An abstract text in English about the study of ground states for a class of nonlinear Schrödinger-Poisson systems with positive potentials.
Ruiz [3] obtained some multiplicity results for $p \in (2, 3)$, and existence results for $p \in [3, 6)$. Later, Ambrosetti and Ruiz [4], and Ambrosetti [5] generalized some existence results of Ruiz [3], and obtained the existence of infinitely solutions for the problem (1.1).

In particular, Sanchel and Soler [6] considered the following Schrodinger-Poisson-Slater systems

$$
\begin{cases}
-\Delta u + \omega u + \Phi u - |u|^2 u = 0, & x \in \mathbb{R}^3, \\
-\Delta \Phi = u^2, & x \in \mathbb{R}^3,
\end{cases}
$$

(1.2)

where $\omega \in \mathbb{R}$. The problem (1.2) was introduced as the model of the Hartree-Fock theory for a one-component plasma. The solution is obtained by using the minimization argument and $\omega$ as a Lagrange multiplier.

However, it is not known if the solution for the problem (1.2) is radial. Mugani [7] considered the following generalized Schrodinger-Poisson systems

$$
\begin{cases}
-\Delta u + \omega u - \lambda \Phi u + W(x, u) = 0, & x \in \mathbb{R}^3, \\
-\Delta \Phi = u^2, & x \in \mathbb{R}^3,
\end{cases}
$$

(1.3)

where $\omega \in \mathbb{R}$, $\lambda > 0$ and $W(x, s) = W(|s|, s)$, and proved the existence of radially symmetric solitary waves for the problem (1.3).

In this paper, without requiring any symmetry assumptions on $V(x)$, $\rho(x)$ and $Q(x)$, we obtain the existence of positive radial ground state solution for the problem (1.1). In particular, the positive potential $Q(x)$ implies that we are dealing with systems of particles having positive mass. It is interesting in physical applications.

The paper is organized as following. In Section 2, we collect some results and state our main result. In Section 3, we prove some lemmas and consider the problem (1.1) at infinity. Section 4 is devoted to our main theorem.

2. Preliminaries and Main Results

Let $L^s\left(\mathbb{R}^3\right)$, $1 \leq s < +\infty$ denotes a Lebesgue space, the norm in $L^s\left(\mathbb{R}^3\right)$ is $|u|_s = \left(\int_{\mathbb{R}^3} |u|^s \, dx\right)^{\frac{1}{s}}$, $D^{1,2}\left(\mathbb{R}^3\right)$ is the completion of $C^\infty_0\left(\mathbb{R}^3\right)$ with respect to the norm

$$
\|u\|_{D^{1,2}} = \left(\int_{\mathbb{R}^3} \left|\nabla u\right|^2 \, dx\right)^{\frac{1}{2}}
$$

$H^1\left(\mathbb{R}^3\right)$ be the usual Sobolev space with the usual norm

$$
\|u\|_{H^1} = \left(\int_{\mathbb{R}^3} \left(\left|\nabla u\right|^2 + |u|^2\right) \, dx\right)^{\frac{1}{2}}.
$$

Assume that the potential $V(x)$ satisfies

$H1) \quad V(x) \in C^1\left(\mathbb{R}^3, \mathbb{R}\right)$, $\inf_{x \in \mathbb{R}^3} V(x) = 1$, $V(x) \leq V_\infty = \lim_{R \to +\infty} V(x) < \infty$.

Let $H^s_0\left(\mathbb{R}^3\right)$ be the Hilbert subspace of $u \in H^1\left(\mathbb{R}^3\right)$ such that

$$
\|u\|_{H^s_0} = \left(\int_{\mathbb{R}^3} \left(\left|\nabla u\right|^2 + V(x) |u|^2\right) \, dx\right)^{\frac{1}{2}}
$$

(2.1)

Then $H^s_0\left(\mathbb{R}^3\right) \subset H^1\left(\mathbb{R}^3\right) \subset L^s\left(\mathbb{R}^3\right)$, $2 \leq s \leq 6$ with the corresponding embeddings being continuous (see [8]). Furthermore, assume the potential $\rho(x)$ satisfies

$H2) \quad \rho(x) > 0$, $\lim_{|x| \to +\infty} \rho(x) = \rho_\infty > 0$, $\rho_0(x) = \rho(x) - \rho_\infty \in L^2\left(\mathbb{R}^3\right)$. 

