3} $$

$$ I_0=\max_{1\le i\le n}|I_i|,\quad J_0=\max_{1\le j\le m}|J_j|,\quad L_0=\max_{1\le i\le n}L_i,\quad l_0=\max_{1\le j\le m}l_j, \tag{4} $$

$$ \eta_i=\left(\sum_{j=1}^{m}|a_{ij}|^{q/(q-1)}\right)^{(q-1)/q},\quad \xi_j=\left(\sum_{i=1}^{n}|b_{ji}|^{p/(p-1)}\right)^{(p-1)/p}. \tag{5} $$

Theorem 3. *Under assumption* ($H1$), *the system* (1) *has a unique solution on* $[0,T]$, *if there exist two real numbers* $p,q>1$ *such that*

$$B=\max\left\{\frac{c_0 n^{1/p}}{\Gamma(\alpha+1)}(T-\sigma)^{\alpha}+\frac{l_0 T^{\beta}}{\Gamma(\beta+1)}\left(\sum_{j=1}^{m}\xi_j^q\right)^{1/q},\frac{d_0 m^{1/q}}{\Gamma(\beta+1)}(T-\sigma)^{\beta}+\frac{L_0 T^{\alpha}}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{n}\eta_i^p\right)^{1/p}\right\}<1. \tag{6}$$

Proof. Define $(F,G):X\times Y\to X\times Y$ as

$$(F,G)(x,y)(t)=\left(F_1(x,y)(t),\cdots,F_n(x,y)(t),G_1(x,y)(t),\cdots,G_m(x,y)(t)\right)^{\mathrm{T}},$$

where

$$F_i(x,y)(t)=\phi_i(0)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[-c_i x_i(s-\sigma)+\sum_{j=1}^{m}a_{ij}f_j\left(y_j(s)\right)+I_i\right]\mathrm{d}s,$$

$$G_j(x,y)(t)=\psi_j(0)+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\left[-d_j y_j(s-\sigma)+\sum_{i=1}^{n}b_{ji}g_i\left(x_i(s)\right)+J_j\right]\mathrm{d}s.$$

By Lemma 1, we know that the fixed point of (*F*, *G*) is a solution of system (1) with initial conditions (2). In the following, we will using the contraction mapping principle to prove that the operator (*F*, *G*) has a unique fixed point.

Firstly, we prove $(F,G)B_\delta\subset B_\delta$, where $B_\delta=\left\{(x,y)\in X\times Y:\|(x,y)\|\le\delta\right\}$. Set

$$\delta\ge\frac{(1+A)\|(\phi,\psi)\|+C}{1-B}, \tag{7}$$

where

$$A=\max\left\{\frac{c_0 n^{1/p}\sigma^{\alpha}}{\Gamma(\alpha+1)},\frac{d_0 m^{1/q}\sigma^{\beta}}{\Gamma(\beta+1)}\right\}, \tag{8}$$

and

$$C=\frac{I_0 n^{1/p}T^{\alpha}}{\Gamma(\alpha+1)}+\frac{J_0 m^{1/q}T^{\beta}}{\Gamma(\beta+1)}+\frac{f_0 T^{\alpha}}{\Gamma(\alpha+1)}\left(\sum_{i=1}a_{i0}^p\right)^{1/p}+\frac{g_0 T^{\beta}}{\Gamma(\beta+1)}\left(\sum_{j=1}b_{0j}^q\right)^{1/q}. \tag{9}$$

By Minkowski inequality, we have

$$\begin{aligned}\|F(x,y)\|_p&=\sup_{0\le t\le T}\left\{\sum_{i=1}^{n}\left|\phi_i(0)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[-c_i x_i(s-\sigma)+\sum_{j=1}^{m}a_{ij}f_j\left(y_j(s)\right)+I_i\right]\mathrm{d}s\right|^p\right\}^{1/p}\\&\le\left(\sum_{i=1}^{n}\left|\phi_i(0)\right|^p\right)^{1/p}+\sup_{0\le t\le T}\left[\sum_{i=1}^{n}\left(\int_0^t\frac{c_i(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|x_i(s-\sigma)\right|\mathrm{d}s\right)^p\right]^{1/p}\\&\quad+\sup_{0\le t\le T}\left[\sum_{i=1}^{n}\left(\int_0^t\sum_{j=1}^{m}\frac{|a_{ij}|(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f_j(0)\right|\mathrm{d}s\right)^p\right]^{1/p}\\&\quad+\sup_{0\le t\le T}\left[\sum_{i=1}^{n}\left(\int_0^t\sum_{j=1}^{m}\frac{|a_{ij}|(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f_j\left(y_j(s)\right)-f_j(0)\right|\mathrm{d}s\right)^p\right]^{1/p}\\&\quad+\sup_{0\le t\le T}\left[\sum_{i=1}^{n}\left(\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|I_i|\mathrm{d}s\right)^p\right]^{1/p}.\end{aligned} \tag{10}$$

