Embeddings of Almost Hermitian Manifold in Almost Hyper Hermitian Manifold and Complex (Hypercomplex) Numbers in Riemannian Geometry

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Abstract
Tubular neighborhoods play an important role in differential topology. We have applied these constructions to geometry of almost Hermitian manifolds. At first, we consider deformations of tensor structures on a normal tubular neighborhood of a submanifold in a Riemannian manifold. Further, an almost hyper Hermitian structure has been constructed on the tangent bundle $TM$ with help of the Riemannian connection of an almost Hermitian structure on a manifold $M$ then, we consider an embedding of the almost Hermitian manifold $M$ in the corresponding normal tubular neighborhood of the null section in the tangent bundle $TM$ equipped with the deformed almost hyper Hermitian structure of the special form. As a result, we have obtained that any Riemannian manifold $M$ of dimension $n$ can be embedded as a totally geodesic submanifold in a Kaehlerian manifold of dimension $2n$ (Theorem 6) and in a hyper Kaehlerian manifold of dimension $4n$ (Theorem 7). Such embeddings are “good” from the point of view of Riemannian geometry. They allow solving problems of Riemannian geometry by methods of Kaehlerian geometry (see Section 5 as an example). We can find similar situation in mathematical analysis (real and complex).

Keywords
Riemannian Manifolds, Almost Hermitian and Almost Hyper Hermitian Structures, Tangent Bundle

1. Deformations of Tensor Structures on a Normal Tubular Neighborhood of a Submanifold

1. Let $(M', g')$ be a $k$-dimensional Riemannian manifold isometrically embedded in a $n$-dimensional Rie-
mennian manifold \((M, g)\). The restriction of \(g\) to \(M'\) coincides with \(g'\) and for any \(p \in M'\).

\[
T_p (M) = T_p (M') \oplus T_p (M')^\perp.
\]

So, we obtain a vector bundle \(M' \to T (M')^\perp : p \to T_p (M')^\perp\) over the submanifold \(M'\). There exists a neighborhood \(U_0\) of the null section \(O_{M'}\) in \(T (M')^\perp\) such that the mapping

\[
\pi \times \exp : v \to \left(\pi (v), \exp_{\pi(v)} v\right), v \in U_0,
\]

is a diffeomorphism of \(\tilde{U}_0\) onto an open subset \(\tilde{U} \subset M\). The subset \(\tilde{U}\) is called a **tubular neighborhood of the submanifold** \(M'\) in \(M\).

For any point \(p \in M\) we can consider a set \(\{\delta(p)\}\) of positive numbers such that the mapping \(\exp_{\delta(p)}\) is defined and injective on \(U(\delta(p)) \subset T_p (M)\). Let \(\varepsilon(p) = \sup \{\delta(p)\}\).

**Lemma [1].** The mapping \(M \to R : p \to \varepsilon(p)\) is continuous on \(M\).

If we take the restriction of the function \(\varepsilon(p)\) on \(\tilde{U}\) then it is clear that there exists a continuous positive function \(\varepsilon(p)\) on \(M'\) such that for any \(p \in M'\) open geodesic balls \(B\left(p; \frac{\varepsilon(p)}{2}\right) \subset B\left(p; \varepsilon(p)\right) \subset \tilde{U}\). For compact manifolds we can choose a constant function \(\varepsilon(p) = \varepsilon > 0\). We denote \(\tilde{U}_p = \exp\left(\tilde{U}_0 \cap T_p (M')^\perp\right)\),

\[
D\left(p; \frac{\varepsilon(p)}{2}\right) = B\left(p; \frac{\varepsilon(p)}{2}\right) \cap \tilde{U}_p, \quad D(p; \varepsilon(p)) = B(p; \varepsilon(p)) \cap \tilde{U}_p.
\]

It is obvious that

\[
\dim \tilde{U}_p = \dim \left(D(p; \varepsilon(p))\right) = n - k.
\]

For any point \(o \in M'\) we can consider such an orthonormal frame \(\{X_i, \ldots, X_k\}\) that \(T_o (M') = L[X_i, \ldots, X_k]\) and \(T_o (M')^\perp = L[X_{k+1}, \ldots, X_n]\). There exist coordinates \(x_1, \ldots, x_n\) in some neighborhood \(\tilde{V}_0 \subset M'\) of the point \(o\) that

\[
\frac{\partial}{\partial x_i} = X_i, \quad i = 1, k.
\]

We consider orthonormal vector fields \(X_{k+1}, \ldots, X_n\) which are cross-sections of the vector bundle \(p \to T_p (M')^\perp\) over \(\tilde{V}_0\) and the neighborhood \(\tilde{W}_0 = \bigcup \tilde{U}_p\). The basis \(\{X_{k+1}, \ldots, X_n\}\) defines the normal coordinates \(x_{k+1}, \ldots, x_n\) on \(\tilde{U}_p\).