It is easy to reduce the problem (1.1) to a single equation with a non-local term. Indeed, for every \( v \in D^{1,2}(\mathbb{R}^3) \), we have

\[
\int_{\mathbb{R}^3} \rho(x) u^2 \, dx = \int_{\mathbb{R}^3} \left( (\rho(x) - \rho_{\infty}) u^2 + \int_{\mathbb{R}^3} \rho_{\infty} u^2 \, dx \right) \leq \int_{\mathbb{R}^3} \left( |\rho(x) - \rho_{\infty}| \, |u|^2 + \int_{\mathbb{R}^3} \rho_{\infty} |u|^2 \, dx \right) \leq \int_{\mathbb{R}^3} \left( |\rho_{\infty}| \, |u|^2 + \frac{1}{\lambda_{\infty}} \left( \int_{\mathbb{R}^3} (u^2)^{\frac{6}{5}} \, dx \right)^{\frac{5}{3}} + \frac{1}{\lambda_{\infty}} \right) \left( \int_{\mathbb{R}^3} (v^6)^{\frac{5}{3}} \, dx \right)^{\frac{3}{5}} \quad (2.2)
\]

Since \( \rho_{\infty}(x) = \rho(x) - \rho_{\infty} \in L^2(\mathbb{R}^3) \), \( u \in H^1(\mathbb{R}^3) \) and (2.1), by the Lax-Milgram theorem, there exists a unique \( \Phi[u] \) such that

\[
\int_{\mathbb{R}^3} \nabla \Phi[u] \nabla v \, dx = \int_{\mathbb{R}^3} \rho(x) u^2 \, dx, \quad \forall v \in D^{1,2}(\mathbb{R}^3)
\]

It follows that \( \Phi[u] \) satisfies the Poisson equation

\[-\Delta \Phi[u] = \rho(x) u^2
\]

and there holds

\[-\Phi[u] = \int_{\mathbb{R}^3} \frac{\rho(x) u^2(y)}{|x - y|} \, dy = \frac{1}{|x|} \rho(x) u^2.
\]

Because \( \rho(x) > 0 \), we have \( \Phi[u] > 0 \) when \( u \neq 0 \), and \( \|\Phi[u]\|_{L^6} = M \|u\|_{L^6} \), \( M \) is positive constant. Substituting \( \Phi[u] \) in to the problem (1.1), we are lead to the equation with a non-local term

\[-\Delta u + V(x) u - \lambda \rho(x) \Phi[u] u + Q(x) |u|^{p-2} u = 0. \quad (2.4)
\]

In the following, we collect some properties of the functional \( \Phi[u] \), which are useful to study our problem.

**Lemma 2.1.** [9] For any \( u \in H^1(\mathbb{R}^3) \), we have

1) \( \Phi[u] : H^1(\mathbb{R}^3) \rightarrow D^{1,2}(\mathbb{R}^3) \) is continuous, and maps bounded sets into bounded sets;

2) if \( u_n \rightharpoonup u \) weakly in \( H^1(\mathbb{R}^3) \), then \( \Phi[u_n] \rightharpoonup \Phi[u] \) weakly in \( D^{1,2}(\mathbb{R}^3) \);

3) \( \Phi[\delta u(x)] = \delta^2 \Phi[u(x)] \) for all \( t \in \mathbb{R} \).

Now, we state our main theorem in this paper.

**Theorem 2.2.** Assume that \( \lambda > 0 \), \( 2 < p < 4 \), the potential \( V(x) \) satisfies condition \( H1 \), the potential \( \rho(x) \) satisfies condition \( H3 \) and \( \rho(x) \geq \rho_{\infty} \), the potential \( Q(x) \) satisfies

\( H3 \quad Q(x) > 0, \quad \lim_{|x|\to\infty} Q(x) = Q_{\infty} > 0, \quad Q_0(x) = Q(x) - Q_{\infty} \in L^{6-p}(\mathbb{R}^3) \)

and \( Q(x) \leq Q_{\infty}, \quad Q_0(x) \leq Q_{\infty} < 0 \) on positive measure. Then there exists a positive radial ground state solution for the problem (1.1).

