Generalization of Some Problems with s-Separation

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Abstract

In this article we apply and discuss El-Desouky technique to derive a generalization of the problem of selecting k balls from an n-line with no two adjacent balls being s-separation. We solve the problem in which the separation of the adjacent elements is not having odd and even separation. Also we enumerate the number of ways of selecting k objects from n-line objects with no two adjacent being of separations m, m + 1, ···, pm, where p is positive integer. Moreover we discuss some applications on these problems.

Keywords

Probability Function, s-Separation, s-Successions, n-Line, n-Circle

1. Introduction

Kaplansky [1] (see also Riordan ([2] p. 198, lemma) and Moser [3]) studied the problem of selecting k objects from n objects arranged in a line (called n-line) or a circle (called n-circle) with no two selected objects being consecutive. Let \( f(x, y) \) and \( g(x, y) \) denote the number of ways of such selections for n-line and n-circle respectively. Kaplansky proved that

\[
f(n, k) = \begin{cases} \binom{n-k+1}{k}, & 0 \leq k \leq n \\ 0, & \text{otherwise}, \end{cases}
\]

and

\[
g(n, k) = \begin{cases} \binom{n-k-1}{k-1}, & n \geq 2k+1 \\ 0, & 1 \leq n \leq k. \end{cases}
\]

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El-Desouky [4] studied another related problem with different techniques and proved that

$$I(n,k) = \sum_{i=0}^{\nu} \binom{k-1}{i} \binom{n-k+i-1}{k-i}, \quad \nu = \min \left( k-1, \left\lfloor \frac{n-k}{2} \right\rfloor \right), \quad 0 \leq k \leq n \quad (1.3)$$

where \( I(n,k) \) is the number of ways of selecting \( k \) balls from \( n \) balls arranged in a line with no two adjacent balls being unit separation.

In the following we adopt some conventions: \( \left[ x^n \right] f(x) \) denotes the coefficient of \( x^n \) in the formal power series \( f(x) \); \( \left[ x^iy^m \right] f(x,y) \) denotes the coefficient of \( x^iy^m \) in the series \( f(x,y) \); \([x]\) is the largest integer less than or equal to \( x \), \( N = \{0,1,\cdots\} \) and \( N_\ast = \{1,2,3,\cdots\} \).

Also, El-Desouky [5] derived a generalization of the problem given in [4] as follows: let \( l_s(n,k) \) denote the number of ways of selecting \( k \) balls from \( n \) balls arranged in a line with no two adjacent balls from the \( k \) selected balls being \( s \)-separation; two balls have separation \( s \) if they are separated by exactly \( s \) balls. Let \( d_s(n,k) \) denote the number of ways of selecting \( k \) balls from \( n \) balls arranged in a circle with no two adjacent balls from the \( k \) selected balls being \( s \)-separation.

Let \( l_s(n,k) \) be as defined before. Then \( l_s(n,k) \) is equal to the number of \( k \)-subsets of \( N_\ast \) where the difference \( s+1 \) is not allowed, so

$$l_s(n,k) = \sum_{i=0}^{\nu} (-1)^i \binom{k-1}{i} \binom{n-(s+1)i}{k-i} \quad (1.4)$$

where \( \nu = \min \left( k-1, \left\lfloor \frac{n-k}{s} \right\rfloor \right), \ 0 \leq k \leq n, \text{ and } s = 0,1,\cdots,n-k. \)

Let \( d_s(n,k) \) be as defined before. Then the difference \( s+1 \) is not allowed, so

$$d_s(n,k) = \sum_{k=0}^{\beta} (-1)^i \binom{k-1}{i} \binom{n-(s+1)i-1}{k-i-1} \quad (1.5)$$

where \( \beta = \min \left( k, \left\lfloor \frac{n-k}{s} \right\rfloor \right), \ 0 \leq k \leq n, \text{ and } s = 0,1,\cdots,n-k. \)

Let \( l_s(n,k,m) \) be the number of ways of selecting \( k \) balls from \( n \) balls arranged in a line with exactly \( m \) adjacent balls being of separation \( s \) or \((s\text{-successions})\), which gives a generalization of (4.1) in El-Desouky [4].