By direct computation, we obtain by (3) that

$$
\begin{aligned}
&\sup_{0\le t\le T}\left[\sum_{i=1}^{n}\left(\int_0^t \frac{c_i(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|x_i(s-\sigma)\right|\mathrm{d}s\right)^p\right]^{1/p}\\
&\le \frac{c_0}{\Gamma(\alpha)}\sup_{0\le t\le T}\left[\sum_{i=1}^{n}\left(\int_0^{\sigma}(t-s)^{\alpha-1}\left|\phi_i(s-\sigma)\right|\mathrm{d}s\right)^p\right]^{1/p}+\frac{c_0}{\Gamma(\alpha)}\sup_{0\le t\le T}\left[\sum_{i=1}^{n}\left(\int_{\sigma}^{t}(t-s)^{\alpha-1}\left|x_i(s-\sigma)\right|\mathrm{d}s\right)^p\right]^{1/p}\\
&=\frac{c_0}{\Gamma(\alpha)}\sup_{0\le t\le T}\left[\sum_{i=1}^{n}\left(\int_{-\sigma}^{0}(t-\sigma-\theta)^{\alpha-1}\left|\phi_i(\theta)\right|\mathrm{d}\theta\right)^p\right]^{1/p}+\frac{c_0}{\Gamma(\alpha)}\sup_{0\le t\le T}\left[\sum_{i=1}^{n}\left(\int_{0}^{t-\sigma}(t-\sigma-\theta)^{\alpha-1}\left|x_i(\theta)\right|\mathrm{d}\theta\right)^p\right]^{1/p} \qquad (11)\\
&\le \frac{c_0}{\Gamma(\alpha)}\sup_{-\sigma\le\theta\le 0}\left(\sum_{i=1}^{n}\left|\phi_i(\theta)\right|^p\right)^{1/p}\cdot n^{1/p}\sup_{\sigma\le t\le T}\int_{t-\sigma}^{t}\gamma^{\alpha-1}\mathrm{d}\gamma+\frac{c_0}{\Gamma(\alpha)}\sup_{0\le t\le T}\left(\sum_{i=1}^{n}\left|x_i(t)\right|^p\right)^{1/p}\cdot n^{1/p}\sup_{0\le t\le T}\int_0^{t-\delta}\gamma^{\alpha-1}\mathrm{d}\gamma\\
&=\frac{c_0 n^{1/p}}{\Gamma(\alpha+1)}\left(\sigma^{\alpha}\|\phi\|_p+(T-\sigma)^{\alpha}\|x\|_p\right).
\end{aligned}
$$

Similar to (11) and the proof of Theorem 1 in [15], we have by ($H1$), (4) and (5) that

$$
\sup_{0\le t\le T}\left[\sum_{i=1}^{n}\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|I_i\right|\mathrm{d}s\right)^p\right]^{1/p}\le\frac{I_0 n^{1/p}T^{\alpha}}{\Gamma(\alpha+1)}, \qquad (12)
$$

$$
\sup_{0\le t\le T}\left[\sum_{i=1}^{n}\left(\int_0^t\sum_{j=1}^{m}\frac{\left|a_{ij}\right|(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f_j(0)\right|\mathrm{d}s\right)^p\right]^{1/p}\le\frac{f_0T^{\alpha}}{\Gamma(\alpha+1)}\left(\sum_{i=1}a_{i0}^p\right)^{1/p}, \qquad (13)
$$

and

$$
\begin{aligned}
&\sup_{0\le t\le T}\left[\sum_{i=1}^{n}\left(\int_0^t\sum_{j=1}^{m}\frac{\left|a_{ij}\right|(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f_j\left(y_j(s)\right)-f_j(0)\right|\mathrm{d}s\right)^p\right]^{1/p}\le\sup_{0\le t\le T}\left[\sum_{i=1}^{n}\left(\int_0^t\sum_{j=1}^{m}\frac{\left|a_{ij}\right|(t-s)^{\alpha-1}}{\Gamma(\alpha)}L_j\left|y_j(s)\right|\mathrm{d}s\right)^p\right]^{1/p}\\
&\le\sup_{0\le t\le T}\left\{\sum_{i=1}^{n}\left[\int_0^t\left(\sum_{j=1}^{m}\left(\frac{\left|a_{ij}\right|L_j(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right)^{q/(q-1)}\right)^{(q-1)/q}\cdot\left(\sum_{j=1}^{m}\left|y_j(s)\right|^q\right)^{1/q}\mathrm{d}s\right]^p\right\}^{1/p}\le\frac{L_0T^{\alpha}}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{n}\eta_i^p\right)^{1/p}\|y\|_q.
\end{aligned} \qquad (14)
$$

Substitute (11)-(14) into (10), we get

$$
\begin{aligned}
\|F(x,y)\|_p\le\|\phi\|_p&+\frac{I_0n^{1/p}T^{\alpha}}{\Gamma(\alpha+1)}+\frac{f_0T^{\alpha}}{\Gamma(\alpha+1)}\left(\sum_{i=1}a_{i0}^p\right)^{1/p}\\
&+\frac{c_0n^{1/p}}{\Gamma(\alpha+1)}\left(\sigma^{\alpha}\|\phi\|_p+(T-\sigma)^{\alpha}\|x\|_p\right)+\frac{L_0T^{\alpha}}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{n}\eta_i^p\right)^{1/p}\|y\|_q.
\end{aligned} \qquad (15)
$$

