On Elliptic Problem with Singular Cylindrical Potential, a Concave Term, and Critical Caffarelli-Kohn-Nirenberg Exponent

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Abstract
In this paper, we establish the existence of at least four distinct solutions to an elliptic problem with singular cylindrical potential, a concave term, and critical Caffarelli-Kohn-Nirenberg exponent, by using the Nehari manifold and mountain pass theorem.

Keywords
Singular Cylindrical Potential, Concave Term, Critical Caffarelli-Kohn-Nirenberg Exponent, Nehari Manifold, Mountain Pass Theorem

1. Introduction
In this paper, we consider the multiplicity results of nontrivial nonnegative solutions of the following problem \( (P_{\lambda, \mu}) \)

\[
\begin{align*}
L_{\mu, \nu} u &= 2 \left| y \right|^{-2a} \Delta u + \mu \left| y \right|^{-2a} u + \lambda \left| y \right|^{q-2} u, \quad u \neq 0 \\
u &\in \mathcal{D}^{1,2}_a
\end{align*}
\]

where \( L_{\mu, \nu} := -\text{div} \left( \left| y \right|^{-2a} \nabla v \right) - \mu \left| y \right|^{-2a} v \), where each point \( x \in \mathbb{R}^N \) is written as a pair \( y, z \in \mathbb{R}^k \times \mathbb{R}^{N-k} \) where \( k \) and \( N \) are integers such that \( N \geq 3 \) and \( k \) belongs to \( \{ 1, \cdots, N \} \), \( -\infty < a < (k-2)/2 \), \( a \leq b < a + 1 \), \(1 < q < 2 \), \( 2_s = 2N/(N - 2 + 2(b - a)) \) is the critical Caffarelli-Kohn-Nirenberg exponent.

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and

$$0 < c = q(a+1)+N(1-q/2), \quad -\infty < \mu < \overline{\mu}_{a,k} = \left(\frac{(k-2(a+1))}{2}\right)^2, \quad \lambda \text{ is a real parameter, } f \in \mathcal{H}_\mu' \cap C \left( \mathbb{R}^N \right),$$

$h$ is a bounded positive function on $\mathbb{R}^N$. $\mathcal{H}_\mu'$ is the dual of $\mathcal{H}_\mu$, where $\mathcal{H}_\mu$ and $\mathcal{D}_0^{1,2}$ will be defined later.

Some results are already available for $\left( P_{a,\mu} \right)$ in the case $k = N$, see for example [1] [2] and the references therein. Wang and Zhou [1] proved that there exist at least two solutions for $\left( P_{a,\mu} \right)$ with $a = 0$,

$$0 < \mu \leq \overline{\mu}_{a,k} = \left(\frac{(N-2)/2}{2}\right)^2 \quad \text{and} \quad h \equiv 1,$$

under certain conditions on $f$. Boucheikif and Matallah [3] showed the existence of two solutions of $\left( P_{a,\mu} \right)$ under certain conditions on functions $f$ and $h$, when $0 < \mu \leq \overline{\mu}_{a,k}$, $\lambda \in (0, \Lambda_*)$, $-\infty < a < (N-2)/2$ and $a \leq b < a + 1$, with $\Lambda_*$ a positive constant.

Concerning existence results in the case $k < N$, we cite [4] [5] and the references therein. Musina [5] considered $\left( P_{a,\mu} \right)$ with $-a/2$ instead of $a$ and $\lambda = 0$, also $\left( P_{a,\mu} \right)$ with $a = 0, b = 0, \lambda = 0$, with $h \equiv 1$ and $a \neq 2 - k$. She established the existence of a ground state solution when $2 < k \leq N$ and

$$0 < \mu < \overline{\mu}_{a,k} = \left(\frac{(k-2+a)/2}{2}\right)^2 \quad \text{for} \quad \left( P_{a,\mu} \right) \quad \text{with} \quad -a/2 \quad \text{instead of} \quad a \quad \text{and} \quad \lambda = 0.$$

She also showed that $\left( P_{a,\mu} \right)$ with $a = 0, b = 0, \lambda = 0$ does not admit ground state solutions. Badiale et al. [6] studied $\left( P_{a,\mu} \right)$ with $a = 0, b = 0, \lambda = 0$ and $h \equiv 1$. They proved the existence of at least a nonzero nonnegative weak solution $u$, satisfying $u(y,z) = u(|y|,z)$ when $2 \leq k < N$ and $\mu \leq 0$. Boucheikif and El Mokhtar [7] proved that $\left( P_{a,\mu} \right)$ admits two distinct solutions when $2 < k \leq N, b = N - p(N-2)/2$ with $p \in \left( 2, 2^* \right], \mu < \overline{\mu}_{a,k}$ and $\lambda \in (0, \Lambda_*)$ where $\Lambda_*$ is a positive constant. Terracini [8] proved that there is no positive solutions of $\left( P_{a,\mu} \right)$ with $b = 0, \lambda = 0$ when $a \neq 0, h \equiv 1$ and $\mu < 0$. The regular problem corresponding to $a = b = \mu = 0$ and $h \equiv 1$ has been considered on a regular bounded domain $\Omega$ by Tarantello [9]. She proved that, for $f \in H^{-1}(\Omega)$, the dual of $H_0'(\Omega)$, not identically zero and satisfying a suitable condition, the problem considered admits two distinct solutions.