Thus,

$$l_s(n,k,m) = \sum_{i=m}^{\mu'} \sum_{j=0}^{i} (-1)^j \binom{k-1}{i} \binom{n-(s+1)i-j}{k-m} \quad (1.6)$$

where \( \mu' = \min \left( k-1, \left\lfloor \frac{n-k+s+m}{s+1} \right\rfloor \right), \ m = 0,1,\cdots,k-1, \ s = 0,1,\cdots,n-k. \)

For more details on such problems, see [3] [6] [7].

2. Main Results

We use El-Desouky technique to solve two problems in the linear case, with new restrictions. That is if the separation of any two adjacent elements from the \( k \) selected elements being of odd separation and of even separation. Moreover, we enumerate \( M_s(n,k,m,pm) \) which denotes the number of ways of selecting \( k \) objects from \( n \) objects arrayed in a line where any two adjacent objects from the \( k \) selected objects are not being of \( m, m+1, \cdots, pm \) separations, where \( p \) is positive integer.

2.1. No Two Adjacent Being Odd Separation

Let \( y_o(n,k) \) denote the number of ways of selecting \( k \) balls from \( n \) balls arranged in a line, where the separa-
tion of any two adjacent balls from the \( k \) selected balls being of odd separation. say \( s \), i.e. \( s = 1,3,5,\ldots \). This means that no two adjacent being of \( 2, 4, 6, \cdots \) differences, see Table 1.

So, following Decomposition (2.3.14) see [8] (p. 55), \( y_s(n,k) \) is equal to the number of \( k \)-subsets of \( N_n \) where the differences \( s+1, s = 1,3,5,\ldots \) are not allowed, hence \( y_s(n,k) = \left[ x^s \right] f(x) \), where

\[
f(x) = \left( x + x^2 + \cdots \right) \left[ x + x^2 + \cdots - \left( x^2 + x^4 + \cdots \right) \right]^{k-1} \left( 1 + x + \cdots \right)
\]

\[
= \frac{x}{1-x} \left[ 1 - x^k \right]^{k-1} \frac{1}{1-x}
\]

\[
= \frac{x}{(1-x)^2} \left( 1-x \right)^{k-1} \left[ 1 - (1-x)(x^3 + \cdots) \right]^{k-1}
\]

\[
= x^k \left( 1-x \right)^{(k+1)} \left( 1-x \right)^{(k+1)},
\]

hence

\[
f(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j x^i \binom{k+j-2}{j} x^j = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^i \binom{k+i}{i} \binom{k+j-2}{j} x^{i+j+k}.
\]

Setting \( n = i + j + k \quad j = n - i - k \) we have

\[
f(x) = \sum_{n=1}^{\infty} \sum_{i=n-k}^{n-1} (-1)^{n-i-k} \binom{k+i}{i} \binom{k+n-i-k-2}{n-i-k} x^n = \sum_{n=1}^{\infty} \sum_{i=n-k}^{n-1} (-1)^{n-i-k} \binom{k+i}{i} \binom{n-i-2}{k-2} x^n.
\]

Therefore, the coefficient of \( x^n \) gives

\[
y_o(n,k) = \sum_{i=0}^{n-k} (-1)^{n-i-k} \binom{k+i}{i} \binom{n-i-2}{k-2}.
\]

A calculated table for the values of \( y_o(n,k) \) is given in Table 1, where \( 1 \leq n, k \leq 10 \).

**Remark 1.** It is easy to conclude that \( y_o(n,k) \) satisfies the following recurrence relation

\[
y_o(n,k) = y_o(n-1,k-1) + y_o(n-2,k), \quad n, k \geq 2 \quad \text{and} \quad y_o(n,k) = 0 \quad \text{for} \quad k > n \quad (2.1)
\]

with the convention \( y_o(n,1) = n \), \( n \geq 1 \).

**Table 1.** A calculated table for the values of \( y_o(n,k) \).

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</table>
2.2. No Two Adjacent Being Even Separation

Let \( y_s(n,k) \) denote the number of ways of selecting \( k \) balls from \( n \) balls arranged in a line, where the separation of any two adjacent balls from the \( k \) selected balls are not being of even separation, say i.e. \( s = 0, 2, 4, \cdots \). This means that no two adjacent being of \( 1, 3, 5, \cdots \) differences.

