Regular Elements of the Complete Semigroups $B_X(D)$ of Binary Relations of the Class $\Sigma_2(X,8)$

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Abstract
As we know if $D$ is a complete $X$-semilattice of unions then semigroup $B_X(D)$ possesses a right unit iff $D$ is an $XI$-semilattice of unions. The investigation of those $\alpha$-idempotent and regular elements of semigroups $B_X(D)$ requires an investigation of $XI$-subsemilattices of semilattice $D$ for which $V(D,\alpha) = Q \in \Sigma_2(X,8)$. Because the semilattice $Q$ of the class $\Sigma_2(X,8)$ are not always $XI$-semilattices, there is a need of full description for those idempotent and regular elements when $V(D,\alpha) = Q$. For the case where $X$ is a finite set we derive formulas by calculating the numbers of such regular elements and right units for which $V(D,\alpha) = Q$.

Keywords
Semilattice, Semigroup, Binary Relation

1. Introduction
In this paper we characterize the elements of the class $\Sigma_2(X,8)$. This class is the complete $X$-semilattice of unions every elements of which are isomorphic to $Q$. So, we characterize the class for each element which is isomorphic to $Q$ by means of the characteristic family of sets, the characteristic mapping and the generate set of $D$.

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Let $X$ be an arbitrary nonempty set, recall that the set of all binary relations on $X$ is denoted $B_X$. The binary operation "" on $B_X$ defined by for $\alpha, \beta \in B_X$, $(x, z) \in \alpha \circ \beta \iff (x, y) \in \alpha$ and $(y, z) \in \beta$, for some $y \in X$ is associative and hence $B_X$ is a semigroup with respect to the operation "". This semigroup is called the semigroup of all binary relations on the set $X$. By $\emptyset$ we denote an empty binary relation or empty subset of the set $X$.

Let $D$ be a $X$-semilattice of unions, i.e. a nonempty set of subsets of the set $X$ that is closed with respect to the set-theoretic operations of unification of elements from $D$, $f$ be an arbitrary mapping from $X$ into $D$. To each such a mapping $f$ there corresponds a binary relation $\alpha_f$ on the set $X$ that satisfies the condition $(x) = \bigcup \{f(x) \times x\}$. The set of all such $\alpha_f$ $(f : X \to D)$ is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation of multiplication of binary relations, which is called a complete semigroup of binary relations defined by a $X$-semilattice of unions $D$ (see ([1], Item 2.1), ([2], Item 2.1)).

Let $x, y, z \in X$, $Y \subseteq X$, $\alpha \in B_X(D)$, $T \subseteq D$, $\emptyset \neq D' \subseteq D$ and $t \in \bar{D} = \bigcup_{y \in D} Y$. We use the notations:

$\gamma \alpha = \{x \in X | y x \} \subseteq \alpha \subseteq \gamma \beta$, $Y \alpha = \bigcup_{y \in Y} \gamma \alpha$, $\nu(D, \alpha) = \{Y \alpha | Y \subseteq D\}$, $\gamma_0 = \{x \in X | \alpha x = T\}$, $\gamma_T = \{x \in X | \alpha x = T\}$, $\gamma_T' = \{x \in X | \alpha x = T\}$, $\gamma_T = \gamma_0 \cup \{0\}$, $\gamma_T = \nu(D, \alpha)$, $\nu(D, \alpha) \subseteq \gamma_0$ and $\nu(D, \alpha) \subseteq \gamma_T$.

In general, a representation of a binary relation $\alpha$ of the form $\alpha = \bigcup_{x \alpha \subseteq} (\nu_x \times T)$ is called quasinormal.

Note that for a quasinormal representation of a binary relation $\alpha$, not all sets $Y_\alpha$ ($T \in \nu[\alpha]$) can be different from an empty set. But for this representation the following conditions are always fulfilled:

(a) $Y_\alpha \cap Y_\beta = \emptyset$, for any $T, \alpha = T' \subseteq$;

(b) $X = \bigcup_{\alpha \subseteq T} Y_\alpha$ (see ([1], Definition 1.11.1), ([2], Definition 1.11.1)).

Let $\varepsilon \in B_X(D)$. $\varepsilon$ is called right unit of the semigroup $B_X(D)$. If $\alpha \circ \varepsilon = \alpha$ for any $\alpha \in B_X(D)$. An element $\alpha$ taken from the semigroup $B_X(D)$ called a regular element of the semigroup $B_X(D)$ if in $B_X(D)$ there exists an element $\beta$ such that $\alpha \circ \beta \circ \alpha = \alpha$ (see [1]-[3]).