For any point \(x \in \tilde{W}_0\) there exists such unique point \(p \in \tilde{V}_0\) that \(x = \exp_p (t \xi), \|\xi\| = 1, \xi \in T_p (M')^\perp\). A point \(x \in \tilde{W}_0\) has the coordinates \(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n\) where \(x_1, \ldots, x_k\) are coordinates of the point \(p\) in \(\tilde{V}_0\) and \(x_{k+1}, \ldots, x_n\) are normal coordinates of \(x\) in \(\tilde{U}_p\). We denote \(X_i = \frac{\partial}{\partial x_i}, i = 1, n\) on \(\tilde{W}_0\). Thus, we can consider **tubular neighborhoods** \(Tb\left(M'; \frac{\varepsilon(p)}{2}\right) = \bigcup_{p \in M'} D\left(p; \frac{\varepsilon(p)}{2}\right)\) and \(Tb\left(M'; \varepsilon(p)\right) = \bigcup_{p \in M'} D(p; \varepsilon(p))\) of the submanifold \(M'\).

2'. Let \(K\) be a smooth tensor field of type \((r, s)\) on the manifold \(M\) and for \(x \in \tilde{W}_0\), let

\[
K_x = \sum_{\lambda_1, \ldots, \lambda_r, \gamma_1, \ldots, \gamma_s} k_{\lambda_1, \ldots, \lambda_r, \gamma_1, \ldots, \gamma_s} (x) X_{\lambda_1} \otimes \cdots \otimes X_{\lambda_r} \otimes X_{\gamma_1} \otimes \cdots \otimes X_{\gamma_s},
\]

where \(\{X_1, \ldots, X_n\}\) is the dual basis of \(T_x (M), x = \exp_p (t \xi), \|\xi\| = 1, \xi \in T_p (M')^\perp\). We define a tensor field \(K\) on \(M\) in the following way.

a) \(x \in D\left(p; \frac{\varepsilon(p)}{2}\right)\), then

\[
\tilde{K}_x = \sum_{\lambda_1, \ldots, \lambda_r, \gamma_1, \ldots, \gamma_s} k_{\lambda_1, \ldots, \lambda_r, \gamma_1, \ldots, \gamma_s} (p) X_{\lambda_1} \otimes \cdots \otimes X_{\lambda_r} \otimes X_{\gamma_1} \otimes \cdots \otimes X_{\gamma_s};
\]

b) \(x \in D(p; \varepsilon(p)) \setminus D\left(p; \frac{\varepsilon(p)}{2}\right)\), then

\[
\tilde{K}_x = \sum_{\lambda_1, \ldots, \lambda_r, \gamma_1, \ldots, \gamma_s} k_{\lambda_1, \ldots, \lambda_r, \gamma_1, \ldots, \gamma_s} (p) X_{\lambda_1} \otimes \cdots \otimes X_{\lambda_r} \otimes X_{\gamma_1} \otimes \cdots \otimes X_{\gamma_s};
\]
\[
\tilde{K}_s = \sum_{h_i \cdot \cdots \cdot h_i} k_{h_i \cdot \cdots \cdot h_i}^A \left( \exp_p \left( \left( 2t - e(p) \right) \xi \right) \right) X_{a_i} \otimes \cdots \otimes X_{a_i} \otimes X_{b_i} \otimes \cdots \otimes X_{b_i} ;
\]

c) \( x \in M \setminus \bigcup_{\omega} D \left( p; e(p) \right) \), then

\[\tilde{K}_s = K_s.\]

It is easy to see the independence of the tensor field \( \tilde{K} \) on a choice of coordinates in \( \bar{W}_0 \) for every point \( o \in M' \).

**Definition 1.** The tensor field \( \tilde{K} \) is called a deformation of the tensor field \( K \) on the normal tubular neighborhood \( M' \).

**Remark.** The obtained tensor field \( \tilde{K} \) is continuous but is not smooth on the boundaries of the normal tubular neighborhoods \( T_b \left( M'; e(p) \right) \) and \( T_b \left( M'; e(p) \right) \); \( \tilde{K} \) is smooth in other points of the manifold \( M \).

3°. We consider a deformation \( \bar{g} \) of the Riemannian metric \( g \) on the normal tubular neighborhood \( T_b \left( M'; e(p) \right) \) of a submanifold \( M' \). For \( x \in \bar{W}_0, \ x = \exp_p (t \xi), \| \xi \| = 1, \xi \in T_p (M') \), we define the Riemannian metric \( \bar{g} \) by the following way.

a) \( \bar{g}_p = g_p \) for any \( p \in M' \);

b) \( \bar{g}_s \left( X_j, X_j \right) = \bar{g}_y (x) = \bar{g}_y (p), \) where \( X_i = \partial \), \( i = 1, n \), \( X_j = \partial \), \( j = 1, n \), on \( \bar{W}_0 \), \( x \in D \left( p; \frac{e(p)}{2} \right) \);

c) \( \bar{g}_s \left( X_j, X_j \right) = \bar{g}_y (x) = \bar{g}_y \left( \exp_p \left( \left( 2t - e(p) \right) \xi \right) \right), \) for any \( x \in D \left( p; e(p) \right) / D \left( p; \frac{e(p)}{2} \right) \);

d) \( \bar{g}_s = g_s \) for each point \( x \in M \setminus \bigcup_{p \in M} D \left( p; e(p) \right) \).