Before formulating our results, we give some definitions and notation.

We denote by $\mathcal{D}_a^{1,2} = \mathcal{D}_a^{1,2} \left( \mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k} \right)$ and $\mathcal{H}_\mu = \mathcal{H}_\mu \left( \mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k} \right)$, the closure of $C_0^\infty \left( \mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k} \right)$ with respect to the norms

$$\|u\|_{0,\mu} = \left( \int_{\mathbb{R}^k} |\nabla u|^2 + \mu |u|^2 \, dx \right)^{1/2}$$

and

$$\|u\|_{a,\mu} = \left( \int_{\mathbb{R}^k} \left( |\nabla u|^2 - \mu |u|^{2(a+1)} \right) \, dx \right)^{1/2},$$

respectively, with $\mu < \overline{\mu}_{a,k} = \left(\frac{(k-(2a+1))/2}{2}\right)^2$ for $k \neq 2(a+1)$.

From the Hardy-Sobolev-Maz’ya inequality, it is easy to see that the norm $\|u\|_{a,\mu}$ is equivalent to $\|v\|_{0,0}$. More explicitly, we have

$$\left( 1 - \left( \frac{1}{\overline{\mu}_{a,k}} \right) \max(\mu, 0) \right)^{1/2} \|u\|_{0,\mu} \leq \|u\|_{a,\mu} \leq \left( 1 - \left( \frac{1}{\overline{\mu}_{a,k}} \right) \min(\mu, 0) \right)^{1/2} \|u\|_{0,\mu},$$

for all $u \in \mathcal{H}_\mu$.

We list here a few integral inequalities.

The starting point for studying $\left( P_{a,\mu} \right)$, is the Hardy-Sobolev-Maz’ya inequality that is particular to the cylindrical case $k < N$ and that was proved by Maz’ya in [4]. It states that there exists positive constant $C_{a,2}$, such that

$$C_{a,2} \left( \int_{\mathbb{R}^k} |y|^{-2b} |v|^2 \, dx \right)^{1/2} \leq \int_{\mathbb{R}^k} \left( |y|^{-2a} |\nabla v|^2 - \mu |y|^{-2(a+1)} |v|^2 \right) \, dx,$$

for any $v \in C_0^\infty \left( \left( \mathbb{R}^k \setminus \{0\} \right) \times \mathbb{R}^{N-k} \right)$. 

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The second one that we need is the Hardy inequality with cylindrical weights\cite{5}. It states that

\[ P_{a,\lambda} \int_{\mathbb{R}^N} |y|^{-2(a+1)} v^2 \, dx \leq \int_{\mathbb{R}^N} |y|^{-2a} |\nabla v|^2 \, dx, \quad \text{for all } v \in \mathcal{H}_\mu. \]  

(1.2)

It is easy to see that (1.1) hold for any \( u \in \mathcal{H}_\mu \) in the sense

\[ \left( \int_{\mathbb{R}^N} |y|^{-2} |u|^2 \, dx \right)^{1/p} \leq C_{a,p} \left( \int_{\mathbb{R}^N} |y|^{-2a} |\nabla u|^2 \, dx \right)^{1/p} \]  

(1.3)

where \( C_{a,p} \) positive constant, \( 1 \leq p \leq 2N/(N-2) \), \( c \leq p(a+1)+N(1-p/2) \), and in \cite{10}, if \( p < 2N/(N-2) \) the embedding \( \mathcal{H}_\mu \rightarrow L_p\left(\mathbb{R}^N, |y|^{-2} \right) \) is compact, where \( L_p\left(\mathbb{R}^N, |y|^{-2} \right) \) is the weighted \( L_p \) space with norm

\[ |u|_{p,\mu} = \left( \int_{\mathbb{R}^N} |y|^{-2a} |u|^p \, dx \right)^{1/p}. \]

Since our approach is variational, we define the functional \( J \) on \( \mathcal{H}_\mu \) by

\[ J(u) := \frac{1}{2} \|u\|_{a,\mu}^2 - P(u) - Q(u), \]

with

\[ P(u) := 2 \int_{\mathbb{R}^N} |y|^{-2a} h |\nabla|^\mu \, dx, \quad Q(u) := (1/q) \int_{\mathbb{R}^N} |y|^{-q} \lambda f |u|^q \, dx. \]