In [1] [2] they show that $\beta$ is regular element of $B_X(D)$ iff $V[\beta] = V(D, \beta)$ is a complete $XI$-semilattice of unions.

A complete $X$-semilattice of unions $D$ is an $XI$-semilattice of unions if it satisfies the following two conditions:

(a) $\bigwedge(D, D_t) \subseteq D$ for any $t \in \bar{D}$;

(b) $Z = \bigwedge_{t \in Z} (D, D_t)$ for any nonempty element $Z$ of $D$ (see ([1], Definition 1.14.2), ([2], Definition 1.14.2) or [4]). Under the symbol $\bigwedge(D, D_t)$ we mean an exact lower bound of the set $D_t$ in the semilattice $D$.

Let $D'$ be an arbitrary nonempty subset of the complete $X$-semilattice of unions $D$. A nonempty element $T$ is a nonlimiting element of the set $D'$ if $T \cap \nu(D', T) \neq \emptyset$ and a nonempty element $T$ is a limiting element of the set $D'$ if $T \cap \nu(D', T) = \emptyset$ (see ([1], Definition 1.13.1 and Definition 1.13.2), ([2], Definition 1.13.1 and Definition 1.13.2)).

Let $D = \{D_1, D_2, \ldots, D_{m-1}\}$ be some finite $X$-semilattice of unions and $C(D) = \{P_0, P_1, P_2, \ldots, P_m\}$. be
the family of sets of pairwise nonintersecting subsets of the set \( X \). If \( \varphi \) is a mapping of the semilattice \( D \) on the family of sets \( C(D) \) which satisfies the condition \( \varphi(D) = P_0 \) and \( \varphi(Z_i) = P_i \) for any \( i = 1, 2, \cdots, m-1 \) and \( \tilde{D}_z = D \setminus \{ T \in D \mid Z \subseteq T \} \), then the following equalities are valid:

\[
\tilde{D} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{m-1}, \quad Z_i = P_0 \cup \bigcup_{T \in D_{z}} \varphi(T)
\]

\[\bullet\]

In the sequel these equalities will be called formal.

It is proved that if the elements of the semilattice \( D \) are represented in the form \( \bullet \), then among the parameters \( P_i \) \((i = 0, 1, 2, \cdots, m-1)\) there exist such parameters that cannot be empty sets for \( D \). Such sets \( P_i \) \((0 < i \leq m-1)\) are called basis sources, whereas sets \( P_j \) \((0 \leq j \leq m-1)\) which can be empty sets too are called completeness sources.

It is proved that under the mapping \( \varphi \) the number of covering elements of the pre-image of a basis source is always equal to one, while under the mapping \( \varphi \) the number of covering elements of the pre-image of a completeness source either does not exist or is always greater than one (see ([1], Item 11.4), ([2], Item 11.4) or [5]).

The one-to-one mapping \( \varphi \) between the complete \( X \)-semilattices of unions \( \varphi(Q) \) and \( D' \) is called a complete isomorphism if the condition

\[\varphi(D') = \bigcup_{T \in D'} \varphi(T)\]

is fulfilled for each nonempty subset \( D' \) of the semilattice \( D' \) (see ([1], definition 6.3.2), ([2], definition 6.3.2) or [6]) and the complete isomorphism \( \varphi \) between the complete semilattices of unions \( Q \) and \( D' \) is a complete \( \alpha \)-isomorphism if (b)

(a) \( Q = V(D,\alpha) \);

(b) \( \varphi(\emptyset) = \emptyset \) for \( \emptyset \in V(D,\alpha) \) and \( \varphi(T) \alpha = T \) for all \( T \in V(D,\alpha) \) (see ([1], Definition 6.3.3), ([2], Definition 6.3.3)).

**Lemma 1.1.** Let \( D \) be a complete \( X \)-semilattice of unions. If a binary relation \( \varepsilon \) of the form \( \varepsilon = \bigcup_{i \in D} \big((i \times (D, D_1)) \cup ((X \setminus D) \times D)\big) \) is right unit of the semigroup \( B_X(D) \), then \( \varepsilon \) is the greatest right unit of that semigroup (see ([1], Lemma 12.1.2), ([2], Lemma 12.1.2)).