**Remark 2.3.** If \( \lambda \leq 0 \), \( V(x) \), \( \rho(x) \) and \( Q(x) \) are positive potentials defined in \( \mathbb{R}^3 \), and \( 2 < p < 6 \), \( (u, \Phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \) be a solution for the problem (1.1). Then \( (u, \Phi) = (0, 0) \), Indeed, we have

\[
0 = \int_{\mathbb{R}^3} \left( |\nabla u|^2 + V(x) u^2 \right) \, dx - \lambda \int_{\mathbb{R}^3} \rho(x) \Phi[u] u^2 \, dx + \int_{\mathbb{R}^3} Q(x) |u|^p \, dx \geq \int_{\mathbb{R}^3} \left( |\nabla u|^2 + V(x) u^2 \right) \, dx.
\]
Since \( V(x) > 0 \), this implies \( u = 0 \). By Lemma 2.1, we have \( \Phi = 0 \).

### 3. Some Lemmas and the Problem (1.1) at Infinity

Now, we consider the functional \( I_\lambda : H^1_v (\mathbb{R}^3) \to \mathbb{R} \) given by

\[
I_\lambda (u) = \frac{1}{2} \int_{\mathbb{R}^3} \left( | \nabla u |^2 + V(x) u^2 \right) \, dx - \frac{\lambda}{4} \int_{\mathbb{R}^3} \rho(x) \Phi [u] u^2 \, dx + \frac{1}{p} \int_{\mathbb{R}^3} Q(x) |u|^p \, dx
\]

\[
= \frac{1}{2} \| u \|_{H^1_v}^2 - \frac{\lambda}{4} \int_{\mathbb{R}^3} \rho(x) \Phi [u] u^2 \, dx + \frac{1}{p} \int_{\mathbb{R}^3} Q(x) |u|^p \, dx.
\]

Since \( \rho(x) \) satisfies condition H2), by (2.2), the Holder inequality and Sobolev inequality, we have

\[
\int_{\mathbb{R}^3} \rho(x) \Phi [u] u^2 \, dx \leq \mathcal{S}^{-2} \left[ \| u \|_{L^2_v}^2 + \rho \right] S_{\rho}^{\frac{2}{2}} \| u \|_{L^2_v} \| u \|_{L^{\frac{4}{3}}}^{\frac{4}{3}},
\]

where \( \mathcal{S} = \inf_{u \in H^1_v (\mathbb{R}^3)} \| u \|_{H^1_v}^2 \) and \( S = \inf_{u \in H^1_v (\mathbb{R}^3)} \| u \|_{L^2_v} \| u \|_{L^{\frac{4}{3}}}^{\frac{4}{3}} \). Since the potential \( Q(x) \) satisfies condition Q,

\( 2 < p < 4 \), we have

\[
\int_{\mathbb{R}^3} Q(x) |u|^p \, dx \leq \int_{\mathbb{R}^3} \left( Q(x) - Q_* \right) |u|^p \, dx + \int_{\mathbb{R}^3} Q_* |u|^p \, dx
\]

\[
\leq \int_{\mathbb{R}^3} \left( Q_0(x) \right) |u|^p \, dx + \int_{\mathbb{R}^3} Q_* |u|^p \, dx
\]

\[
\leq \left( \int_{\mathbb{R}^3} Q_0(x) \left( \frac{6}{5-p} \right) \, dx \right)^{\frac{6-p}{3}} \left( \int_{\mathbb{R}^3} |u|^p \, dx \right)^{\frac{5}{3}} + Q_* \int_{\mathbb{R}^3} |u|^p \, dx.
\]

By Sobolev inequality, we obtain that

\[
\int_{\mathbb{R}^3} Q(x) |u|^p \, dx \leq M \| u \|_{H^1_v}^p
\]

Combining (3.2) and (3.3), we obtain that the functional \( I_\lambda \) is a well defined \( C^1 \) functional, and if \( u \in H^1_v (\mathbb{R}^3) \) is critical point of it, then the pair \( (u, \Phi [u]) \) is a weak solution of the problem (1.1).