Similarly, we obtain

$$
\begin{aligned}
\|G(x,y)\|_q\le\|\psi\|_q&+\frac{J_0m^{1/q}T^{\beta}}{\Gamma(\beta+1)}+\frac{g_0T^{\beta}}{\Gamma(\beta+1)}\left(\sum_{j=1}b_{0j}^q\right)^{1/q}\\
&+\frac{d_0m^{1/q}}{\Gamma(\beta+1)}\left(\sigma^{\beta}\|\psi\|_q+(T-\sigma)^{\beta}\|y\|_p\right)+\frac{l_0T^{\beta}}{\Gamma(\beta+1)}\left(\sum_{j=1}^{m}\xi_j^q\right)^{1/q}\|x\|_p.
\end{aligned} \qquad (16)
$$

Thus, from (15), (16) and (7), we have

$$\begin{aligned}\|(F,G)(x,y)\| &= \|F(x,y)\|_p + \|G(x,y)\|_q \le (1+A)\|(\phi,\psi)\| + \frac{I_0 n^{1/p} T^\alpha}{\Gamma(\alpha+1)} + \frac{J_0 m^{1/q} T^\beta}{\Gamma(\beta+1)} \\ &+ \frac{f_0 T^\alpha}{\Gamma(\alpha+1)}\left(\sum_{i=1} a_{i0}^p\right)^{1/p} + \frac{g_0 T^\beta}{\Gamma(\beta+1)}\left(\sum_{j=1} b_{0j}^q\right)^{1/q} + B\|(x,y)\| \\ &\le (1+A)\|(\phi,\psi)\| + C + B\delta \le \delta.\end{aligned}$$

Secondly, we prove that $(F,G): X\times Y \to X\times Y$ is a contraction mapping. Let $(x,y),(u,v)\in X\times Y$, similar to the above process, we get

$$\begin{aligned}&\|(F,G)(x,y)-(F,G)(u,v)\| = \|F(x,y)-F(u,v)\|_p + \|G(x,y)-G(u,v)\|_q \\ &= \sup_{0\le t\le T}\left\{\sum_{i=1}^n\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[-c_i\left[x_i(s-\sigma)-u_i(s-\sigma)\right]+\sum_{j=1}^m a_{ij}\left(f_j\left(y_j(s)\right)-f_j\left(v_j(s)\right)\right)\right]\mathrm{d}s\right|^p\right\}^{1/p} \\ &+ \sup_{0\le t\le T}\left\{\sum_{j=1}^m\left|\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\left[-d_j\left[y_j(s-\sigma)-v_j(s-\sigma)\right]+\sum_{i=1}^n b_{ji}\left(g_i\left(x_i(s)\right)-g_i\left(u_i(s)\right)\right)\right]\mathrm{d}s\right|^q\right\}^{1/q} \\ &\le \sup_{0\le t\le T}\left[\sum_{i=1}^n\left(\int_0^t \frac{(t-s)^{\alpha-1} c_i\left|x_i(s-\sigma)-u_i(s-\sigma)\right|}{\Gamma(\alpha)}\mathrm{d}s\right)^p\right]^{1/p} \\ &+ \sup_{0\le t\le T}\left[\sum_{i=1}^n\left(\int_0^t\sum_{j=1}^m \frac{(t-s)^{\alpha-1}|a_{ij}|L_j}{\Gamma(\alpha)}\left|y_j(s)-v_j(s)\right|\mathrm{d}s\right)^p\right]^{1/p} \\ &+ \sup_{0\le t\le T}\left[\sum_{j=1}^m\left(\int_0^t \frac{(t-s)^{\beta-1} d_j\left|y_j(s-\sigma)-v_j(s-\sigma)\right|}{\Gamma(\beta)}\mathrm{d}s\right)^q\right]^{1/q} \\ &+ \sup_{0\le t\le T}\left[\sum_{j=1}^m\left(\int_0^t\sum_{i=1}^n \frac{(t-s)^{\beta-1}|b_{ji}|l_i}{\Gamma(\beta)}\left|x_i(s)-u_i(s)\right|\mathrm{d}s\right)^q\right]^{1/q} \\ &\le \max\left\{\frac{c_0 n^{1/p}}{\Gamma(\alpha+1)}(T-\sigma)^\alpha + \frac{l_0 T^\beta}{\Gamma(\beta+1)}\left(\sum_{j=1}^m \xi_j^q\right)^{1/q}, \frac{d_0 m^{1/q}}{\Gamma(\beta+1)}(T-\sigma)^\beta + \frac{L_0 T^\alpha}{\Gamma(\alpha+1)}\left(\sum_{i=1}^n \eta_i^p\right)^{1/p}\right\}\|(x,y)-(u,v)\| \\ &= B\|(x,y)-(u,v)\|,\end{aligned}$$

By (6), we conclude that (F,G) is a contraction mapping. It follows from the contraction mapping principle that system (1) has a unique solution. The proof is completed.