**Theorem 1.1.** Let \( D_j = \{ T_1, \cdots, T_j \} \), \( X \) and \( Y \) be three such sets, that \( \emptyset \neq Y \subseteq X \). If \( f \) is such mapping of the set \( X \), in the set \( D_j \), for which \( f(y) = T_j \) for some \( y \in Y \), then the numbers of all those mappings \( f \) of the set \( X \) in the set \( D_j \) is equal to \( s = j^{j-1} \left( j^{j-1} - (j-1)^{j-1} \right) \) (see ([1], Theorem 1.18.2), ([2], Theorem 1.18.2)).

**Theorem 2.1.** Let \( D \) be a finite \( X \)-semilattice of unions and \( \alpha \circ \sigma \circ \alpha = \alpha \) for some \( \alpha \) and \( \sigma \) of the semigroup \( B_X(D) \); \( D(\alpha) \) be the set of those elements \( T \) of the semilattice \( Q = B_X(D) \setminus \{ \emptyset \} \) which are nonlimiting elements of the set \( Q_T \). Then a binary relation \( \alpha \) having a quasinormal representation of the form \( \alpha = \bigcup_{T \in V(D,\alpha)} Y_T^{\alpha} \times T \) is a regular element of the semigroup \( B_X(D) \) iff the set \( V(D,\alpha) \) is a \( XI \)-semilattice of unions and for \( \alpha \)-isomorphism \( \varphi \) of the semilattice \( V(D,\alpha) \) on some \( X \)-subsemilattice \( D' \) of the semilattice \( D \) the following conditions are fulfilled:

(a) \( \varphi(T) = T \sigma \) for any \( T \in V(D,\alpha) \);

(b) \( \bigcup_{T \in D(\alpha)} Y_T^{\alpha} \supseteq \varphi(T) \) for any \( T \in D(\alpha) \);

(c) \( Y_T^{\alpha} \cap \varphi(T) \neq \emptyset \) for any element \( T \) of the set \( \tilde{D}(\alpha) \) (see ([1], Theorem 6.3.3), ([2], Theorem 6.3.3) or [6]).

**Theorem 3.1.** Let \( D \) be a complete \( X \)-semilattice of unions. The semigroup \( B_X(D) \) possesses a right unit iff \( D \) is an \( XI \)-semilattice of unions (see ([1], Theorem 6.1.3), ([2], Theorem 6.1.3) or [7]).
2. Results

Let \( D \) is any \( X \)-semilattice of unions and \( Q = \{T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0\} \subseteq D \), which satisfies the following conditions:

\[
\begin{align*}
T_7 &\subseteq T_6 \subseteq T_5 \subseteq T_4 \subseteq T_3 \subseteq T_2 \subseteq T_1 \subseteq T_0, \\
T_6 &\subseteq T_4 \subseteq T_3 \subseteq T_2 \subseteq T_1 \subseteq T_0, \\
T_7 \cup T_6 &\subseteq T_4, \\
T_7 &\cap T_6 \neq \emptyset, \\
T_7 &\cap T_5 \neq \emptyset, \\
T_7 &\cap T_3 \neq \emptyset, \\
T_7 &\cap T_1 \neq \emptyset,
\end{align*}
\]

(1)

The semilattice \( Q \), which satisfying the conditions (1) is shown in Figure 1. By the symbol \( \Sigma_\{X,8\} \) we denote the set of all \( X \)-semilattices of unions whose every element is isomorphic to \( Q \).

Let \( C(Q) = \{P_7, P_6, P_5, P_4, P_3, P_2, P_1, P_0\} \) is a family sets, where \( P_7, P_6, P_5, P_4, P_3, P_2, P_1, P_0 \) are pairwise disjoint subsets of the set \( X \) and

\[
\psi = \left( \begin{array}{c}
T_7 \\
T_6 \\
T_5 \\
T_4 \\
T_3 \\
T_2 \\
T_1 \\
T_0 \\
P_7 \\
P_6 \\
P_5 \\
P_4 \\
P_3 \\
P_2 \\
P_1 \\
P_0
\end{array} \right)
\]

is a mapping of the semilattice \( Q \) into the family sets \( C(Q) \). Then for the formal equalities of the semilattice \( Q \) we have a form:

\[
\begin{align*}
T_0 &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7, \\
T_1 &= P_0 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7, \\
T_2 &= P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7, \\
T_3 &= P_0 \cup P_1 \cup P_2 \cup P_4 \cup P_5 \cup P_6 \cup P_7, \\
T_4 &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_5 \cup P_6 \cup P_7, \\
T_5 &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_6 \cup P_7, \\
T_6 &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_7, \\
T_7 &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6.
\end{align*}
\]

(2)

here the elements \( P_1, P_2, P_3, P_5 \) are basis sources, the element \( P_0, P_4, P_6, P_7 \) are sources of completenes of the semilattice \( Q \). Therefore \( |X| \geq 4 \) and \( \delta = 4 \).