Now, we define the Nehari manifold ([10]) of the functional \( I_\lambda \)

\[
N_\lambda = \left\{ u \in H^1_v (\mathbb{R}^3) \backslash \{ 0 \} : H^1_\lambda (u) = 0 \right\},
\]

where

\[
H^1_\lambda (u) = I'_\lambda (u) [u] = \| u \|_{H^1_v}^2 - \lambda \int_{\mathbb{R}^3} \rho(x) \Phi [u] u^2 \, dx + \int_{\mathbb{R}^3} Q(x) |u|^p \, dx
\]

Hence, we have

\[
I'_\lambda (u) \big|_{N_\lambda} = \frac{1}{4} \| u \|_{H^1_v}^4 + \left( \frac{1}{p} - \frac{1}{4} \right) \int_{\mathbb{R}^3} Q(x) |u|^p \, dx
\]

\[
\left( \frac{1}{2} - \frac{1}{p} \right) \| u \|_{H^1_v}^2 + \lambda \left( \frac{1}{p} - \frac{1}{4} \right) \int_{\mathbb{R}^3} \rho(x) \Phi [u] u^2 \, dx
\]

\[
\left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^3} Q(x) |u|^p \, dx - \frac{\lambda}{4} \int_{\mathbb{R}^3} \rho(x) \Phi [u] u^2 \, dx.
\]
Lemma 3.1. 1) For any $\lambda > 0$, $u \in H^{1}_p \left( \mathbb{R}^3 \right) \setminus \{0\}$, there exists a unique $t(u) > 0$ such that $t(u)u \in N_{\lambda}$. Moreover, we have $I_{\lambda} \left( t(u)u \right) = \max_{t \geq 0} I_{\lambda} \left( tu \right)$.

2) $I_{\lambda}(u)$ is bounded from below on $N_{\lambda}$ by a positive solution.

Proof. 1) Taking any $u \in H^{1}_p \left( \mathbb{R}^3 \right) \setminus \{0\}$ and $\|u\|_{H^1} = 1$, we obtain that there exists a unique $t(u) > 0$ such that $t(u)u \in N_{\lambda}$. Indeed, we define the function $g(t) = I_{\lambda} \left( tu \right)$. We note that $g'(t) = (I'_{\lambda} \left( tu \right), v) = 0$ if only if $tu \in N_{\lambda}$. Since $g'(t) = 0$ is equivalent to

$$t^2 \|u\|_{H^1}^2 - \lambda t^2 \int_{\mathbb{R}^3} \rho(x) \Phi \left( u^2 \right) dx + t^\rho \int_{\mathbb{R}^3} Q(x) |u|^\rho dx = 0.$$ 

By $\rho(x) > 0$ and $\Phi[u] > 0$, we have

$$b = \int_{\mathbb{R}^3} \rho(x) \Phi \left( u^2 \right) dx > 0,$$

$$c = \int_{\mathbb{R}^3} Q(x) |u|^\rho dx > 0.$$ 

By $\lambda > 0$, $2 < p < 4$, the equation $1 - \lambda \hbar t^2 + ct^{p-2} = 0$ has a unique $t(u) > 0$ and the corresponding point $t(u)u \in N_{\lambda}$ and $I_{\lambda} \left( t(u)u \right) = \max_{t \geq 0} I_{\lambda} \left( tu \right)$.

2) Let $u \in N_{\lambda}$, by (3.4) and $2 < p < 4$, we have

$$I_{\lambda}(u) = \left( \frac{1}{2} - \frac{1}{p} \right) \|u\|_{H^1}^2 + \lambda \left( \frac{1}{p} - \frac{1}{4} \right) \int_{\mathbb{R}^3} \rho(x) \Phi \left( u^2 \right) dx > \left( \frac{1}{2} - \frac{1}{p} \right) \|u\|_{H^1}^2 > C > 0.$$ 

By the definition of Nehari manifold $N_{\lambda}$ of the functional $I_{\lambda}$, we obtain that $u$ is a critical point of $I_{\lambda}$ if and only if $u$ is a critical point of $I_{\lambda}$ constrained on $N_{\lambda}$ (3.5).