Theorem 4. *Assume that* (*H*2) *holds. If there exist real numbers* $p,q>1$ *such that*

$$\max\left\{\frac{c_0 n^{1/p}}{\Gamma(\alpha+1)}(T-\sigma)^\alpha, \frac{d_0 m^{1/q}}{\Gamma(\beta+1)}(T-\sigma)^\beta\right\} < 1, \tag{17}$$

then the system (1) has at least one solution on $[0,T]$.

Proof. Let

$$\delta \ge \frac{(1+A)\|(\phi,\psi)\| + \frac{I_0 n^{1/p} T^\alpha}{\Gamma(\alpha+1)} + \frac{J_0 m^{1/q} T^\beta}{\Gamma(\beta+1)} + \frac{M_2 T^\beta}{\Gamma(\beta+1)}\left(\sum_{j=1}^m \xi_j^q\right)^{1/q} + \frac{M_1 T^\alpha}{\Gamma(\alpha+1)}\left(\sum_{i=1}^n \eta_i^p\right)^{1/p}}{1-\max\left\{\frac{c_0 n^{1/p}}{\Gamma(\alpha+1)}(T-\sigma)^\alpha, \frac{d_0 m^{1/q}}{\Gamma(\beta+1)}(T-\sigma)^\beta\right\}}. \tag{18}$$

Define two operators $(L,\bar{L})$ and $(N,\bar{N})$ on B_δ as follows

$$(L,\bar{L})(x,y)(t)=\left(L_1(x,y)(t),\cdots,L_n(x,y)(t),\bar{L}_1(x,y)(t),\cdots,\bar{L}_m(x,y)(t)\right)^{\mathrm{T}},$$

$$(N,\bar{N})(x,y)(t)=\left(N_1(x,y)(t),\cdots,N_n(x,y)(t),\bar{N}_1(x,y)(t),\cdots,\bar{N}_m(x,y)(t)\right)^{\mathrm{T}},$$

where

$$L_i(x,y)(t)=\phi_i(0)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[-c_ix_i(s-\sigma)+I_i\right]\mathrm{d}s,$$

$$N_i(x,y)(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sum_{j=1}^m a_{ij}f_j\left(y_j(s)\right)\mathrm{d}s,\quad i=1,\cdots,n,$$

$$\bar{L}_j(x,y)(t)=\psi_j(0)+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\left[-d_jy_j(s-\sigma)+J_j\right]\mathrm{d}s,$$

$$\bar{N}_j(x,y)(t)=\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\sum_{i=1}^n b_{ji}g_i\left(x_i(s)\right)\mathrm{d}s,\quad j=1,\cdots,m.$$

Firstly, we will prove $(L,\bar{L})(x,y)+(N,\bar{N})(x,y)\in B_\delta$. In fact, using Minkowski inequality and (18) gives that

$$\begin{aligned}
&\left\|(L,\bar{L})(x,y)+(N,\bar{N})(x,y)\right\|\\
&=\sup_{0\le t\le T}\left\{\sum_{i=1}^n\left|\phi_i(0)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[-c_ix_i(s-\sigma)+\sum_{j=1}^m a_{ij}f_j\left(y_j(s)\right)+I_i\right]\mathrm{d}s\right|^p\right\}^{1/p}\\
&\quad+\sup_{0\le t\le T}\left\{\sum_{j=1}^m\left|\psi_i(0)+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\left[-d_jy_j(s-\sigma)+\sum_{i=1}^n b_{ji}g_i\left(x_i(s)\right)+J_j\right]\mathrm{d}s\right|^q\right\}^{1/q}\\
&\le\left(\sum_{i=1}^n\left|\phi_i(0)\right|^p\right)^{1/p}+\left(\sum_{j=1}^m\left|\psi_i(0)\right|^q\right)^{1/q}+\sup_{0\le t\le T}\left[\sum_{i=1}^n\left(\int_0^t\frac{c_i(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|x_i(s-\sigma)\right|\mathrm{d}s\right)^p\right]^{1/p}\\
&\quad+\sup_{0\le t\le T}\left[\sum_{j=1}^m\left(\int_0^t\frac{d_j(t-s)^{\beta-1}}{\Gamma(\beta)}\left|y_j(s-\sigma)\right|\mathrm{d}s\right)^q\right]^{1/q}+\sup_{0\le t\le T}\left[\sum_{i=1}^n\left(\int_0^t\sum_{j=1}^m\frac{\left|a_{ij}\right|(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f_j\left(y_j(s)\right)\right|\mathrm{d}s\right)^p\right]^{1/p}\\
&\quad+\sup_{0\le t\le T}\left[\sum_{j=1}^m\left(\int_0^t\sum_{i=1}^n\frac{\left|b_{ji}\right|(t-s)^{\beta-1}}{\Gamma(\beta)}\left|g_i\left(x_i(s)\right)\right|\mathrm{d}s\right)^q\right]^{1/q}+\sup_{0\le t\le T}\left[\sum_{i=1}^n\left(\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|I_i\right|\mathrm{d}s\right)^p\right]^{1/p}\\
&\quad+\sup_{0\le t\le T}\left[\sum_{j=1}^m\left(\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\left|J_j\right|\mathrm{d}s\right)^q\right]^{1/q}\\
&\le(1+A)\left\|(\phi,\psi)\right\|+\frac{I_0n^{1/p}T^\alpha}{\Gamma(\alpha+1)}+\frac{J_0m^{1/q}T^\beta}{\Gamma(\beta+1)}+\max\left\{\frac{c_0n^{1/p}}{\Gamma(\alpha+1)}(T-\sigma)^\alpha,\frac{d_0m^{1/q}}{\Gamma(\beta+1)}(T-\sigma)^\beta\right\}\delta\\
&\quad+\frac{M_2T^\beta}{\Gamma(\beta+1)}\left(\sum_{j=1}^m\xi_j^q\right)^{1/q}+\frac{M_1T^\alpha}{\Gamma(\alpha+1)}\left(\sum_{i=1}^n\eta_i^p\right)^{1/p}\le\delta.
\end{aligned}$$