**Theorem 2.1.** Let \( Q = \{T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0\} \in \Sigma_\{X,8\} \). Then \( Q \) is \( XI \)-semilattice, when \( T_5 \cap T_7 = \emptyset \).

**Proof.** Let \( t \in T_0 \), \( Q_t = \{T \in Q \mid t \in T\} \) and \( \land(Q_t, Q_t) \) is the exact lower bound of the set \( Q_t \) in \( Q \). Then of the formal equalities (2) follows, that

\[
Q_t = \begin{cases}
\emptyset, & \text{if } t \in P_0, \\
\{T_7, T_2, T_0\}, & \text{if } t \in P_1, \\
T_7, & \text{if } t \in P_2, \\
\{T_5, T_4, T_3, T_1, T_0\}, & \text{if } t \in P_3, \\
\{T_6, T_4, T_2, T_1, T_0\}, & \text{if } t \in P_4, \\
\{T_6, T_5, T_4, T_2, T_1, T_0\}, & \text{if } t \in P_5, \\
\{T_6, T_5, T_4, T_3, T_2, T_1, T_0\}, & \text{if } t \in P_6, \\
\{T_6, T_5, T_4, T_3, T_2, T_1, T_0\}, & \text{if } t \in P_7,
\end{cases}
\]

\[
\land(Q_t, Q_t) = \begin{cases}
T_7, & \text{if } t \in P_1, \\
T_6, & \text{if } t \in P_2, \\
T_7, & \text{if } t \in P_3, \\
T_6, & \text{if } t \in P_4, \\
T_6, & \text{if } t \in P_5, \\
T_6, & \text{if } t \in P_6, \\
T_7, & \text{if } t \in P_7.
\end{cases}
\]
We have $Q^\ast = \{ \land (Q, Q) | t \in T_0 \} = \{ T_7, T_6, T_5, T_4 \}$ and $\land (Q, Q) \not\in Q$ if $t \in P_0 \cup P_2 \cup P_6 \cup P_3$. So, from the definition, XI-semilattice follows that $Q$ is not XI-semilattice.

If $P_0 = P_2 = P_6 = P_3 = \emptyset$ (since they are completeness sources), then $\land (Q, Q) \in Q$ for all $t \in T_0$ and $T_4 = T_5 \cup T_6$, $T_1 = T_7 \cup T_8$, $T_2 = T_9 \cup T_3$. Of the last conditions and from the Definition XI-semilattice follows that $Q$ is XI-semilattice. Of the equality $P_0 = P_2 = P_6 = P_3 = \emptyset$ follows that $Q$ is XI-semilattice.

Theorem is proved.

Lemma 2.1. Let $Q = \{ T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0 \} \in \Sigma_2 (X, 8)$ and $T_9 \cap T_3 = \emptyset$. Then following equalities are true:

$$P_1 = T_7, \quad P_2 = T_8, \quad P_3 = T_9 \setminus T_2, \quad P_4 = T_1 \setminus T_1$$

Proof. The given Lemma immediately follows from the formal equalities (2) of the semilattice $Q$. The lemma is proved.

Lemma 2.2. Let $Q = \{ T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0 \} \in \Sigma_2 (X, 8)$ and $T_9 \cap T_3 = \emptyset$. Then the binary relation

$$\varepsilon = (T_7 \times T_1) \cup (T_6 \times T_6) \cup ((T_5 \setminus T_1) \times T_1) \cup ((T_5 \setminus T_2) \times T_2) \cup ((X \setminus T_0) \times T_0)$$

is the largest right unit of the semigroup $B_X (D)$.

Proof. By preposition and from Theorem 2.1 follows that $Q$ is XI-semilattice. Of this, from Lemma 1.1, from Lemma 2.1 and from Theorem 1.3 we have that the binary relation

$$\varepsilon = \bigcup_{i \in \mathbb{N}} (\{ i \} \times \land (Q, Q)) \cup ((X \setminus T_0) \times T_0) = (P_1 \times T_7) \cup (P_2 \times T_6) \cup (P_3 \times T_5) \cup (P_4 \times T_4) \cup (P_5 \times T_3) \cup (P_6 \times T_2) \cup (P_7 \times T_1)$$

is the largest right unit of the semigroup $B_X (D)$.

The lemma is proved.