Now, we set

$$m_{\lambda} = \inf \{ I_{\lambda}(u) : u \in N_{\lambda} \}$$

By 2) of Lemma 3.1, we have $m_{\lambda} > 0$.

Since $\lim_{|x| \to +\infty} V(x) = V_{\infty}$, $\lim_{|x| \to +\infty} \rho(x) = \rho_{\infty}$, $\lim_{|x| \to +\infty} Q(x) = Q_{\infty}$, we consider the problem (1.1) at infinity

$$\begin{cases}
-\Delta u + V_{\infty} u - \lambda \rho_{\infty} \Phi u + Q_{\infty} |u|^{p-2} u = 0, & x \in \mathbb{R}^3, \\
-\Delta \Phi = \rho_{\infty} u^2, & x \in \mathbb{R}^3.
\end{cases}$$

(3.6)

Similar to (2.2), we obtain that there exists a unique $\Phi[u]$ such that

$$\int_{\mathbb{R}^3} \nabla \Phi \left[ u \right] \nabla v dx = \int_{\mathbb{R}^3} \rho_{\infty} u^2 v dx, \quad \forall v \in D^{1,2} \left( \mathbb{R}^3 \right).$$

It follows that $\Phi[u]$ satisfies the Poisson equation

$$-\Delta \Phi \left[ u \right] = \rho_{\infty} u^2.$$ 

(3.7)

Hence substituting $\Phi[u]$ into the first equation of (3.6) we have to study the equivalent problem

$$-\Delta u + V_{\infty} u - \lambda \rho_{\infty} \Phi[u] u + Q_{\infty} |u|^{p-2} u = 0$$

(3.8)

The weak solution of the problem (3.8) is the critical point of the functional

$$I_{\lambda}^w(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left( |u|^2 + V_{\infty} u^2 \right) dx - \frac{\lambda}{4} \int_{\mathbb{R}^3} \rho_{\infty} \Phi \left[ u \right] u^2 dx + \frac{1}{p} \int_{\mathbb{R}^3} Q_{\infty} |u|^p dx$$

$$= \frac{1}{2} \|u\|_{H^1}^2 - \frac{\lambda}{4} \int_{\mathbb{R}^3} \rho_{\infty} \Phi \left[ u \right] u^2 dx + \frac{1}{p} \int_{\mathbb{R}^3} Q_{\infty} |u|^p dx.$$
where $H^1_v$ is endowed with the norm
\[
\|u\|_{H^1_v} = \left( \int \|\nabla u\|^2 + V_u u^2 \right)^{\frac{1}{2}} dx
\]

Define the Nehari manifold of the functional $I^\omega_\lambda$
\[
N^\omega_\lambda = \{ u \in H^1_v(\mathbb{R}^3) \setminus \{0\} : H^\omega_\lambda (u) = 0 \},
\]
where
\[
H^\omega_\lambda (u) = I^\omega_\lambda (u)[u] = \|u\|_{H^1_v}^2 - \lambda \int \rho_\omega \Phi [u] u^2 dx + \int Q_\omega |u|^\rho^\omega dx
\]
and
\[
m^\omega_\lambda = \inf \{ I^\omega_\lambda (u) : u \in N^\omega_\lambda \} > 0
\]
The Nehari manifold $N^\omega_\lambda$ has properties similar to those of $N_\lambda$.

**Lemma 3.2.** The problem (3.8) has a positive radial ground state solution $\omega_\lambda \in N^\omega_\lambda$ such that
\[
I^\omega_\lambda (\omega_\lambda) = m^\omega_\lambda
\]

For the proof of Lemma 3.2, we make use of Schwarz symmetric method. We begin by recalling some basic properties.