Thus, we conclude that $(L,\bar{L})(x,y)+(N,\bar{N})(x,y)\in B_\delta$.

Secondly, for any $(x,u),(y,v)\in B_\delta$, we have

$$\begin{aligned}
\left\|(L,\bar{L})(x,y)\right\| &= \left\|L(x,y)-L(u,v)\right\|_p + \left\|\bar{L}(x,y)-\bar{L}(x,y)\right\|_q \\
&\le \sup_{0\le t\le T}\left[\sum_{i=1}^{n}\left(\int_0^t \frac{c_i(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|x_i(s-\sigma)-u_i(s-\sigma)\right|\mathrm{d}s\right)^p\right]^{1/p} \\
&\quad + \sup_{0\le t\le T}\left[\sum_{j=1}^{m}\left(\int_0^t \frac{d_j(t-s)^{\beta-1}}{\Gamma(\beta)}\left|y_j(s-\sigma)-v_j(s-\sigma)\right|\mathrm{d}s\right)^q\right]^{1/q} \\
&\le \max\left\{\frac{c_0 n^{1/p}}{\Gamma(\alpha+1)}(T-\sigma)^\alpha, \frac{d_0 m^{1/q}}{\Gamma(\beta+1)}(T-\sigma)^\beta\right\}\left\|(x,u)-(y,v)\right\|,
\end{aligned}$$

which implies that $(L,\bar{L})$ is a contraction mapping by (17).

Thirdly, we prove that $(N,\bar{N})$ is continuous and compact. Since f_j, g_i, $j=1,\cdots,m$, $i=1,\cdots,n$, are continuous, it is obvious that $(N,\bar{N})$ is also continuous. Let $(x,y)\in B_\delta$, we get by ($H2$) that

$$\begin{aligned}
&\left\|(N,\bar{N})(x,y)\right\| = \left\|N(x,y)\right\|_p + \left\|\bar{N}(x,y)\right\|_q \\
&= \sup_{0\le t\le T}\left[\sum_{i=1}^{n}\left(\int_0^t\sum_{j=1}^{m}\frac{|a_{ij}|(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f_j(y_j(s))\right|\mathrm{d}s\right)^p\right]^{1/p} + \sup_{0\le t\le T}\left[\sum_{j=1}^{m}\left(\int_0^t\sum_{i=1}^{n}\frac{|b_{ji}|(t-s)^{\beta-1}}{\Gamma(\beta)}\left|g_i(x_i(s))\right|\mathrm{d}s\right)^q\right]^{1/q} \\
&\le \frac{M_2T^\beta}{\Gamma(\beta+1)}\left(\sum_{j=1}^{m}\xi_j^q\right)^{1/q} + \frac{M_1T^\alpha}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{n}\eta_i^p\right)^{1/p},
\end{aligned}$$

which implies that $(N,\bar{N})$ is uniformly bounded on B_δ. Moreover, we can show that $(N,\bar{N})(x,y)(t)$ is equicontinuous. In fact, for $(x,y)\in B_\delta$, $0<t_2<t_1$, we obtain