The independence of \( \bar{g} \) on a choice of local coordinates follows and the correctly defined Riemannian metric \( \bar{g} \) on \( M \) has been obtained.

It is known from [3] that every autoparallel submanifold of \( M \) is a totally geodesic submanifold and a submanifold \( M' \) is autoparallel if and only if \( \nabla_x Y \in T (M') \) for any \( X, Y \in \mathcal{X} (M') \), where \( \nabla \) is the Riemannian connection of \( g \).

**Theorem 1.** Let \( M' \) be a submanifold of a Riemannian manifold \( (M, g) \) and \( \bar{g} \) be the deformation of \( g \) on the normal tubular neighborhood \( T_b \left( M'; e(p) \right) \) of \( M' \) constructed above. Then \( M' \) is a totally geodesic submanifold of \( T_b \left( M'; e(p) \right) \big/ \bar{g} \).

**Proof.** For any point \( x \in D \left( p; \frac{e(p)}{2} \right) \subset \bar{W}_0 \) the functions \( \bar{g}_y (x) = g_y (p) \) and \( \frac{\partial \bar{g}_y}{\partial x_l} = 0, l = k + 1, n \) on \( D \left( p; \frac{e(p)}{2} \right) \) because the vector fields \( X_i = \partial \) are tangent to \( D \left( p; \frac{e(p)}{2} \right) \). By the formula of the Riemannian connection \( \bar{\nabla} \) of the Riemannian metric \( \bar{g} \), [2], we obtain for \( i, j = 1, k, l = k + 1, n \)

\[
2\bar{g}_p \left( \bar{\nabla}_x X_j, X_i \right) = X_j \bar{g} \left( X_j, X_i \right) + X_j \bar{g} \left( X_j, X_i \right) - X_j \bar{g} \left( X_j, X_i \right) + \bar{g}_p \left[ \left( X_j, X_i \right), X_i \right] + \bar{g}_p \left[ X_j, X_i \right] + \bar{g}_p \left[ X_j, X_i \right] \right] = \frac{\partial \bar{g}_y}{\partial x_l} = 0.
\]

Here we use the fact that \( \left[ X_j, X_i \right] = [X_j, X_i] = [X_j, X_i] = 0 \) and that \( \bar{g} \left( X_j, X_i \right) = \bar{g} \left( X_j, X_i \right) = 0 \) because \( X_i \in T (M') \).

Thus, \( \bar{\nabla}_x X_j \in T (M') \) and from the remarks above the theorem follows.

**QED.**

**Corollary 1.1.** Let \( \bar{R} \) be the Riemannian curvature tensor field of \( \bar{\nabla} \). Then \( \bar{R} \) vanishes on every
Proof. From the formula (1.1) it is clear that \( \nabla_{X_l}X_m = 0 \) for \( l,m = k+1,n \). The rest is obvious. QED.

2. Almost Hyper Hermitian Structures (ahHs) on Tangent Bundles

0°. We follow especially close to [4].

Let \((M,g)\) be a \(n\)-dimensional Riemannian manifold and \(TM\) be its tangent bundle. For a Riemannian connection \( \nabla \) we consider the connection map \( K \) of \( \nabla \) [5], [1], defined by the formula
\[
\nabla_X Z = KZ.X, \quad (2.1)
\]
where \( Z \) is considered as a map from \( M \) into \( TM \) and the right side means a vector field on \( M \) assigning to \( pM \in M \) the vector \( KpM \) \( Z_pM \in pM \).

If \( U \in TM \), we denote by \( H_U \) the kernel of \( K_{\pi M} \) and this \( n \)-dimensional subspace of \( TM_U \) is called the horizontal subspace of \( TM_U \).