Lemma 2.3. Let $Q = \{ T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0 \} \in \Sigma_2 (X, 8)$ and $T_9 \cap T_3 = \emptyset$. Binary relation $\alpha$ having quasi-normal representation of the form

$$\alpha = \{ Y_1 \times T_7 \} \cup \{ \ldots \} \cup \{ \ldots \} \cup \{ \ldots \} \cup \{ \ldots \} \cup \{ \ldots \} \cup \{ \ldots \} \cup \{ \ldots \}$$

where $Y_1, Y_2, Y_3 \not\in \emptyset$ and $V (D, \alpha) = Q \in \Sigma_2 (X, 8)$ is a regular element of the semigroup $B_X (D)$ isomorphic for some complete $\alpha$-isomorphism $\varphi = \{ T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0 \}$ of the semilattice $Q$ on some $X$-subsemilattice $Q' = \{ T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0 \}$ (see Figure 2) of the semilattice $Q$ satisfies the following conditions:

$$Y_1 \not\subseteq T_7, \quad Y_2 \not\subseteq T_6, \quad Y_3 \not\subseteq T_5, \quad Y_4 \not\subseteq T_4, \quad Y_5 \not\subseteq T_3, \quad Y_6 \not\subseteq T_2, \quad Y_7 \not\subseteq T_1, \quad Y_8 \not\subseteq \emptyset, \quad Y_9 \not\subseteq \emptyset$$

Proof. It is easy to see, that the set $Q (\alpha) = \{ T_7, T_6, T_5, T_4, T_3, T_2, T_1 \}$ is a generating set of the semilattice $Q$.

Then the following equalities are hold:
By Statement b) of the Theorem 1.2 follows that the following conditions are true:

\[ Y^a_7 \supseteq \bar{T}_1, \quad Y^a_6 \supseteq \bar{T}_6, \quad Y^a_5 \cup Y^a_6 \supseteq \bar{T}_5 \cup \bar{T}_6, \quad Y^a_7 \cup Y^a_6 \cup Y^a_4 \cup Y^a_2 \supseteq \bar{T}_7 \cup \bar{T}_6 \cup \bar{T}_4 \cup \bar{T}_2, \quad Y^a_7 \cup Y^a_6 \cup Y^a_4 \cup Y^a_2 \supseteq \bar{T}_7 \cup \bar{T}_6 \cup \bar{T}_4 \cup \bar{T}_2, \]

\[ Y^a_7 \cup Y^a_6 \cup Y^a_5 \cup Y^a_4 \cup Y^a_2 = (Y^a_7 \cup Y^a_5) \cup (Y^a_7 \cup Y^a_6 \cup Y^a_4) \cup Y^a_2 \supseteq \bar{T}_7 \cup \bar{T}_5 \cup \bar{T}_6 \cup \bar{T}_2, \quad Y^a_7 \cup Y^a_6 \cup Y^a_5 \cup Y^a_4 \cup Y^a_2 = \bar{T}_7 \cup \bar{T}_5 \cup \bar{T}_6 \cup \bar{T}_2, \]

i.e., the inclusions \( Y^a_7 \cup Y^a_6 \cup Y^a_5 \cup Y^a_4 \cup Y^a_2 \supseteq \bar{T}_7 \), \( Y^a_7 \cup Y^a_6 \cup Y^a_5 \cup Y^a_4 \cup Y^a_2 \supseteq \bar{T}_7 \), \( Y^a_7 \cup Y^a_6 \cup Y^a_5 \cup Y^a_4 \cup Y^a_2 \supseteq \bar{T}_7 \), \( Y^a_7 \cup Y^a_6 \cup Y^a_5 \cup Y^a_4 \cup Y^a_2 \supseteq \bar{T}_7 \) are always hold. Further, it is to see, that the following conditions are true:

\[ l(\bar{T}_7 \setminus \{T_1\}) = \emptyset, \quad \bar{T}_7 \setminus l(\bar{T}_6 \setminus \{T_1\}) = T_6 \setminus \emptyset \neq \emptyset; \]

\[ l(\bar{T}_7 \setminus \{T_5\}) = \emptyset, \quad \bar{T}_7 \setminus l(\bar{T}_6 \setminus \{T_5\}) = T_6 \setminus \emptyset \neq \emptyset; \]

\[ l(\bar{T}_7 \setminus \{T_4\}) = \emptyset, \quad \bar{T}_7 \setminus l(\bar{T}_6 \setminus \{T_4\}) = T_6 \setminus \emptyset \neq \emptyset; \]