A point \( u \in \mathcal{H}_\mu \) is a weak solution of the equation \((P_{a,\mu})\) if it satisfies

\[ \left\langle J'(u), \varphi \right\rangle = R(u) \varphi - S(u) \varphi - T(u) \varphi = 0, \quad \text{for all } \varphi \in \mathcal{H}_\mu, \]

with

\[ R(u) \varphi := \int_{\mathbb{R}^N} \left( |y|^{-2a} (\nabla u \nabla \varphi) - \mu |y|^{-2(a+1)} (u \varphi) \right), \]

\[ S(u) \varphi := 2 \int_{\mathbb{R}^N} |y|^{-2a} h |\nabla|^\mu \varphi, \]

\[ T(u) \varphi := \int_{\mathbb{R}^N} |y|^{-q} \left( \lambda f |u|^{q-1} \varphi \right). \]

Here \( \langle \cdot, \cdot \rangle \) denotes the product in the duality \( \mathcal{H}_\mu^* \), \( \mathcal{H}_\mu^* \) dual of \( \mathcal{H}_\mu \).

Let

\[ S_\mu := \inf_{u \in \mathcal{H}_\mu \setminus \{0\}} \frac{\|u\|_{a,\mu}^2}{\left( \int_{\mathbb{R}^N} |y|^{-2a} |\nabla|^\mu \, dx \right)^{2/2^*}}. \]

From \cite{11}, \( S_\mu \) is achieved.

Throughout this work, we consider the following assumptions:

(F) there exist \( \nu_0 > 0 \) and \( \delta_0 > 0 \) such that \( f(x) \geq \nu_0 \), for all \( x \) in \( B(0,2\delta_0) \).

(H) \( \lim_{|y|\rightarrow\infty} h(y) = \lim_{|y|\rightarrow\infty} h_0(y) = h_y > 0, \quad h(y) \geq h_y, \quad y \in \mathbb{R}^N \).

Here, \( B(a,r) \) denotes the ball centered at \( a \) with radius \( r \).

In our work, we research the critical points as the minimizers of the energy functional associated to the problem \((P_{a,\mu})\) on the constraint defined by the Nehari manifold, which are solutions of our system.

Let \( \Lambda_{\mu} \) be positive number such that

\[ \Lambda_{\mu} := (C_{\mu})^{-q} \left( h_0 \right)^{\frac{1}{2(2^*-2)}} \left( S_\mu \right)^{2/(2^*-2)} \left( L(q) \right)^{\frac{1}{2(2^*-2)}}. \]

where \( L(q) := \left( \frac{2 - 2q}{2q - 2} \right) \left[ \left( \frac{2 - q}{2(2 - q)} \right) \right]^{\frac{1}{2(2^*-2)}}. \)
Now we can state our main results.

**Theorem 1.** Assume that $-\infty < a < (k - 2)/2$, $0 < c = q(a + 1) + N(1 - q/2)$, $-\infty < \mu < \overline{\mu}_{a,k}$, (F) satisfied and $\lambda$ verifying $0 < \lambda < \Lambda$, then the system $(P_{\lambda,\mu})$ has at least one positive solution.

**Theorem 2.** In addition to the assumptions of the Theorem 1, if (H) hold and $\lambda$ satisfying $0 < \lambda < (1/2)\Lambda_0$, then $(P_{\lambda,\mu})$ has at least two positive solutions.

**Theorem 3.** In addition to the assumptions of the Theorem 2, assuming $(H)$ hold and $\lambda$ satisfying $\lambda_0 < \lambda < \Lambda$, there exists a positive real $\Lambda$ such that $\lambda < \Lambda$, then $(P_{\lambda,\mu})$ has at least two positive solution and two opposite solutions.

This paper is organized as follows. In Section 2, we give some preliminaries. Sections 3 and 4 are devoted to the proofs of Theorems 1 and 2. In the last Section, we prove the Theorem 3.

### 2. Preliminaries

**Definition 1.** Let $c \in \mathbb{R}$, $E$ a Banach space and $I \in C^1(E,\mathbb{R})$.

i) $(u_n)_n$ is a Palais-Smale sequence at level $c$ (in short $(PS)_c$) in $E$ for $I$ if

$$I(u_n) = c + o_n(1) \quad \text{and} \quad I'(u_n) = o_n(1),$$

where $o_n(1)$ tends to 0 as $n$ goes at infinity.

ii) We say that $I$ satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence in $E$ for $I$ has a convergent subsequence.

**Lemma 1.** Let $X$ Banach space, and $J \in C^1(X,\mathbb{R})$ verifying the Palais-Smale condition. Suppose that $J(0) = 0$ and that:

i) there exist $R > 0$, $r > 0$ such that if $\|u\| = R$, then $J(u) \geq r$;

ii) there exist $(u_0) \in X$ such that $\|u_0\| > R$ and $J(u_0) \leq 0$;

let $c = \inf_{r \in [0,1]} \max_{\|u\|=r} J(\gamma(1))$ where

$$\Gamma = \{ \gamma \in C([0,1];X) \text{ such that } \gamma(0) = 0 \text{ and } \gamma(1) = u_0 \},$$

then $c$ is critical value of $J$ such that $c \geq r$.