Let \( \pi \) denote the natural projection of \( TM \) onto \( M \), then \( \pi \) is a \( C^\infty \)-map of \( TTM \) onto \( TM \). If \( U \in TM \), we denote by \( V_U \) the kernel of \( \pi_{\pi M} \) and this \( n \)-dimension subspace of \( TM_U \) is called the vertical subspace of \( TM_U \) (\( \dim TM_U = 2(\dim M = 2n) \)). The following maps are isomorphisms of corresponding vector spaces \( (p = \pi(U)) \)
\[
\pi_{\pi M} : H_U \rightarrow M_p, \quad K_{\pi M} : V_U \rightarrow M_p
\]
and we have
\[
TM_U = H_U \oplus V_U
\]
If \( X \in \chi(M) \), then there exists exactly one vector field on \( TM \) called the “horizontal lift” (resp. “vertical lift”) of \( X \) and denoted by \( \widetilde{X}^h(\widetilde{X}^v) \), such that for all \( U \in TM : \)
\[
\pi_{\pi M}X^h_U = X_{\pi(U)}, \quad K\widetilde{X}^v_U = 0_{\pi(U)}, \quad \pi_{\pi M}X^v_U = 0_{\pi(U)}, \quad K\widetilde{X}^h_U = X_{\pi(U)}, \quad (2.2)
\]
\[
(2.3)
\]
Let \( R \) be the curvature tensor field of \( \nabla \), then following [5] we write
\[
\left[ \widetilde{X}^h, \widetilde{Y}^v \right] = 0, \quad (2.4)
\]
\[
\left[ \widetilde{X}^h, \widetilde{Y}^h \right] = (\nabla_{\widetilde{X}^h} \widetilde{Y}^h)^{\pi}, \quad (2.5)
\]
\[
\pi_{\pi M}\left[ \left[ \widetilde{X}^h, \widetilde{Y}^h \right]_{\pi} \right] = [X,Y], \quad (2.6)
\]
\[
K\left[ \left[ \widetilde{X}^h, \widetilde{Y}^h \right]_{\pi} \right] = R(X,Y)U. \quad (2.7)
\]
For vector fields \( \widetilde{X} = \widetilde{X}^h \oplus \widetilde{X}^v \) and \( \widetilde{Y} = \widetilde{Y}^h \oplus \widetilde{Y}^v \) on \( TM \) the natural Riemannian metric \( \hat{g} = \left\langle \cdot, \cdot \right\rangle \) is defined on \( TM \) by the formula
\[
\left\langle \widetilde{X}, \widetilde{Y} \right\rangle = g(\pi_{\pi M}\widetilde{X},\pi_{\pi M}\widetilde{Y}) + g(K\widetilde{X}^h,K\widetilde{Y}^v). \quad (2.8)
\]
It is clear that the subspaces \( H_U \) and \( V_U \) are orthogonal with respect to \( \left\langle \cdot, \cdot \right\rangle \).

1°. We define a tensor field \( J_1 \) on \( TM \) by the equalities
\[
J_1\widetilde{X}^h = \widetilde{X}^v, \quad J_1\widetilde{X}^v = -\widetilde{X}^h, \quad X \in \chi(M). \quad (2.9)
\]
For \( X \in \chi(M) \) we get
\[ J_i^2 \bar{X} = J_i \left( J_i \left( \bar{X}^\perp \oplus \bar{X}^\perp \right) \right) = J_i \left( -\bar{X}^\perp \oplus -\bar{X}^\perp \right) = -\left( \bar{X}^\perp \oplus \bar{X}^\perp \right) = -I\bar{X} \]

and

\[ J_i^2 = -I. \]

For \( X, Y \in \chi(M) \) we obtain

\[ \langle J_i \bar{X}, J_i \bar{Y} \rangle = \langle -\bar{X}^\perp \oplus -\bar{X}^\perp, -\bar{Y}^\perp \oplus -\bar{Y}^\perp \rangle = \langle -\bar{X}^\perp, -\bar{Y}^\perp \rangle + \langle \bar{X}^\perp, \bar{Y}^\perp \rangle, \]

\[ \langle \bar{X}, \bar{Y} \rangle = \langle \bar{X}^\perp \oplus \bar{X}^\perp, \bar{Y}^\perp \oplus \bar{Y}^\perp \rangle = \langle \bar{X}^\perp, \bar{Y}^\perp \rangle + \langle \bar{X}^\perp, \bar{Y}^\perp \rangle \]

and it follows that \( \langle J_i \bar{X}, J_i \bar{Y} \rangle = \langle \bar{X}, \bar{Y} \rangle, (TM, J_i, \langle \cdot, \cdot \rangle) \) is an almost Hermitian manifold.

Further, we want to analyze the second fundamental tensor field \( h^1 \) of the pair \((J_i, \langle \cdot, \cdot \rangle)\) where \( h^1 \) is defined by (2.11), [6].