\[ l(\bar{T}_7 \setminus \{T_2\}) = \emptyset, \quad \bar{T}_7 \setminus l(\bar{T}_6 \setminus \{T_2\}) = T_6 \setminus \emptyset \neq \emptyset; \]

\[ l(\bar{T}_7 \setminus \{T_1\}) = \emptyset, \quad \bar{T}_7 \setminus l(\bar{T}_6 \setminus \{T_1\}) = T_6 \setminus \emptyset \neq \emptyset; \]

i.e., \( T_7, T_6, T_5, T_3 \) are nonlimiting elements of the sets \( \bar{Q}(\alpha)_{T_7} \), \( \bar{Q}(\alpha)_{T_6} \), \( \bar{Q}(\alpha)_{T_5} \) and \( \bar{Q}(\alpha)_{T_3} \) respectively. By Statement c) of the Theorem 1.2 it follows, that the conditions \( Y^a_7 \cap \bar{T}_7 \neq \emptyset \), \( Y^a_6 \cap \bar{T}_6 \neq \emptyset \), \( Y^a_5 \cap \bar{T}_5 \neq \emptyset \) and \( Y^a_4 \cap \bar{T}_4 \neq \emptyset \) are hold. Since \( Z_7 \subset Z_5 \), \( Z_6 \subset Z_3 \) we have \( Y^a_5 \cap \bar{T}_5 \neq \emptyset \) and \( Y^a_4 \cap \bar{T}_4 \neq \emptyset \).

Therefore the following conditions are hold:

\[ Y^a_7 \supseteq \bar{T}_1, \quad Y^a_6 \supseteq \bar{T}_6, \quad Y^a_5 \cup Y^a_6 \supseteq \bar{T}_5, \quad Y^a_7 \cup Y^a_6 \cup Y^a_4 \supseteq \bar{T}_7 \cup \bar{T}_6 \cup \bar{T}_4, \quad Y^a_7 \cup Y^a_6 \cup Y^a_4 \cup Y^a_2 \supseteq \bar{T}_7 \cup \bar{T}_6 \cup \bar{T}_4 \cup \bar{T}_2, \quad Y^a_7 \cup Y^a_6 \cup Y^a_4 \cup Y^a_2 \supseteq \bar{T}_7 \cup \bar{T}_6 \cup \bar{T}_4 \cup \bar{T}_2, \]

The lemma is proved.

**Definition 2.1.** Assume that \( Q' \in \Sigma_7(X,8) \). Denote by the symbol \( R(Q') \) the set of all regular elements \( \alpha \) of the semigroup \( B_X(D) \), for which the semilattices \( Q' \) and \( Q \) are mutually \( \alpha \)-isomorphic and \( V(D,\alpha) = Q' \).

It is easy to see the number \( q \) of automorphism of the semilattice \( Q \) is equal to 2.

**Theorem 2.2.** Let \( \bar{Q} = \{T_7, T_6, T_5, T_1, T_3, T_2, T_4, T_0\} \in \Sigma_7(X,8) \), \( T_5 \cap \bar{T}_3 = \emptyset \) and \( |\Sigma_7(X,8)| = m_0 \). If \( X \) be finite set, and the \( XI \)-semilattice \( Q \) and \( Q' = \{\bar{T}_7, \bar{T}_6, \bar{T}_5, \bar{T}_3, \bar{T}_2, \bar{T}_1, \bar{T}_0\} \) are \( \alpha \)-isomorphic, then
\[ |R(Q)| = 2 \cdot m_0 \cdot \left(2^{|T|} - 1\right) \cdot \left(2^{|T|} - 1\right) \cdot 8^{|T|} \]

**Proof.** Assume that \( \alpha \in R(Q) \). Then a quasinormal representation of a regular binary relation \( \alpha \) has the form

\[ \alpha = (Y_a \times T_a) \cup (Y_a \times T_b) \cup (Y_a \times T_c) \cup (Y_a \times T_d) \cup (Y_a \times T_e) \cup (Y_a \times T_f) \cup (Y_a \times T_g) \]

where \( Y_a, Y_b, Y_c, Y_d, Y_e, Y_f, Y_g \) and by Lemma 2.3 satisfies the conditions: \( X \)

\[ Y_a \supseteq T_a, \ Y_a \supseteq T_b, \ Y_a \cup Y_a \supseteq T_c, \ Y_a \cup Y_a \supseteq T_d, \ Y_a \cup T_a \supseteq T_e, \ Y_a \cup T_a \supseteq T_f, \ Y_a \cup T_a \supseteq T_g \] (3)

Let \( f_\alpha \) is a mapping the set \( X \) in the semilattice \( Q \) satisfying the conditions \( f_\alpha(t) = \alpha t \) for all \( t \in X \).