### Nehari Manifold

It is well known that $J$ is of class $C^1$ in $H_{\mu}$ and the solutions of $(P_{\lambda,\mu})$ are the critical points of $J$ which is not bounded below on $H_{\mu}$. Consider the following Nehari manifold

$$\mathcal{N} = \{ u \in H_{\mu} \setminus \{0\} : \langle J'(u), u \rangle = 0 \}.$$

Thus, $u \in \mathcal{N}$ if and only if

$$\|u\|_{\mu}^2 - 2J(u) = 0. \quad (2.1)$$

Note that $\mathcal{N}$ contains every nontrivial solution of the problem $(P_{\lambda,\mu})$. Moreover, we have the following results.

**Lemma 2.** $J$ is coercive and bounded from below on $\mathcal{N}$.

**Proof.** If $u \in \mathcal{N}$, then by (2.1) and the Hölder inequality, we deduce that

$$J(u) = ((2, -2)/2, 2)\|u\|_{\mu}^2 - ((2, -q)/2, 2)Q(u) \geq ((2, -2)/2, 2)\|u\|_{\mu}^2 - \left(\frac{2, -q}{2, q}\right)(\|u\|_{H_{\mu}}^2)^{1/(2-q)}(\|u\|_{H_{\mu}})^q.$$

Thus, $J$ is coercive and bounded from below on $\mathcal{N}$. Define

$$\phi(u) = \langle J'(u), u \rangle.$$
Then, for \( u \in \mathcal{N} \)

\[
\langle \phi'(u), u \rangle = 2\| u \|_{\alpha,a}^2 - (2, -q) P(u) - qQ(u)
\]

\[
= (2-q)\| u \|_{\alpha,a}^2 - 2, (2-q) P(u)
\]

\[
= (2-q)Q(u) - (2-2)\| u \|_{\alpha,a}^2.
\]  

(2.3)

Now, we split \( \mathcal{N} \) in three parts:

\[
\mathcal{N}^+ = \{ u \in \mathcal{N} : \langle \phi'(u), u \rangle > 0 \}
\]

\[
\mathcal{N}^0 = \{ u \in \mathcal{N} : \langle \phi'(u), u \rangle = 0 \}
\]

\[
\mathcal{N}^- = \{ u \in \mathcal{N} : \langle \phi'(u), u \rangle < 0 \}.
\]

We have the following results.

**Lemma 3.** Suppose that \( u_0 \) is a local minimizer for \( J \) on \( \mathcal{N} \). Then, if \( u_0 \notin \mathcal{N}^0 \), \( u_0 \) is a critical point of \( J \).

**Proof.** If \( u_0 \) is a local minimizer for \( J \) on \( \mathcal{N} \), then \( u_0 \) is a solution of the optimization problem

\[
\min_{\{u: \langle \phi'(u), u \rangle = 0\}} J(u).
\]

Hence, there exists a Lagrange multipliers \( \theta \in \mathbb{R} \) such that

\[
J'(u_0) = \theta \phi'(u_0) \text{ in } \mathcal{H}.
\]

Thus,

\[
\langle J'(u_0), u_0 \rangle = \theta \langle \phi'(u_0), u_0 \rangle.
\]

But \( \langle \phi'(u_0), u_0 \rangle \neq 0 \), since \( u_0 \notin \mathcal{N}^0 \). Hence \( \theta = 0 \). This completes the proof.

**Lemma 4.** There exists a positive number \( \Lambda_0 \) such that for all \( \lambda \), verifying

\[
0 < \lambda < \Lambda_0,
\]

we have \( \mathcal{N}^0 = \emptyset \).

**Proof.** Let us reason by contradiction. Suppose \( \mathcal{N}^0 \neq \emptyset \) such that \( 0 < \lambda < \Lambda_0 \). Then, by (2.3) and for \( u \in \mathcal{N}^0 \), we have

\[
\| u \|_{\alpha,a}^2 = 2, (2-q)\|(2-q) P(u) = ((2-q)/(2-q))Q(u)
\]

(2.4)

Moreover, by the Hölder inequality and the Sobolev embedding theorem, we obtain

\[
\| u \|_{\alpha,a}^2 \geq (S_{\mu})^{2/(2\mu-2)}[(2-q)/2, (2-q)\| \lambda \|^q]^{-q/(2-q)}
\]

(2.5)

and

\[
\| u \|_{\alpha,a}^2 \leq \left[ \frac{2-q}{2+q} \right]^{-q/(2-q)} \left( \lambda \right)^{-q/(2-q)}(C_{\alpha,q})^q.
\]

(2.6)

From (2.5) and (2.6), we obtain \( \lambda \geq \Lambda_0 \), which contradicts an hypothesis. Thus \( \mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^- \). Define \( \Lambda \geq \Lambda_0 \), which contradicts an hypothesis.