The Riemannian connection \( \hat{\nabla} \) of the metric \( \hat{g} = \langle \cdot, \cdot \rangle \) on \( TM \) is defined by the formula (see [1])

\[ \langle \hat{\nabla}_X Y, Z \rangle = \frac{1}{2} \left( \langle X, [Y, Z] \rangle + \langle [X, Y], Z \rangle - \langle X, \nabla X, Y \rangle + \langle X, [Y, Z] \rangle \right) \]

(2.10)

For orthonormal vector fields \( \bar{X}, \bar{Y}, \bar{Z} \) on \( TM \) we obtain

\[ h^1_\bar{X} \bar{Y} \bar{Z} = \langle h^1_\bar{X} \bar{Y}, \bar{Z} \rangle = \frac{1}{2} \langle \hat{\nabla}_\bar{X} \bar{Y}, \bar{Z} \rangle + J_i \langle \hat{\nabla}_\bar{X} \bar{Y}, \bar{Z} \rangle \]

(2.11)

Using (2.4)-(2.7) and (2.11) we consider the following cases for the tensor field \( h^1 \) assuming all the vector fields to be orthonormal.

\[ h^1_{\bar{X}\bar{Y}\bar{Z}} = \frac{1}{4} \left( \langle [\bar{X}^\perp, \bar{Y}^\perp], \bar{Z}^\perp \rangle + \langle [\bar{Y}^\perp, \bar{Z}^\perp], \bar{X}^\perp \rangle + \langle [\bar{Z}^\perp, \bar{X}^\perp], \bar{Y}^\perp \rangle \right) - \langle [\bar{X}^\perp, \bar{Y}^\perp], \bar{Z}^\perp \rangle - \langle [\bar{X}^\perp, \bar{Y}^\perp], \bar{Z}^\perp \rangle \]

\[ = \frac{1}{4} \left( g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X) - \langle [\bar{X}^\perp, \bar{Y}^\perp], \bar{Z}^\perp \rangle \right) - \langle [\bar{X}^\perp, \bar{Y}^\perp], \bar{Z}^\perp \rangle \]

(1.1)

\[ = \frac{1}{2} g(V_x, Y, Z) - \frac{1}{2} \left( g(V_x, Y, Z) - g(V_x, Z, Y) \right) \]

\[ = \frac{1}{2} \left( g(V_x, Y, Z) - g(V_x, Y, Z) \right) = 0. \]

\[ h^1_{\bar{X}\bar{Y}\bar{Z}} = \frac{1}{4} \left( \langle [\bar{X}^\perp, \bar{Y}^\perp], \bar{Z}^\perp \rangle + \langle [\bar{Y}^\perp, \bar{Z}^\perp], \bar{X}^\perp \rangle + \langle [\bar{Z}^\perp, \bar{X}^\perp], \bar{Y}^\perp \rangle \right) - \langle [\bar{X}^\perp, \bar{Y}^\perp], \bar{Z}^\perp \rangle - \langle [\bar{X}^\perp, \bar{Y}^\perp], \bar{Z}^\perp \rangle \]

(2.1)

\[ = \frac{1}{4} \left( g(R(X, Y), Z) + \langle [\bar{Z}^\perp, \bar{X}^\perp], \bar{Y}^\perp \rangle \right) \]

\[ = \frac{1}{4} \left( g(R(X, Y), Z) + g(R(Z, X), Y) \right) \]

\[ = -\frac{1}{4} \left( g(R(X, Y) Z, U) + g(R(Z, X), Y) \right). \]
By similar arguments we obtain

\( h^1_{X^1 + Z} = -\frac{1}{4} \left( g(R(Z, X)Y, U) + g(R(X, Y)Z, U) \right) \). \quad (3.1')

\( h^1_{X^1 + Z} = -\frac{1}{4} \left( g(R(Z, Y)X, U) \right) \). \quad (4.1')

\( h^1_{X^1 + Z} = \frac{1}{4} \left( g(R(Z, Y)X, U) \right) \). \quad (5.1')

\( h^1_{X^1 + Z} = 0 \). \quad (6.1')

\( h^1_{X^1 + Z} = 0 \). \quad (7.1')

\( h^1_{X^1 + Z} = 0 \). \quad (8.1')

It is obvious that \((J, g)\) is a Kaehlerian structure if and only if \( h^1 = 0 \).

Now assume additionally that we have an almost Hermitian structure \( J \) on \((M, g)\). We define a tensor field \( J_2 \) on \( TM \) by the equalities

\( J_2 X^\# = (JX)^\#, \quad J_2 X^* = -(JX)^* \), \( X \in \chi(M) \). \quad (2.12)

For \( X \in \chi(M) \) we get

\( J_2^2 X = J_2 \left( J_2 \left( X^\# + X^* \right) \right) = J_2 \left( JX^\# + -(JX)^* \right) = -(X^\# + X^*) - JX \)

and

\( J_2^2 = -I. \)

For \( X, Y \in \chi(M) \) we obtain

\[ \langle J_2 X, J_2 Y \rangle = \left( \langle (JX)^\#, -(JX)^* \rangle \right) = \left( \langle JX^\#, (JY)^* \rangle + \langle (JX)^*, (JY)^* \rangle \right) = g(JX, JY) + g(JX, JY) = g(X, Y) + g(X, Y) \]

\[ = \langle X^\#, F^\# \rangle + \langle X^*, F^* \rangle = \langle X^\# + X^*, F^# + F^* \rangle = \langle X, F \rangle. \]

Further, we obtain

\[ J_1 \left( J_2 X \right) = J_1 \left( JX^\# + -(JX)^* \right) = \left( JX^\# \right) + \left( (JX)^* \right), \]

\[ J_2 \left( J_1 X \right) = J_2 \left( -(X^\# + X^*) \right) = -\left( (JX)^\# \right) + \left( -(JX)^* \right). \]

Thus, we get \( J_1 J_2 = -J_2 J_1 = J_3 \) and all \( \langle J_1, J_2, J_3, J_4 \rangle \) on \( TM \) has been constructed.