Let $f \in L^r(\mathbb{R}^3)$ such that $f \geq 0$, then there is a unique nonnegative function $f^* \in L^r(\mathbb{R}^3)$, called the Schwarz symmetric of $f$, such that it depends only on $|x|$, whose level sets
\[
\{ x \in \mathbb{R}^3 : f(x) > t \} = \{ x \in \mathbb{R}^3 : f^*(x) > t \}.
\]

We consider the following Poisson equation
\[
-\Delta \phi = f \quad \text{and} \quad -\Delta v = f^*
\]

From Theorem 1 of [11], we have
\[
\int \|v\|^{s} dx \leq \int \|\phi\|^s dx, \quad \forall 0 < s \leq 2.
\]

Hence, let $\phi = \tilde{\Phi}[u]$, $f = \rho_\omega u^2$ and $v = \tilde{\Phi}[u^*]$, $f^* = \rho_\omega (u^*)^2$, we have
\[
\int \rho_\omega \tilde{\Phi}[u] u^2 dx \leq \int \rho_\omega \tilde{\Phi}[u^*] (u^*)^2 dx.
\]

**The Proof of Lemma 3.2.** Let $u_\omega \in N^\omega_\lambda$ be such that $I^\omega_\lambda (u_\omega) \to m^\omega_\lambda$. Let $t_\omega > 0$ such that $t_\omega |u_\omega| \in N^\omega_\lambda$ then we have
\[
\int \left( \|\nabla u_\omega\|^2 + V_\omega (u_\omega)^2 \right) dx - \lambda \int \rho_\omega \tilde{\Phi}[u_\omega](u_\omega)^2 dx + \int Q_\omega |u_\omega|^\rho^\omega dx = 0,
\]
and
\[
(t_\omega)^2 \int \left( \|\nabla u_\omega\|^2 + V_\omega (u_\omega)^2 \right) dx - \lambda (t_\omega)^2 \int \rho_\omega \tilde{\Phi}[u_\omega](u_\omega)^2 dx + (t_\omega)^\rho \int Q_\omega |u_\omega|^\rho^\omega dx = 0.
\]

Hence, we obtain that
\[
\left( (t_\omega)^2 - (t_\omega)^\rho \right) \int \left( \|\nabla u_\omega\|^2 + V_\omega (u_\omega)^2 \right) dx + \left( (t_\omega)^\rho - (t_\omega)^2 \right) \int Q_\omega |u_\omega|^\rho^\omega dx = 0.
\]
Since \( t_n > 0 \) and \( 2 < p < 4 \), (3.10) implies that \( t_n = 1 \). Therefore, we can assume that \( u_n \geq 0 \).

On the other hand, let \( (t_n, u_n) \) be the Schwartz symmetric function associated to \( u_n \), then we have
\[
\int_{\mathbb{R}^3} |t_n| \, dx = \int_{\mathbb{R}^3} |u_n| \, dx, \quad \text{and} \quad \int_{\mathbb{R}^3} |\nabla u_n| \, dx \geq \int_{\mathbb{R}^3} |\nabla (u_n)| \, dx
\]
(3.11)

Let \( (t_n, u_n) \) be such that \( (t_n, u_n) \in \mathcal{N}^\ast \), and \( u_n \in \mathcal{N}^\ast \), by (3.9) and (3.11), we have
\[
0 = \left[ (t_n)^{a_t} \int_{\mathbb{R}^3} |\nabla (t_n)|^2 + V_n(t_n)^{a_t} \right] \, dx - \lambda \left[ (t_n)^{a_t} \int_{\mathbb{R}^3} \rho_n \Phi(t_n) |u_n|^2 \, dx \right] + \left[ (t_n)^{a_t} \int_{\mathbb{R}^3} Q_n \, |u_n| \right] \, dx
\]
\[
\leq \left[ (t_n)^{a_t} \int_{\mathbb{R}^3} |\nabla u_n|^2 + V_n(u_n)^{a_t} \right] \, dx - \lambda \left[ (t_n)^{a_t} \int_{\mathbb{R}^3} \rho_n \Phi(u_n) |u_n|^2 \, dx \right] + \left[ (t_n)^{a_t} \int_{\mathbb{R}^3} Q_n \, |u_n| \right] \, dx
\]
\[
= \left[ (t_n)^{a_t} \right] \left[ (t_n)^{a_t} \right] \int_{\mathbb{R}^3} |\nabla u_n|^2 + V_n(u_n)^{a_t} \, dx + \left[ (t_n)^{a_t} \right] \int_{\mathbb{R}^3} Q_n \, |u_n| \right] \, dx.
\]