$$\begin{aligned}
&\left\|(N,\bar{N})(x,y)(t_1)-(N,\bar{N})(x,y)(t_2)\right\| \\
&= \left\|N(x,y)(t_1)-N(x,y)(t_2)\right\|_p + \left\|\bar{N}(x,y)(t_1)-\bar{N}(x,y)(t_2)\right\|_q \\
&\le \left[\sum_{i=1}^{n}\left(\int_0^{t_1}\sum_{j=1}^{m}\frac{|a_{ij}|(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f_j(y_j(s))\right|\mathrm{d}s - \int_0^{t_2}\sum_{j=1}^{m}\frac{|a_{ij}|(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f_j(y_j(s))\right|\mathrm{d}s\right)^p\right]^{1/p} \\
&\quad + \left[\sum_{j=1}^{m}\left(\int_0^{t_1}\sum_{i=1}^{n}\frac{|b_{ji}|(t_1-s)^{\beta-1}}{\Gamma(\beta)}\left|g_i(x_i(s))\right|\mathrm{d}s - \int_0^{t_2}\sum_{i=1}^{n}\frac{|b_{ji}|(t_2-s)^{\beta-1}}{\Gamma(\beta)}\left|g_i(x_i(s))\right|\mathrm{d}s\right)^q\right]^{1/q} \\
&\le \left[\sum_{i=1}^{n}\left(\sum_{j=1}^{m}M_1|a_{ij}|\left(\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s + \int_{t_2}^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s\right)\right)^p\right]^{1/p} \\
&\quad + \left[\sum_{j=1}^{m}\left(\sum_{i=1}^{n}M_2|b_{ji}|\left(\int_0^{t_2}\frac{(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}}{\Gamma(\beta)}\mathrm{d}s + \int_{t_2}^{t_1}\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\mathrm{d}s\right)\right)^q\right]^{1/q} \\
&\le \frac{M_2T^\beta}{\Gamma(\beta+1)}\left(\sum_{j=1}^{m}\xi_j^q\right)^{1/q}\left|2(t_1-t_2)^\beta+t_2^\beta-t_1^\beta\right| + \frac{M_1T^\alpha}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{n}\eta_i^p\right)^{1/p}\left|2(t_1-t_2)^\alpha+t_2^\alpha-t_1^\alpha\right| \to 0,
\end{aligned}$$

as $t_2\to t_1$. Hence, $(N,\bar{N})(B_\delta)$ is relatively compact. By the Arzela-Ascoli theorem, $(N,\bar{N})$ is compact. So, by Lemma 2 we have that system (1) has at least one solution.

Theorem 5. *Assume that* ($H1$) *and condition* (6) *hold. Then the solution of system* (1) *is uniformly stable on* $[0,T]$.

Proof. Assume that $(x(t),y(t))^{\mathrm{T}}$ and $(u(t),v(t))^{\mathrm{T}}$ are any two solutions of system (1) with the initial conditions (ϕ,ψ) and $(\bar{\phi},\bar{\psi})$, respectively. Then

$$(x(t),y(t))^{\mathrm{T}}=(F,G)(x,y)(t),\quad (u(t),v(t))^{\mathrm{T}}=(F,G)(u,v)(t),$$

that is,

$$\begin{cases} x_i(t)=\phi_i(0)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[-c_ix_i(s-\sigma)+\sum_{j=1}^m a_{ij}f_j(y_j(s))+I_i\right]\mathrm{d}s,\\ y_j(t)=\psi_j(0)+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\left[-d_jy_j(s-\sigma)+\sum_{i=1}^n b_{ji}g_i(x_i(s))+J_j\right]\mathrm{d}s,\end{cases}$$

$$\begin{cases} u_i(t)=\bar{\phi}_i(0)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[-c_iu_i(s-\sigma)+\sum_{j=1}^m a_{ij}f_j(v_j(s))+I_i\right]\mathrm{d}s,\\ v_j(t)=\bar{\psi}_j(0)+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\left[-d_jv_j(s-\sigma)+\sum_{i=1}^n b_{ji}g_i(u_i(s))+J_j\right]\mathrm{d}s,\end{cases}$$

which implies

$$\begin{aligned}
&\|(x,y)-(u,v)\|=\|x-u\|_p+\|y-v\|_q\\
&\le\|\phi-\bar{\phi}\|_p+\sup_{0\le t\le T}\left[\sum_{i=1}^n\left(\int_0^t\frac{c_i(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x_i(s-\sigma)-u_i(s-\sigma)|\mathrm{d}s\right)^p\right]^{1/p}\\
&\quad+\sup_{0\le t\le T}\left[\sum_{i=1}^n\left(\int_0^t\sum_{j=1}^m\frac{|a_{ij}|(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f_j(y_j(s))-f_j(v_j(s))|\mathrm{d}s\right)^p\right]^{1/p}\\
&\quad+\|\psi-\bar{\psi}\|_q+\sup_{0\le t\le T}\left[\sum_{j=1}^m\left(\int_0^t\frac{d_j(t-s)^{\beta-1}}{\Gamma(\beta)}|y_j(s-\sigma)-v_j(s-\sigma)|\mathrm{d}s\right)^q\right]^{1/q}\\
&\quad+\sup_{0\le t\le T}\left[\sum_{j=1}^m\left(\int_0^t\sum_{i=1}^n\frac{|b_{ji}|(t-s)^{\beta-1}}{\Gamma(\beta)}|g_i(x_i(s))-g_i(u_i(s))|\mathrm{d}s\right)^q\right]^{1/q}\\
&\le\|(\phi,\psi)-(\bar{\phi},\bar{\psi})\|+\left(\frac{c_0n^{1/p}}{\Gamma(\alpha+1)}(T-\sigma)^\alpha+\frac{l_0T^\beta}{\Gamma(\beta+1)}\left(\sum_{j=1}^m\xi_j^q\right)^{1/q}\right)\|x-u\|_p\\
&\quad+\left(\frac{d_0m^{1/q}}{\Gamma(\beta+1)}(T-\sigma)^\beta+\frac{L_0T^\alpha}{\Gamma(\alpha+1)}\left(\sum_{i=1}^n\eta_i^p\right)^{1/p}\right)\|y-v\|_q\\
&\le\|(\phi,\psi)-(\bar{\phi},\bar{\psi})\|+B\|(x,u)-(y,v)\|.
\end{aligned}$$