**Lemma 5.**

i) For all \( \lambda \) such that \( 0 < \lambda < \Lambda_0 \), one has \( c \leq c^+ < 0 \).
ii) For all \( \lambda \) such that \( 0 < \lambda < (1/2) \Lambda_0 \), one has

\[
c^- > C_0 \left( \lambda, \lambda_2, S_\mu, \|f\|_{v_\mu} \right)
= \left( \frac{(2, -2)}{2, 2} \right)^{2(2, -2)} \left( \frac{(2-q)}{(2, 2-q) \mu_0} \right) \left( S_\mu \right)^{2(2-q)} \left( C_\sigma, \sigma \right)^{\frac{1}{q}}.
\]

**Proof.** i) Let \( u \in \mathcal{N}^+ \). By (2.3), we have

\[
\left( (2-q)/2, (2, -1) \right) \mu_0^2 > P(u)
\]

and so

\[
J(u) = (-1/2) \|u\|_{\mathcal{H}, u}^2 + (2, -1) P(u) < \left( \frac{2, (2-q) - (2, 1)(2-q)}{2, 2-q} \right) \|u\|_{\mathcal{H}, u}^2.
\]

We conclude that \( c^- \leq c^- < 0 \).

ii) Let \( u \in \mathcal{N}^- \). By (2.3), we get

\[
\left( (2-q)/2, (2, -q) \right) \mu_0^2 < P(u).
\]

Moreover, by (H) and Sobolev embedding theorem, we have

\[
P(u) \leq \left( S_\mu \right)^{2/2} \|u\|_{\mathcal{H}, u}^2.
\]

This implies

\[
\|u\|_{\mathcal{H}, u}^2 \geq \left( S_\mu \right)^{2(2, -2)} \left( \frac{(2-q)}{2, 2-q} \right)^{1(2-q)} \mu_0^2, \text{ for all } u \in \mathcal{N}^-.
\]

(2.7)

By (2.2), we get

\[
J(u) \geq \left( (2, -2)/2, 2 \right) \|u\|_{\mathcal{H}, u}^2 \left( \frac{(2-q)}{2, q} \right) \left( \mu_0^2 \right)^{1(2-q)} \left( C_\sigma, \sigma \right)^{\frac{1}{q}} \|u\|_{\mathcal{H}, u}^2.
\]

Thus, for all \( \lambda \) such that \( 0 < \lambda < (1/2) \Lambda_0 \), we have \( J(u) \geq C_0 \).

For each \( u \in \mathcal{H} \) with \( \int_{\mathbb{R}^N} |\gamma|^{2-b} \mu [\gamma]^{b} \mu dx > 0 \), we write

\[
t_m := t_{\text{max}}(u) = \left[ \frac{(2-q)}{2, 2-q} \int_{\mathbb{R}^N} |\gamma|^{2-b} \mu [\gamma]^{b} \mu dx \right]^{1(2-q)} > 0.
\]

**Lemma 6.** Let \( \lambda \) real parameters such that \( 0 < \lambda < \Lambda_0 \). For each \( u \in \mathcal{H} \) with \( \int_{\mathbb{R}^N} |\gamma|^{2-b} \mu [\gamma]^{b} \mu dx > 0 \), one has the following:

i) If \( Q(u) \leq 0 \), then there exists a unique \( t^- > t_m \) such that \( t^- u \in \mathcal{N}^- \) and

\[
J(t^- u) = \sup_{t \in \mathbb{R}} J(tu).
\]

ii) If \( Q(u) > 0 \), then there exist unique \( t^+ \) and \( t^- \) such that \( 0 < t^- < t_m < t^+ \), \( (t^+ u) \in \mathcal{N}^+ \), \( t^- u \in \mathcal{N}^- \),

\[
J(t^+ u) = \inf_{t \in \mathbb{R}} J(tu) \text{ and } J(t^- u) = \sup_{t \in \mathbb{R}} J(tu).
\]

**Proof.** With minor modifications, we refer to [12].

**Proposition 1** (see [12])

i) For all \( \lambda \) such that \( 0 < \lambda < \Lambda_0 \), there exists a \( \{PS\}_\epsilon \) sequence in \( \mathcal{N}^+ \).
ii) For all $\lambda$ such that $0 < \lambda < (1/2)\Lambda_0$, there exists a $\left(PS\right)_c^-$ sequence in $\mathcal{N}^-$. 

3. Proof of Theorems 1

Now, taking as a starting point the work of Tarantello [13], we establish the existence of a local minimum for $J$ on $\mathcal{N}^+$. 

**Proposition 2.** For all $\lambda$ such that $0 < \lambda < \Lambda_0$, the functional $J$ has a minimizer $u_0^* \in \mathcal{N}^+$ and it satisfies:

i) $J(u_0^*) = c = c^+$,

ii) $(u_0^*)$ is a nontrivial solution of $(\mathcal{P}_{\lambda,\mu})$.