This implies that \( (t_n) \leq 1 \). Therefore, we have \( I_\lambda(u_n) \leq I_\lambda(u_n) \), and we can suppose that \( u_n \) is radial in \( H^1_0(\mathbb{R}^3) \). Since \( H^1_0(\mathbb{R}^3) \) is compactly embedded into \( L^p(\mathbb{R}^3) \) for \( 2 < p < 4 \), we obtain that \( m_\lambda \) is achieved at some \( \omega_\lambda \in \mathcal{N}^\ast \), which is positive and radial. Therefore, Lemma 3.2 is proved.

4. The Proof of Main Theorem

In this section, we prove Theorem 2.2. Firstly, we consider a compactness result and obtain the behavior of the (PS) sequence of the functional \( I_\lambda \).

**Lemma 4.1.** Let \( u_n \) be a (PS) sequence of the functional \( I_\lambda \) constrained on \( \mathcal{N}_\lambda \), that is
\[
 u_n \in \mathcal{N}_\lambda, \quad I_\lambda(u_n) \to d, \quad \text{and} \quad I'_\lambda(u_n) \big|_{\mathcal{N}_\lambda} \to 0, \quad \text{as} \quad n \to \infty
\]
(4.1)

Then there exists a solution \( u \) of the problem (2.4), a number \( k \in \mathbb{N} \) such that \( u \) functions \( u^1, u^2, \ldots, u^k \) of \( H^1_0(\mathbb{R}^3) \) and \( k \) sequences of points \( y^i \), \( \lambda \leq 0 \), \( \lambda \leq k \) such that
1. \( |u^i| \to \infty \), \( |y^i - y^j| \to \infty \), if \( i \neq j \), \( n \to \infty \);
2. \( u_n - \sum_{i=1}^k u^i (\cdot - y^i) \to \overline{u} \);
3. \( I_\lambda(u_n) \to I_\lambda(\overline{u}) + \sum_{i=1}^k I_\lambda(u^i) \);
4. \( u^i \) are non-trivial weak solution of the problem (3.8).

**Proof.** The proof is similar to that of Lemma 4.1 in [9].

By Lemma 4.1, taking into account that \( I_\lambda(u^i) \geq m_\lambda \) for all \( j \) and \( d \in (0, m_\lambda) \), we obtain that \( d = 0 \) and \( u_n \to \overline{u} \) in \( H^1_0(\mathbb{R}^3) \) (strongly), i.e. \( u_n \) is relatively compact for all \( d \in (0, m_\lambda) \). Hence we only need to prove that the energy of a solution of the problem (2.4) cannot overcome the energy of a ground state solution of the problem (3.8).

**The proof of Theorem 2.2.** By Lemma 4.1, we only prove that \( m_\lambda < m_{\lambda} \). Indeed, let \( \omega_n \in \mathcal{N}_\lambda \) such that \( I_\lambda(\omega_n) = m_{\lambda} \), and let \( t > 0 \) such that \( t \omega_n \in \mathcal{N}_\lambda \). Since \( V(x) \leq V_\infty \), \( \rho(x) \geq \rho_\infty \) and \( Q(x) \leq Q_\infty \), we have
\[
m_\lambda \leq I_\lambda(t \omega_n) = \frac{t^2}{2} \int_{\mathbb{R}^3} |\nabla \omega_n|^2 + V(x) \omega_n^2 \, dx - \frac{\lambda t^2}{4} \int_{\mathbb{R}^3} \rho(x) \Phi(\omega_n) \omega_n^2 \, dx + \frac{t^p}{p} \int_{\mathbb{R}^3} Q(x) |\omega_n|^p \, dx
\]
(4.2)