So, following Decomposition (2.3.14) see [8] (p. 55) then \( y_s(n,k) \) is equal to the number of \( k \)-subsets of \( N_n \) where the differences \( 1, s, 2, s, 3, \cdots \) are not allowed, hence \( y_s(n,k) = \left[ x^n \right] f(x) \), where

\[
f(x) = \left( x + x^2 + \cdots \right)^{k-1} \left( 1 + x + \cdots \right)
\]

\[
= \frac{x}{1-x} \left[ \frac{x}{1-x} \left( x + x^2 + \cdots \right) \right]^{k-1} \frac{1}{1-x}
\]

\[
= \frac{x}{1-x} \left[ \frac{1}{1-x} x^{k-1} \right] \left[ 1 - \left( 1-x \right) \left( 1 + x^2 + \cdots \right) \right]^{k-1}
\]

\[
= \frac{x}{1-x} \left[ \frac{1}{1-x} x^{k-1} \right] \left( 1 + x^2 + \cdots \right)^{k-1}
\]

\[
= \frac{x^{2k-1}}{1-x} \left( 1 + x^2 + \cdots \right)^{k-1},
\]

hence

\[
f(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j x^{2k-1+i} \left( \begin{array}{c} k+i \hline i \end{array} \right) \left( \begin{array}{c} k+j-2 \hline j \end{array} \right) x^i = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j \left( \begin{array}{c} k+i \hline i \end{array} \right) \left( \begin{array}{c} k+j-2 \hline j \end{array} \right) x^{2k-1+i+j}.
\]

Setting \( n = 2k - 1 + i + j \), \( j = n - 2k + 1 - i \) we get

\[
f(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{n+i-1} \left( \begin{array}{c} k+i \hline i \end{array} \right) \left( \begin{array}{c} k+n-2k+1-i-2 \hline n-2k+1-i \end{array} \right) x^i = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{n+i-1} \left( \begin{array}{c} k+i \hline i \end{array} \right) \left( \begin{array}{c} n-k-i-1 \hline k-2 \end{array} \right) x^i.
\]

Therefore, the coefficient of \( x^n \) gives

\[
y_s(n,k) = \sum_{i=0}^{\infty} (-1)^{n+i-1} \left( \begin{array}{c} k+i \hline i \end{array} \right) \left( \begin{array}{c} n-k-i-1 \hline k-2 \end{array} \right) \left[ x^n \right] f(x).
\]

Moreover in the next subsection, we use our technique to enumerate \( M_s(n,k;m,pm) \) the number of ways of selecting \( k \) objects from \( n \) objects arrayed in a line such that no two adjacent elements have the differences \( m + 1, m + 2, \cdots, pm + 1 \) i.e. no two adjacent element being of \( m, m + 1, \cdots, pm \) separations, where \( p \) is positive integer.

2.3. Explicit Formula for \( M_s(n,k;m,pm) \)

Let \( M_s(n,k;m,pm) \) be the number of ways of selecting \( k \) objects from \( n \) objects arrayed in a line where any two adjacent objects from the \( k \) selected objects are not being of \( m, m + 1, \cdots, pm \) separations, where \( p \) is positive integer, hence \( M_s(n,k;m,pm) = \left[ x^n \right] f(x) \), where

\[
f(x) = \left( x + x^2 + \cdots \right)^{k-1} \left( x^{m+1} + x^{m+2} + \cdots + x^{pm+1} \right)^{k-1} \frac{1}{1-x}
\]

\[
= \frac{x^k}{1-x^k} \left[ \frac{1}{1-x^k} x^m \right]^{k-1} \left( 1 - \left( 1-x^k \right) x^{m+1} \right)^{k-1}
\]

\[
= \frac{x^k}{1-x^k} \left[ \frac{1}{1-x^k} x^m \right]^{k-1} \left[ 1 - \left( 1-x^k \right) x^{m+1} \right]^{k-1}
\]

\[
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} (-1)^{i+j+l} \left( \begin{array}{c} k-1 \hline i \end{array} \right) \left( \begin{array}{c} j \hline j \end{array} \right) x^{i+j+l} x^{i+j+l} x^{k+l} i^{x^i} x^j.
\]
Setting \( n = j(pm-m+1)+mi+l+k \) it is easy to find the coefficient of \( x^n \) hence

\[
M_j(n,k;m,pm) = \sum_{i=0}^{k-1} \sum_{j=0}^{i} (-1)^{i+j} \binom{k-1}{i} \binom{n-j(pm-m+1)-mi}{k}
\]

(2.3)

3. Some Applications

Let \( n \) urns be set out along a line, that is, one-dimensional.