**Proof.** If $0 < \lambda < \Lambda_0$, then by Proposition 1 (i) there exists a $(u_n)_n$ $(PS)_c^-$ sequence in $\mathcal{N}^+$, thus it bounded by Lemma 2. Then, there exists $u_0^* \in \mathcal{H}$ and we can extract a subsequence which will denoted by $(u_n)_n$ such that

$$
\begin{align*}
&u_n \rightharpoonup u_0^* \text{ weakly in } \mathcal{H} \\
&u_n \to u_0^* \text{ strongly in } L^2(\mathbb{R}^N, |x|^{-2}) \\
&u_n \to u_0^* \text{ strongly in } L^2(\mathbb{R}^N, |x|^{-2}) \\
&u_n \to u_0^* \text{ a.e. in } \mathbb{R}^N
\end{align*}
$$

(3.1)

Thus, by (3.1), $u_0^*$ is a weak nontrivial solution of $(\mathcal{P}_{\lambda,\mu})$. Now, we show that $u_n$ converges to $u_0^*$ strongly in $\mathcal{H}$. Suppose otherwise. By the lower semi-continuity of the norm, then either 

$$
\liminf_{n \to \infty} \|u_n\|_{\mu,\alpha} < \liminf_{n \to \infty} \|u_0^*\|_{\mu,\alpha}
$$

and we obtain

$$
c \leq J(u_0^*) = \frac{1}{2} \left( (2-q)/2, 2 \right) \|u_0^*\|_{\mu,\alpha}^2 - \left( (2-q)/2, 2 \right) Q(u_0^*) < \liminf_{n \to \infty} J(u_n) = c.
$$

We get a contradiction. Therefore, $u_n$ converge to $u_0^*$ strongly in $\mathcal{H}$. Moreover, we have $u_0^* \in \mathcal{N}^+$. If not, then by Lemma 6, there are two numbers $t_0^*$ and $t_0^*$, uniquely defined so that $(t_0^* u_n^*) \in \mathcal{N}^+$ and $(t_0^* u_n^*) \in \mathcal{N}^-$. In particular, we have $t_0^* < t_0^* = 1$. Since

$$
\frac{d}{dt} J(t u_n^*) \bigg|_{t=t_0^*} = 0 \quad \text{and} \quad \frac{d^2}{dt^2} J(t u_n^*) \bigg|_{t=t_0^*} > 0,
$$

there exists $t_0^* < t < t_0^*$ such that $J(t_0^* u_n^*) < J(t u_n^*)$. By Lemma 6, we get

$$
J(t_0^* u_0^*) < J(t u_0^*) < J(t_0^* u_0^*) = J(u_0^*),
$$

which contradicts the fact that $J(u_0^*) = c^*$. Since $J(u_0^*) = J(\|u_0^*\|_{\mu,\alpha})$ and $\|u_0^*\|_{\mu,\alpha} \in \mathcal{N}^+$, then by Lemma 3, we may assume that $u_0^*$ is a nontrivial nonnegative solution of $(\mathcal{P}_{\lambda,\mu})$. By the Harnack inequality, we conclude that $u_0^* > 0$ and $v_0^* > 0$, see for example [14].

4. Proof of Theorem 2

Next, we establish the existence of a local minimum for $J$ on $\mathcal{N}^-$. For this, we require the following Lemma.

**Lemma 7.** For all $\lambda$ such that $0 < \lambda < (1/2)\Lambda_0$, the functional $J$ has a minimizer $u_0^* \in \mathcal{N}^-$ and it satisfies:

i) $J(u_0^*) = c^- > 0$,

ii) $u_0^*$ is a nontrivial solution of $(\mathcal{P}_{\lambda,\mu})$ in $\mathcal{H}$.

**Proof.** If $0 < \lambda < (1/2)\Lambda_0$, then by Proposition 1 ii) there exists a $(u_n)_n$ $(PS)_c^-$ sequence in $\mathcal{N}^-$, thus it bounded by Lemma 2. Then, there exists $u_0^* \in \mathcal{H}$ and we can extract a subsequence which will denoted by $(u_n)_n$ such that
\[ u_n \to u_0^\pm \text{ weakly in } \mathcal{H} \]
\[ u_n \to u_0^\pm \text{ weakly in } L^2 \left( \mathbb{R}^N, |x|^{-2+b} \right) \]
\[ u_n \to u_0^\pm \text{ strongly in } L^q \left( \mathbb{R}^N, |x|^e \right) \]
\[ u_n \to u_0^\pm \text{ a.e. in } \mathbb{R}^N \]

This implies
\[ P(u_n) \to P(u_0^\pm), \text{ as } n \to \infty. \]

Moreover, by (H) and (2.3) we obtain
\[ P(u_n) > A(q) \| u_n \|_{\mu,a}^2, \tag{4.1} \]
where, \[ A(q) := \frac{(2-q)}{2}(2-q). \]
By (2.5) and (4.1) there exists a positive number
\[ C_1 := \left[ A(q) \right]^{\frac{2}{2-2q}} \left( S_{\mu} \right)^{\frac{2}{2-2q}}, \]
such that
\[ P(u_n) > C_1. \tag{4.2} \]