Since \( \omega_n \in \mathcal{N}_\lambda \) and \( t \omega_n \in \mathcal{N}_\lambda \), we have
Therefore, we have

\[
t^4 \int_{\mathbb{R}^3} \left( |\nabla \omega_\infty|^2 + V_\infty \omega_\infty^2 \right) dx + t^4 \int_{\mathbb{R}^3} |Q_\infty| \omega_\infty^p dx = \lambda t^4 \int_{\mathbb{R}^3} \rho_\infty \Phi[\omega_\infty] \omega_\infty^2 dx \\
\leq \lambda t^4 \int_{\mathbb{R}^3} \rho(x) \Phi[\omega_\infty] \omega_\infty^2 dx \\
= t^4 \int_{\mathbb{R}^3} \left( |\nabla \omega_\infty|^2 + V(x) \omega_\infty^2 \right) dx + t^p \int_{\mathbb{R}^3} Q(x) |\omega_\infty|^p dx \\
\leq t^4 \int_{\mathbb{R}^3} \left( |\nabla \omega_\infty|^2 + V_\infty \omega_\infty^2 \right) dx + t^p \int_{\mathbb{R}^3} Q_\infty |\omega_\infty|^p dx.
\]

By \(2 < p < 4\), we have \(t \leq 1\). If \(t = 1\), we have \(\omega_\infty \in N_{\lambda}^e\) and \(\omega_\infty \in N_\lambda\). Hence, by \(\omega_\infty \in N_{\lambda}^e\), we have

\[
\int_{\mathbb{R}^3} \left( |\nabla \omega_\infty|^2 + V_\infty \omega_\infty^2 \right) dx + \int_{\mathbb{R}^3} Q_\infty |\omega_\infty|^p dx = \lambda \int_{\mathbb{R}^3} \rho_\infty \Phi[\omega_\infty] \omega_\infty^2 dx
\]

and by \(\omega_\infty \in N_\lambda\), we have

\[
\int_{\mathbb{R}^3} \left( |\nabla \omega_\infty|^2 + V(x) \omega_\infty^2 \right) dx + \int_{\mathbb{R}^3} Q(x) |\omega_\infty|^p dx = \lambda \int_{\mathbb{R}^3} \rho(x) \Phi[\omega_\infty] \omega_\infty^2 dx.
\]

Combining (4.3) and (4.4), we have

\[
\int_{\mathbb{R}^3} \left( V_\infty - V(x) \right) \omega_\infty^2 dx + \int_{\mathbb{R}^3} \left( Q_\infty - Q(x) \right) |\omega_\infty|^p dx - \lambda \int_{\mathbb{R}^3} \rho_\infty \Phi[\omega_\infty] \omega_\infty^2 dx + \lambda \int_{\mathbb{R}^3} \rho(x) \Phi[\omega_\infty] \omega_\infty^2 dx = 0
\]

Since \(V(x) \leq V_\infty\), \(\rho(x) \geq \rho_\infty\), \(Q(x) \leq Q_\infty\), and \(Q(x) - Q_\infty < 0\) on a positive measure, we have

\[
\int_{\mathbb{R}^3} \left( Q_\infty - Q(x) \right) |\omega_\infty|^p dx
\]

which is not identically zero, and is contradiction. Hence, we have \(t < 1\). By (4.2), we have

\[
m_{\lambda} < \frac{1}{2} \int_{\mathbb{R}^3} \left( |\nabla \omega_\infty|^2 + V_\infty \omega_\infty^2 \right) dx - \frac{\lambda}{4} \int_{\mathbb{R}^3} \rho_\infty \Phi[\omega_\infty] \omega_\infty^2 dx + \frac{1}{p} \int_{\mathbb{R}^3} Q_\infty |\omega_\infty|^p dx
\]

\[
= \int_{\mathbb{R}^3} \left( \omega_\infty \right) = m_{\lambda}^e.
\]

Then there exists a positive radial ground state solution for the problem (1.1).

Acknowledgements

This research is supported by Shanghai Natural Science Foundation Project (No. 15ZR1429500), Shanghai Leading Academic Discipline Project (No. XTKX2012) and National Project Cultivate Foundation of USST (No. 13XGM05).

References


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