For orthonormal vector fields \( X, Y, Z \) on \( TM \) we obtain

\[ h^2_{X^1 + Y} = \frac{1}{2} \left( \langle \hat{\nabla}_X Y + J_2 \hat{\nabla}_X J_2 Y, Z \rangle \right) = \frac{1}{2} \left( \langle \hat{\nabla}_X Y, Z \rangle \right) \]

\[ = \frac{1}{2} \left( \langle X, F \rangle, Z \rangle \right) + \langle \langle Z, X \rangle, F \rangle + \langle \langle Z, F \rangle, X \rangle = \frac{1}{2} \left( \langle X, J_2 Z \rangle, J_2 Y \rangle \right. \]

\[ - \left. \langle \langle J_2 Z, X \rangle, J_2 Y \rangle \right) \]

Using (2.4)-(2.7) and (2.13) we consider the following cases for the tensor field \( h^2 \) assuming all the vector fields to be orthonormal.
\[ h_{e}^{2} = \frac{1}{4} \left( \left[ [X^{\alpha}, \bar{Y}^{\alpha}], Z^{\beta} \right] + \left[ [Z^{\alpha}, \bar{X}^{\alpha}], Y^{\beta} \right] + \left[ [Z^{\alpha}, \bar{Y}^{\alpha}], X^{\beta} \right] \right) \]
\[ - \left( \left[ [X^{\alpha}, J_{2} \bar{Y}^{\alpha}], J_{2} \bar{Z}^{\beta} \right] - \left[ [J_{2} \bar{Z}^{\alpha}, \bar{X}^{\alpha}], J_{2} \bar{Y}^{\beta} \right] - \left[ [J_{2} \bar{Y}^{\alpha}, J_{2} \bar{Z}^{\alpha}], \bar{X}^{\beta} \right] \right) \]
\[ = \frac{1}{4} \left( g \left( [X, Y], Z \right) + g \left( [Z, X], Y \right) + g \left( [Z, Y], X \right) \right) - \left( \left( [X, J_{2} Y], J_{2} Z \right) - \left( [J_{2} Z, X], J_{2} Y \right) - \left( [J_{2} Y, J_{2} Z], X \right) \right) \]
\[ = \frac{1}{2} \left( g \left( \nabla_{x} Y, Z \right) - g \left( \nabla_{x} J_{2} Y, J_{2} Z \right) \right) = h_{xyz}. \]
(1.2')

\[ h_{\tau}^{2} = \frac{1}{4} \left( \left[ [X^{\alpha}, \bar{Y}^{\alpha}], Z^{\beta} \right] + \left[ [Z^{\alpha}, \bar{X}^{\alpha}], Y^{\beta} \right] + \left[ [Z^{\alpha}, \bar{Y}^{\alpha}], X^{\beta} \right] \right) \]
\[ - \left( \left[ [X^{\alpha}, J_{2} \bar{Y}^{\alpha}], J_{2} \bar{Z}^{\beta} \right] - \left[ [J_{2} \bar{Z}^{\alpha}, \bar{X}^{\alpha}], J_{2} \bar{Y}^{\beta} \right] - \left[ [J_{2} \bar{Y}^{\alpha}, J_{2} \bar{Z}^{\alpha}], \bar{X}^{\beta} \right] \right) \]
\[ = \frac{1}{4} \left( g \left( R(X, Y)U, Z \right) + g \left( R(X, J_{2} Y)U, J_{2} Z \right) \right) - \frac{1}{4} \left( g \left( R(X, Y)Z, U \right) + g \left( R(X, J_{2} Y)J_{2} Z, U \right) \right). \]
(2.2')

By similar arguments we obtain
\[ h_{e}^{2} = -\frac{1}{4} \left( g \left( R(X, Z)Y, U \right) + g \left( R(X, J_{2} Z)J_{2} Y, U \right) \right). \]
(3.2')

\[ h_{\tau}^{2} = -\frac{1}{4} \left( g \left( R(Z, Y)X, U \right) + g \left( R(J_{2} Z, Y)X, U \right) \right). \]
(4.2')

\[ h_{e}^{2} = 0. \]
(5.2')

\[ h_{\tau}^{2} = 0. \]
(6.2')

\[ h_{e}^{2} = 0. \]
(7.2')

\[ h_{\tau}^{2} = \frac{1}{2} \left( g \left( \nabla_{x} Y, Z \right) - g \left( \nabla_{x} J_{2} Y, J_{2} Z \right) \right) = h_{xyz}. \]
(8.2')

Here \( h \) is the second fundamental tensor field of the pair \((J, g)\) on \( M \).