\( f_{1a}, f_{2a}, f_{3a}, f_{4a}, f_{5a} \) are the restrictions of the mapping \( f_\alpha \) on the sets \( T_a, T_b, T_c, T_d, T_e, X \setminus T_a \) respectively. It is clear, that the intersection disjoint elements of the set \( \{T_a, T_b, T_c, T_d, T_e, X \setminus T_a\} \) are empty set and \( T_a \cup T_b \cup (T_c \setminus T_a) \cup (T_d \setminus T_a) \cup (X \setminus T_a) = X \).

We are going to find properties of the maps \( f_{1a}, f_{2a}, f_{3a}, f_{4a}, f_{5a} \).

1) \( t \in T_a \). Then by Property (3) we have \( t \in T_a \subseteq Y_a \), i.e., \( t \in Y_a \) and \( t \alpha = T_a \) by definition of the set \( Y_a \). Therefore \( f_{1a}(t) = T_a \) for all \( t \in T_a \).

2) \( t \in T_b \). Then by Property (3) we have \( t \in T_b \subseteq Y_b \), i.e., \( t \in Y_b \) and \( t \alpha = T_b \) by definition of the set \( Y_b \). Therefore \( f_{2a}(t) = T_b \) for all \( t \in T_b \).

3) \( t \in T_c \setminus T_a \). Then by Property (3) we have \( t \in T_c \setminus T_a \subseteq Y_b \cup Y_a \), i.e., \( t \in Y_b \cup Y_a \) and \( t \alpha \in \{T_c, T_a\} \) by definition of the sets \( Y_b \) and \( Y_a \). Therefore \( f_{3a}(t) \in \{T_c, T_a\} \) for all \( t \in T_c \setminus T_a \).

Preposition we have that \( Y_c \cap T_a \neq \emptyset \), i.e. \( t \alpha = T_a \) for some \( t \in T_a \). If \( t \in T_a \), then \( t \alpha = T_a \cap T_a \neq \emptyset \), i.e. \( t \alpha = T_a \) and \( t \alpha = T_a \) by definition of the sets \( Y_c \cap T_a \neq \emptyset \), i.e. \( t \alpha = T_a \cap T_a \neq \emptyset \). Therefore \( f_{3a}(t) = T_c \) for some \( t \in T_c \).

4) \( t \in T_c \setminus T_b \). Then by Property (3) we have \( t \in T_c \setminus T_b \subseteq Y_c \cup Y_b \), i.e., \( t \in Y_c \cup Y_b \) and \( t \alpha \in \{T_c, T_b\} \) by definition of the sets \( Y_c \) and \( Y_b \). Therefore \( f_{4a}(t) \in \{T_c, T_b\} \) for all \( t \in T_c \setminus T_b \).

Preposition we have that \( Y_b \cap T_c \neq \emptyset \), i.e. \( t \alpha = T_b \) for some \( t \in T_c \). If \( t \in T_c \), then \( t \alpha = T_b \cap T_c \neq \emptyset \), i.e. \( t \alpha = T_b \) and \( t \alpha = T_c \) by definition of the sets \( Y_b \cap T_c \neq \emptyset \), i.e. \( t \alpha = T_b \cap T_c \neq \emptyset \). Therefore \( f_{4a}(t) = T_b \) for some \( t \in T_b \).

5) \( t \in X \setminus T_0 \). Then by definition quasinormal representation binary relation \( \alpha \) and by Property (3) we have \( t \in X \setminus T_0 \subseteq X = Y_a \cup Y_b \cup Y_c \cup Y_d \cup Y_e \cup Y_f \cup Y_g \), i.e. \( t \alpha \in \{T_0, T_1, T_2, T_3, T_4, T_5, T_6, X \} \) by definition of the sets \( Y_a, Y_b, Y_c, Y_d, Y_e, Y_f, Y_g \). Therefore \( f_{5a}(t) \in \{T_0, T_1, T_2, T_3, T_4, T_5, T_6, X \} \) for all \( t \in X \setminus T_0 \).

Therefore for every binary relation \( \alpha \in R(Q) \) exist ordered system \( (f_{1a}, f_{2a}, f_{3a}, f_{4a}, f_{5a}) \). It is obvious that for different binary relations exist different ordered systems.