Suppose we have \( m \) balls of which \( m_i \) are of colour \( c_i, \ i = 1,2,\cdots,k \) and we assign these balls to urns so that, see Pease [9]:

i) No urn contains more than one ball.

ii) All \( m_i \) balls of colour \( c_i \) are in consecutive urns, \( i = 1,2,\cdots,k \).

El-Desouky proved that if the order of colours of the groups is specified, the number of arrangement is just \( \binom{n-m+k}{k} \). Hence if the total number of balls \( \sum_{i=1}^{k} m_i = 2k-1 \), the number of arrangements is

\( I_o(n,k) = f(n,k) = \binom{n-k+1}{k} \) as a special case of El-Desouky results [5].

It is of practical interest to find the asymptotic behavior of \( f(n,k) \) or the probability \( p(n,k) = f(n,k)/\binom{n}{k} \) for large \( n \) and \( k \).

Let \( X \) be a random variable having the probability function \( p(n,k) \) then

\[
P(X = k) = p(n,k) = \binom{n-k+1}{k} \binom{n}{k},
\]

so

\[
\ln P(X = k) = \ln \left[ \left(1 - \frac{k-1}{n}\right) \left(1 - \frac{k-2}{n}\right) \cdots \left(1 - \frac{2(k-1)}{n}\right) \right] - \ln \left[ \left(1 - \frac{k}{n}\right) \left(1 - \frac{k-2}{n}\right) \cdots \left(1 - \frac{2(k-1)}{n}\right) \right]
\]

\[
= \left( \frac{k-1}{n} - \frac{k}{n} - \frac{2(k-1)}{n} \right) - \left( \frac{k}{n} - \frac{2}{n} - \frac{k-1}{n} \right)
\]

\[
= \frac{3k(k-1)}{2n} - \frac{k(k-1)}{2n} = \frac{k(k-1)}{n},
\]

where we used the first approximation

\[
\ln(1-x) = -x.
\]

Therefore,

\[
P(X = k) = e^{-\frac{k(k-1)}{n}}.
\]

Putting \( Y = \frac{X}{\sqrt{n}} \) we have

\[
P(Y = t) = P\left(\frac{X}{\sqrt{n}} = t\right) = P\left(X = \sqrt{n}t\right)
\]

\[
= e^{-\frac{t^2}{n}}, \text{ hence}
\]
\[
\lim_{n \to \infty} P(Y = t) = e^{-t^2}.
\]

Maosen [10] considered the following problem. Let \( t \) be any nonnegative integer.

If we want to select \( k \) balls from an \( n \)-line or an \( n \)-circle under the restriction that any two adjacent selected balls are not \( t \)-separated, how many ways are there to do it? He solved these problems by means of a direct structural analysis. For the two kinds of problems, he used \( F_t(n,k) \) to denote the number of ways of selecting \( k \) balls from \( n \) balls arranged in a line with no two adjacent selected balls being \( t \)-separation and \( G_t(n,k) \) to denote the number of ways of selecting \( k \) balls from an \( n \)-circle with no two adjacent selected being \( t \)-separation. He proved that

\[
F_t(n,k) = \sum_{l=0}^{\infty} (-1)^l \binom{k-1}{l} \binom{n-l(t+1)}{k-1},
\]

\( t \geq 1, t \leq n \) (3.2)

\[
G_t(n,k) = \frac{n}{k} \left\{ (-1)^{n-j} \binom{k}{j} \binom{n-j(t+1)-1}{k-1-j} + (-1)^{k} \delta[n,k(1+t)] \right\}.
\]

(3.3)

**Remark 2.** In fact El-Desouky [5] has proved (3.2) in 1988.

**References**


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