3. Embeddings of Almost Hermitian Manifolds in Almost Hyper Hermitian Those

For an almost Hermitian manifold \((M, J, g)\) we have constructed in Section 2 ahHs \((J_{1}, J_{2}, J_{3}, \tilde{g})\) on \( TM \). The manifold \( M \) can be considered as the null section \( O_{M} \subset TM \) and it is clear from (2.8) that \( \tilde{g}_{M} = g \). All the results of 1 can be applied to a submanifold \( M \subset (TM, \tilde{g}) \), see [7]. So, we can consider the normal tubular neighborhoods \( Tb \left( M, \frac{\epsilon(p)}{2} \right) \subset Tb \left( M, \epsilon(p) \right) \subset TM \) and the deformations \( \tilde{J}_{1}, \tilde{J}_{2}, \tilde{J}_{3}, \tilde{g} \) of the tensor fields \( J_{1}, J_{2}, J_{3}, \tilde{g} \) respectively.

**Theorem 2.** Let \((M, J, g)\) be an almost Hermitian manifold and \( Tb \left( M, \epsilon(p) \right) \) be the corresponding normal tubular neighborhood with respect to \( \tilde{g} = \{ \} \) on \( TM \). Then \( M(O_{M}) \) is a totally geodesic submanifold of the almost hyper Hermitian manifold \( \left( Tb \left( M, \frac{\epsilon(p)}{2} \right), J_{1}, J_{2}, J_{3}, \tilde{g} \right) \), where the ahHs \((J_{1}, J_{2}, J_{3}, \tilde{g})\) is the deformation of the structure \( \left( J_{1}, J_{2}, J_{3}, \tilde{g} \right) \) obtained in 2', Section 1. The structure \( \left( \tilde{J}_{1}, \tilde{g} \right) \) is Kaehlerian one.

**Proof.** It follows from Theorem 1 that \( M \) is a totally geodesic submanifold of the Riemannian manifold \( \left( Tb \left( M, \frac{\epsilon(p)}{2} \right), \tilde{g} \right) \).
Let $\mathcal{W}$ be a coordinate neighborhood in $\mathcal{M}$ considered in $1^*$, Section 1. A point $x \in \mathcal{W}$ has the coordinates $x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}$ where $x_1, \ldots, x_n$ are coordinates of the point $p$ in $\mathcal{V} \subset \mathcal{M}$ and $x_{n+1}, \ldots, x_{2n}$ are normal coordinates of $x$ in $D\left(p, \frac{\varepsilon(p)}{2}\right)$.

We denote $x_i = \frac{\partial}{\partial x_i}, i = 1, 2n$, $\hat{\nabla}_i X_j = \sum k \hat{\Gamma}^k_{ij} X_k$, $\overline{\nabla}_i X_j = \sum k \overline{\Gamma}^k_{ij} X_k$, $JX_j = \sum k J^k_j X_k$, $J\overline{X}_j = \sum k J^k_{\overline{j}} X_k$, $\hat{g}_{ij} = \hat{g}(X_i, X_j)$, $g_{ij} = g(X_i, X_j)$ where $\hat{\nabla}$ and $\overline{\nabla}$ are Riemannian connections of metrics $\hat{g}$ and $g$, $J$ is any tensor field from $J_1, J_2, J_3$.

Using the construction in $2^*$, Section 1 we have $\hat{g}_{ij}(x) = \hat{g}_{ij}(p)$, $\overline{g}_{ij}(x) = \overline{g}_{ij}(p)$ on $Tb\left(M, \frac{\varepsilon(p)}{2}\right) \cap \mathcal{W}$.

According to [2] we can write

$$\sum \hat{g}_{ij} \Gamma^k_{ij} = \frac{1}{2} \left( \frac{\partial \hat{g}_{ij}}{\partial x_i} + \frac{\partial \hat{g}_{ij}}{\partial x_j} - \frac{\partial \hat{g}_{ij}}{\partial x_k} \right)$$

(3.1)

It follows from (3.1) that $\Gamma^k_{ij}(x) = \Gamma^k_{ij}(p)$ and $\overline{\Gamma}^k_{ij}(x) = \overline{\Gamma}^k_{ij}(p)$ i.e. $\overline{\nabla}_i X_j = 0$ for $i = n + 1, 2n$. Further, we get

$$\left(\overline{\nabla}_i X_j \right)(x) = \sum \left( J^k_j \overline{\Gamma}^k_{ij} + X_j \overline{J}^k_j \right)(x) X_k$$

(1.3)

$\overline{\nabla}_i J_j = 0$ for $i = n + 1, 2n$.