This implies that
\[ P(u_0^\pm) \geq C_1. \]

Now, we prove that \( (u_n) \) converges to \( u_0^\pm \) strongly in \( \mathcal{H} \). Suppose otherwise. Then, either
\[ \| u_0^\pm \|_{\mu,a} < \liminf_{n \to \infty} \| u_n \|_{\mu,a}. \]
By Lemma 6 there is a unique \( t_0^\pm \) such that \( (t_0^\pm u_0^\pm) \in \mathcal{N}^-. \) Since
\[ u_n \in \mathcal{N}^-, J(u_n) \geq J(tu_n), \text{ for all } t \geq 0, \]
we have
\[ J(t_0^\pm u_0^\pm) < \lim_{n \to \infty} J(t_0^\pm u_n) \leq \lim_{n \to \infty} J(u_n) = c^-, \]
and this is a contradiction. Hence,
\[ (u_n) \to u_0^\pm \text{ strongly in } \mathcal{H}. \]

Thus,
\[ J(u_n) \text{ converges to } J(u_0^\pm) = c^- \text{ as } n \text{ tends to } +\infty. \]

Since \( J(u_0^\pm) = J(u_0^-) \) and \( u_0^\pm \in \mathcal{N}^- \), then by (4.2) and Lemma 3, we may assume that \( u_0^\pm \) is a nontrivial nonnegative solution of \( (\mathcal{P}_{\lambda,\mu}) \). By the maximum principle, we conclude that \( u_0^- > 0 \).

Now, we complete the proof of Theorem 2. By Propositions 2 and Lemma 7, we obtain that \( (\mathcal{P}_{\lambda,\mu}) \) has two positive solutions \( u_0^+ \in \mathcal{N}^+ \) and \( u_0^- \in \mathcal{N}^- \). Since \( \mathcal{N}^+ \cap \mathcal{N}^- = \emptyset \), this implies that \( u_0^+ \) and \( u_0^- \) are distinct.

5. Proof of Theorem 3

In this section, we consider the following Nehari submanifold of \( \mathcal{N} \)
\[ \mathcal{N}_0 = \left\{ u \in \mathcal{H} \setminus \{0\} : \langle J'(u), u \rangle = 0 \text{ and } \|u\|_{\mu,a}^2 \geq \varphi > 0 \right\}. \]

Thus, \( u \in \mathcal{N}_0 \) if and only if
\[ \|u\|_{\mu,a}^2 - 2P(u) - Q(u) = 0 \text{ and } \|u\|_{\mu,a}^2 \geq \varphi > 0. \]
Firstly, we need the following Lemmas

**Lemma 8.** Under the hypothesis of theorem 3, there exist \( \varrho_0, \Lambda_z > 0 \) such that \( \mathcal{N}_\varrho \) is nonempty for any \( \lambda \in (0, \Lambda_z) \) and \( \varrho \in (0, \varrho_0) \).

*Proof.* Fix \( u_0 \in \mathcal{H} \setminus \{0\} \) and let

\[
g(t) = \langle J'(tu_0), tu_0 \rangle = t^2 \|u_0\|_{\rho, a}^2 - 2t^2 r^2 P(u_0) - rtQ(u_0).
\]

Clearly \( g(0) = 0 \) and \( g(t) \to -\infty \) as \( n \to +\infty \). Moreover, we have

\[
g(1) = \|u_0\|_{\rho, a}^2 - 2P(u_0) - Q(u_0) \\
\geq \left[ \|u_0\|_{\rho, a}^2 - 2 \left( S_\mu \right)^{-2/2} h_0 \|u_0\|_{\rho, a}^{2/2} \right] - \left( \|f\|_{\rho, a} \right)^{3(2-\varrho)} \|u_0\|_{\rho, a}.
\]

If \( \|u_0\|_{\rho, a} \geq \varrho > 0 \) for \( 0 < \varrho < \varrho_0 \) then there exists \( \lambda_\varrho = \left( S_\mu \right)^{2/2} / (2, (2, -1))^{(2-\varrho)/2} \), then there exists

\[
\Lambda_\varrho := \left[ \left( h_0, 2, (2, -1) \right) \left( S_\mu \right)^{-2/2} \right]^{-3(2-\varrho)} - \Theta \times \Phi,
\]

where

\[
\Theta := (2, (2, -1))^{2-\varrho} \left( h_0 \right)^{2/2}, S_\mu\right)^{-2/2}
\]

and

\[
\Phi := \left[ \left( h_0, 2, (2, -1) \right) \left( S_\mu \right)^{-2/2} \right]^{-3(2-\varrho)}
\]

and there exists \( t_0 > 0 \) such that \( g(t_0) = 0 \). Thus, \( (t, u_0) \in \mathcal{N}_\varrho \) and \( \mathcal{N}_\varrho \) is nonempty for any \( \lambda \in (0, \Lambda_z) \).