Hence, we have

$$\|(x,y)-(u,v)\|\le\frac{1}{1-B}\|(\phi,\psi)-(\bar{\phi},\bar{\psi})\|. \tag{19}$$

For any $\varepsilon>0$, if we take $\|(\phi,\psi)-(\bar{\phi},\bar{\psi})\|<(1-B)\varepsilon$, then we can obtain from (19) that

$$\left\|(x,y)-(u,v)\right\|<\varepsilon,$$

which implies that the solution of system (1) is uniformly stable on $[0,T]$.

4. An Illustrative Example

In this section, we give an example to illustrate the effectiveness of our main results.

Consider the following two-state Caputo fractional BAM type neural networks model with leakage delay

$$\begin{cases} {}^C D^{\alpha} x_1(t) = -0.3x_1(t-\sigma) - 0.2f_1(y_1(t)) + 0.1f_2(y_2(t)) + 0.4, \\ {}^C D^{\alpha} x_2(t) = -0.2x_2(t-\sigma) + 0.3f_1(y_1(t)) + 0.2f_2(y_2(t)) - 0.3, \\ {}^C D^{\beta} y_1(t) = -0.4y_1(t-\sigma) + 0.4g_1(x_1(t)) + 0.2g_2(x_2(t)) - 0.5, \\ {}^C D^{\beta} y_2(t) = -0.5y_2(t-\sigma) + 0.1g_1(x_1(t)) - 0.3g_2(x_2(t)) + 0.2 \end{cases} \tag{20}$$

with the initial condition

$$x_1(t)=\phi_1(t),\quad x_2(t)=\phi_2(t),\quad y_1(t)=\psi_1(t),\quad y_2(t)=\psi_2(t),\quad t\in[-\sigma,0],$$

where $\phi_i,\psi_i \in C([-\sigma,0],\mathbb{R})$, $i=1,2$, $\alpha=0.8,\beta=0.9$, $\sigma=0.3$, $n=m=2$, $f_j(x_j)=\frac{1}{2}\left(\left|x_j+1\right|-\left|x_j-1\right|\right)$, $g_i(x_i)=\tanh(x_i)$ $(i,j=1,2)$. Let $T=1$, $p=q=2$, from (3)-(5), it is easy to check that

$$c_0=0.3,\quad d_0=0.5,\quad a_{10}=0.3,\quad a_{20}=0.5,\quad b_{10}=0.6,\quad b_{10}=0.4,$$

$$L_0=l_0=1,\quad \eta_1=\sqrt{a_{11}^2+a_{12}^2}=0.2236,\quad \eta_2=\sqrt{a_{21}^2+a_{22}^2}=0.3606,$$

$$\xi_1=\sqrt{b_{11}^2+b_{12}^2}=0.4472,\quad \xi_2=\sqrt{b_{21}^2+b_{22}^2}=0.3162.$$

Thus,

$$\begin{aligned} B &= \max\left\{\frac{c_0 n^{1/p}}{\Gamma(\alpha+1)}(T-\sigma)^{\alpha}+\frac{l_0 T^{\beta}}{\Gamma(\beta+1)}\left(\sum_{j=1}^{m}\xi_j^q\right)^{1/q}, \frac{d_0 m^{1/q}}{\Gamma(\beta+1)}(T-\sigma)^{\beta}+\frac{L_0 T^{\alpha}}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{n}\eta_i^p\right)^{1/p}\right\} \\ &= \max\left\{\frac{0.3\times\sqrt{2}\times 0.7^{0.8}}{\Gamma(1.8)}+\frac{\sqrt{0.4472^2+0.3162^2}}{\Gamma(1.9)}, \frac{0.5\times\sqrt{2}\times 0.7^{0.9}}{\Gamma(1.9)}+\frac{\sqrt{0.2236^2+0.3606^2}}{\Gamma(1.9)}\right\} \\ &= 0.9745<1, \end{aligned}$$

that is, condition (6) holds. By utilizing Theorems 3.1 and 3.3, we can obtain that the system (20) has a unique solution which is uniformly stable on $[0,1]$.