Let \( f_1 : T_1 \rightarrow \{T_1\}, \ f_2 : T_2 \rightarrow \{T_2\}, \ f_3 : T_3 \setminus T_2 \rightarrow \{T_3, T_2\}, \ f_4 : T_3 \setminus T_1 \rightarrow \{T_3, T_1\}, \ f_5 : X \setminus T_0 \rightarrow Q \) are such mappings, which satisfying the conditions:

6) \( f_1(t) \in \{T_1\} \) for all \( t \in T_1 \);

7) \( f_2(t) \in \{T_2\} \) for all \( t \in T_2 \);
8) $f_i(t) \in \{T_e, T_i\}$ for all $t \in \overline{T_i} \setminus T_i$ and $f_i(t_i) = T_i$ for some $t_i \in \overline{T_i} \setminus T_i$;  
9) $f_4(t) \in \{T_e, T_i\}$ for all $t \in \overline{T_i} \setminus T_i$ and $f_4(t_4) = Z_4$ for some $t_4 \in \overline{T_i} \setminus T_i$;  
10) $f_3(t) \in \{T_e, T_0, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$ for all $t \in X \setminus \overline{T_0}$.

Now we define a map $f$ of a set $X$ in the semilattice $Q$, which satisfies the following condition:

$$f(t) = \begin{cases} f_1(t), & \text{if } t \in T_0, \\
 f_2(t), & \text{if } t \in \overline{T_2} \setminus T_2, \\
 f_3(t), & \text{if } t \in \overline{T_3} \setminus T_3, \\
 f_4(t), & \text{if } t \in \overline{T_4} \setminus T_4, \\
 f_5(t), & \text{if } t \in \overline{T_5} \setminus T_5. 
\end{cases}$$

Now let $\beta = \bigcup_{x \in X} (\{x\} \times f(x))$, $Y_\beta^i = \{t \in T_i \mid Y \subseteq T_i\}$ for all $i = 1, 2, \ldots, 5$. Then binary relation $\beta$ is written in the form

$$\beta = \bigcup \bigcup Y_\beta^i \times T_i \cup \bigcup \bigcup Y_\beta^i \times T_i \cup \bigcup \bigcup Y_\beta^i \times T_i \cup \bigcup \bigcup Y_\beta^i \times T_i \cup \bigcup \bigcup Y_\beta^i \times T_i \cup \bigcup \bigcup Y_\beta^i \times T_i$$

and satisfying the conditions:

$$Y_\beta^i \supseteq T_i, \quad Y_\beta^i \supseteq T_i, \quad Y_\beta^i \cup Y_\beta^i \supseteq T_i, \quad Y_\beta^i \cup Y_\beta^i \supseteq T_i, \quad Y_\beta^i \cap T_i \cap \emptyset = \emptyset, \quad Y_\beta^i \cap T_i \neq \emptyset$$

From this and by Lemma 2.3 we have that $\beta \in R(Q')$.

Therefore for every binary relation $\alpha \in R(Q')$, and ordered system $(f_{1a}, f_{2a}, f_{3a}, f_{4a}, f_{5a})$ exist one to one mapping.

By Theorem 1.1 the number of the mappings $f_{1a}, f_{2a}, f_{3a}, f_{4a}, f_{5a}$ are respectively:

$$1, 1, 2^{|T_1|} - 1, 2^{|T_2|} - 1, 8^{|T_5|}$$

(see ([1], Corollary 1.18.1), ([2], Corollary 1.18.1)).

The number of ordered system $(f_{1a}, f_{2a}, f_{3a}, f_{4a}, f_{5a})$ or number regular elements can be calculated by the formula

$$|R(Q')| = 2^{|T_1|} \cdot \left(2^{|T_2|} - 1\right) \cdot \left(2^{|T_3|} - 1\right) \cdot 8^{|T_5|}$$

(see ([1], Theorem 6.3.5), ([2], Theorem 6.3.5)).

The theorem is proved.

**Corollary 2.1.** Let $Q = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\} \subseteq \Sigma_2(X, 8)$, $T_3 \cap T_6 = \emptyset$. If $X$ be a finite set and $E^{(e)}(Q)$ be the set of all right units of the semigroup $B_X(Q)$, then the following formula is true

$$|E^{(e)}(Q)| = \left(2^{|T_1|} - 1\right) \cdot \left(2^{|T_2|} - 1\right) \cdot 8^{|T_5|}$$

**Proof:** This corollary immediately follows from Theorem 2.2 and from the ([1], Theorem 6.3.7) or ([2], Theorem 6.3.7).

The corollary is proved.

**References**