**Lemma 9.** There exist \( M, \Lambda_1 \) positive reals such that

\[
\langle \phi'(u), u \rangle < -M < 0, \text{ for } u \in \mathcal{N}_\varrho,
\]

and any \( \lambda \) verifying

\[
0 < \lambda < \min \left( (1/2) \Lambda_\varrho, \Lambda_z \right).
\]

*Proof.* Let \( u \in \mathcal{N}_\varrho \), then by (2.1), (2.3) and the Holder inequality, allows us to write

\[
\langle \phi'(u), u \rangle \leq \|u_0\|_{\rho, a}^2 \left[ \left( \|f\|_{\rho, a} \right)^{3(2-\varrho)} B(\varrho, q) - (2, -2) \right],
\]

where \( B(\varrho, q) := \left( 2, (2, -1) \right) \left( C_{a, q} \right)^{2(2-\varrho)} \). Thus, if

\[
0 < \lambda < \Lambda_1 = \left[ (2, -2) / B(\varrho, q) \right],
\]

and choosing \( \Lambda_1 := \min \left( \Lambda_\varrho, \Lambda_z \right) \) with \( \Lambda_z \) defined in Lemma 8, then we obtain that

\[
\langle \phi'(u), u \rangle < 0, \text{ for any } u \in \mathcal{N}_\varrho.
\]

**Lemma 10.** Suppose \( N \geq \max \{3, 6(a-b+1)\} \) and \( \int_\Omega |x|^{2-b} \eta dx > 0 \). Then, there exist \( r \) and \( \eta \) positive constants such that

i) we have

\[
J(u) \geq \eta > 0, \text{ for } \|u\|_{\rho, a} = r.
\]
ii) there exists $\sigma \in \mathcal{N}_\varepsilon$ when $\|\sigma\|_{\mu,\nu} > r$, with $r = \|\mu\|_{\mu,\nu}$, such that $J(\sigma) \leq 0$.

Proof. We can suppose that the minima of $J$ are realized by $(u_0^+)$ and $u_0^-$. The geometric conditions of the mountain pass theorem are satisfied. Indeed, we have

i) By (2.3), (5.1) and the fact that $P(u) \leq (S_{\mu})^{-2/2} h_0 \|u\|_{\mu,\nu}^2$, we get

$$J(u) \geq [(1/2)-(2, -2)/(2, q)] \|u\|_{\mu,\nu}^2 -(S_{\mu})^{-2/2} h_0 \|u\|_{\mu,\nu}^2,$$

Exploiting the function $l(x) = x(2, -x)$ and if $N \geq \max (3, 6(a-b+1))$, we obtain that $[(1/2)-(2, -2)/(2, q)] > 0$ for $1 < q < 2$. Thus, there exist $\eta$, $r > 0$ such that $J(u) \geq \eta > 0$ when $r = \|\mu\|_{\mu,\nu}$ small.

ii) Let $t > 0$, then we have for all $\phi \in \mathcal{N}_\varepsilon$

$$J(t\phi) := (t^2/2)\|\sigma\|_{\mu,\nu}^2 - (t^q)P(\phi) - (t^q/2)Q(\phi).$$

Letting $\sigma = t\phi$ for $t$ large enough. Since

$$P(\phi) := \int_{\Omega} |\phi|^2 h_0 |\phi|^{2b} \, dx > 0,$$

we obtain $J(\sigma) \leq 0$. For $t$ large enough we can ensure $\|\sigma\|_{\mu,\nu} > r$.

Let $\Gamma$ and $c$ defined by

$$\Gamma := \{\gamma : [0,1] \rightarrow \mathcal{N}_\varepsilon : \gamma(0) = u_0^- \text{ and } \gamma(1) = u_0^+\}$$

and

$$c := \inf_{\gamma \in \Gamma} \max_{\Omega} \inf_{t \in [0,1]} (J(\gamma(t))).$$

Proof of Theorem 3.

If

$$\lambda < \min \left(\{(1/2)\Lambda_0, \Lambda_1\} \right),$$

then, by the Lemmas 2 and Proposition 1 ii), $J$ verifying the Palais-Smale condition in $\mathcal{N}_\varepsilon$. Moreover, from the Lemmas 3, 9 and 10, there exists $u_\varepsilon$ such that

$$J(u_\varepsilon) = c \text{ and } u_\varepsilon \in \mathcal{N}_\varepsilon.$$

Thus $u_\varepsilon$ is the third solution of our system such that $u_\varepsilon \neq u_0^+$ and $u_\varepsilon \neq u_0^-$. Since $(\mathcal{P}_{\lambda,\mu})$ is odd with respect $u$, we obtain that $-u_\varepsilon$ is also a solution of $(\mathcal{P}_{\lambda,\mu}).$

References