In the following, we show the simulation result for model (20). We consider four cases:

Case 1 with the initial values

$$(\phi_1(t),\phi_2(t),\psi_1(t),\psi_2(t))^{\mathrm{T}} \equiv (-0.3,1.5,0.9,-0.8)^{\mathrm{T}} \quad \text{for} \quad t\in[-0.3,0],$$

Case 2 with the initial values

$$(\phi_1(t),\phi_2(t),\psi_1(t),\psi_2(t))^{\mathrm{T}} \equiv (1.2,-0.9,1.4,-1.3)^{\mathrm{T}} \quad \text{for} \quad t\in[-0.3,0],$$

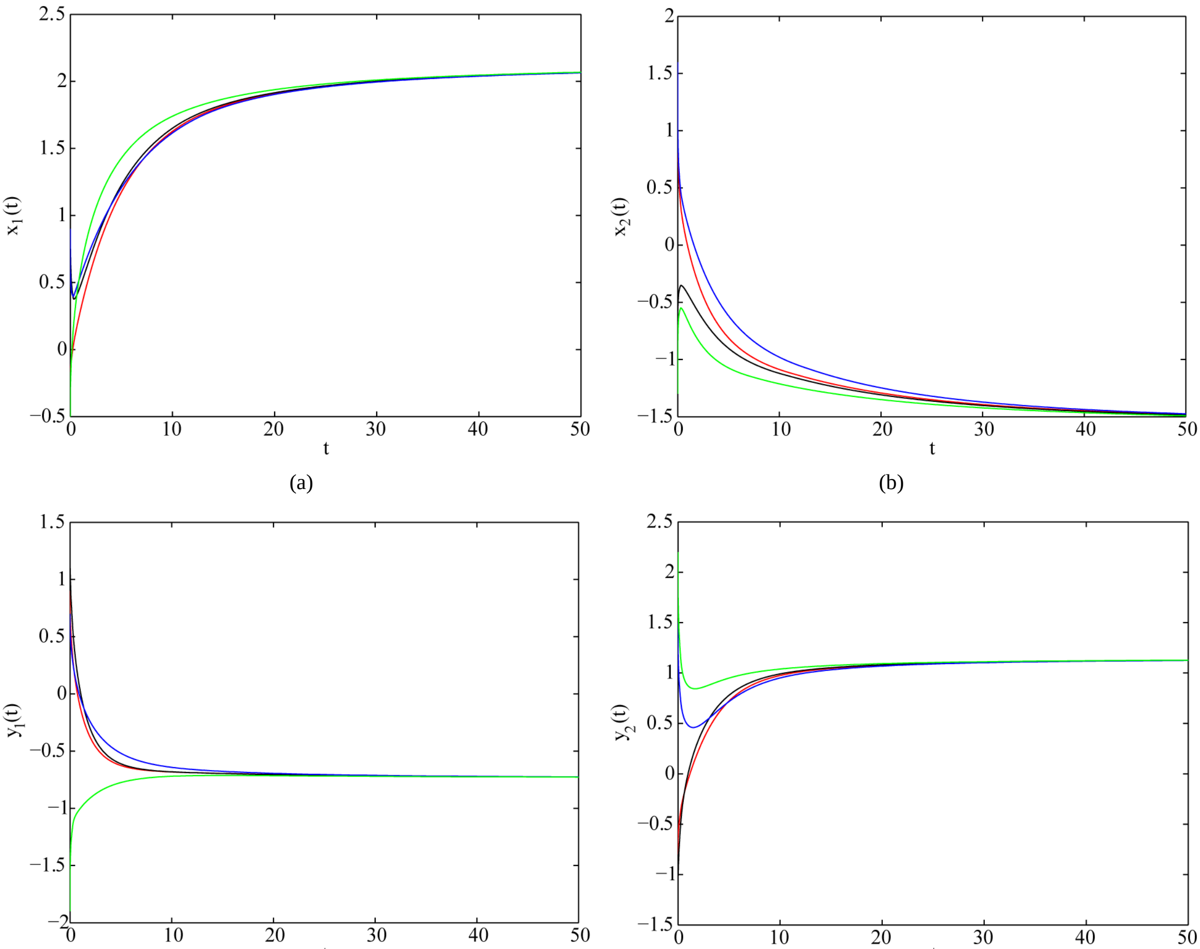
Case 3 with the initial values

$$(\phi_1(t),\phi_2(t),\psi_1(t),\psi_2(t))^{\mathrm{T}} \equiv (0.9,1.6,0.7,1.5)^{\mathrm{T}} \quad \text{for} \quad t\in[-0.3,0],$$

Case 4 with the initial values

$$(\phi_1(t),\phi_2(t),\psi_1(t),\psi_2(t))^{\mathrm{T}} \equiv (-0.5,-1.3,-1.9,2.2)^{\mathrm{T}} \quad \text{for} \quad t\in[-0.3,0].$$

The time responses of state variables are shown in **Figure 1**.



(a) (b)

(c) (d)

Figure 1. Transient states of the fractional-order BAM neural networks (20) with $\alpha = 0.8$, $\beta = 0.9$, and $\sigma = 0.3$.

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