For $i = 1, n$ $(X_i \overline{J}_j)(x) = (X_i \overline{J}_j)(p)$ and we obtain

$$\left(\left(\overline{\nabla}_i J_j \right)(x) = \sum \left( J^k_i \overline{\Gamma}^k_{ij} - \overline{\Gamma}^k_{ij} J^k_i + X_i \overline{J}_j \right)(p) X_k \right).$$

From the other side we can write

$$\left(\left(\overline{\nabla}_i J_j \right)(x) = \sum \left( J^k_i \overline{\Gamma}^k_{ij} - \overline{\Gamma}^k_{ij} J^k_i + X_i \overline{J}_j \right)(p) X_k \right).$$

According to [6] we have $(\overline{\nabla}_i J_j)(x) = (2h_{ij} J_k)(x)$ where the second fundamental tensor field $h$ is defined by (2.11). From (1.1')-(8.1') it follows that $h_p = 0$ for any $p \in M \left( O = O_p \right)$. Thus, we have obtained $\overline{\nabla}_i J_i = 0$ and the structure $(J_1, \overline{g})$ is Kaehlerian one on $Tb\left(M, \frac{\varepsilon(p)}{2}\right)$.

QED.

As a corollary we have got the following:

**Theorem 3 [8].** Let $(M, g)$ be a smooth Riemannian manifold and $Tb\left(M, \frac{\varepsilon(p)}{2}\right)$ be the corresponding normal tubular neighborhood with respect to $g = \left(\overline{g}\right)$ on $\mathcal{M}$. Then $M(O_a)$ is a totally geodesic submanifold of the Kaehlerian manifold $\left( Tb\left(M, \frac{\varepsilon(p)}{2}\right), J_1, \overline{g}\right)$.

The classification given in [9] can be rewritten in terms of the second fundamental tensor field $h$ (Table 1).

---

**A. A. Ermolitski**
Table 1. Classification of almost Hermitian structures.

<table>
<thead>
<tr>
<th>Class</th>
<th>Defining condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>$h = 0$</td>
</tr>
<tr>
<td>$U_1 = NK$</td>
<td>$h_X = 0$</td>
</tr>
<tr>
<td>$U_2 = AK$</td>
<td>$\sigma h_{xyz} = 0$</td>
</tr>
<tr>
<td>$U_3 = SK \cap H$</td>
<td>$h_{xyz} - h_{xyz} = \beta(Z) = 0$</td>
</tr>
<tr>
<td>$U_4$</td>
<td>$h_{xyz} = \frac{1}{2(n-1)} \left[ (X,Y)\beta(Z) - (X,Z)\beta(Y) + (X,JY)\beta(JZ) + (X,JZ)\beta(JY) \right]$</td>
</tr>
<tr>
<td>$U_5$</td>
<td>$h_{xyz} = \frac{1}{2(n-1)} \left[ (X,Y)\beta(Z) - (X,Z)\beta(Y) + (X,JY)\beta(JZ) + (X,JZ)\beta(JY) \right]$</td>
</tr>
<tr>
<td>$U_6 = OK$</td>
<td>$h_{xyz} = h_{xyz}$</td>
</tr>
<tr>
<td>$U_7 = H$</td>
<td>$N(J) = 0$ or $h_{xyz} = -h_{xyz}$</td>
</tr>
<tr>
<td>$U_9 = SK \cap H$</td>
<td>$h_{xyz} = \beta(Z) = 0$</td>
</tr>
<tr>
<td>$U_10$</td>
<td>$\sigma \left[ h_{xyz} - \frac{1}{(n-1)} (X,Y)\beta(Z) \right] = 0$</td>
</tr>
<tr>
<td>$U_11$</td>
<td>$h_{xyz} = -\frac{1}{2(n-1)} \left[ (X,Y)\beta(X) - |Y|^2 \beta(Y) - (X,JY)\beta(JX) \right]$</td>
</tr>
<tr>
<td>$U_12$</td>
<td>$\sigma (h_{xyz} + h_{xyz}) = \beta(Z) = 0$</td>
</tr>
<tr>
<td>$U_13$</td>
<td>$\sigma (h_{xyz} + h_{xyz}) = 0$</td>
</tr>
<tr>
<td>$U_14$</td>
<td>$\sigma (h_{xyz} + h_{xyz}) = 0$</td>
</tr>
<tr>
<td>$U$</td>
<td>No condition</td>
</tr>
</tbody>
</table>

see chapter 5 of monograph [6].

Let $\dim M \geq 6$ and $2\beta(X) = \partial\Phi(JX)$, where $\Phi(X,Y) = g(JX,Y)$, then we have Table 1.

**Proposition 4.** Let $(J, g)$ be from some class from the Table 1. Then the structure $(\tilde{J}_2, \tilde{g})$ has the analogous class on $\left( M, \tilde{g}(p) \right)$.

**Proof.** From (1.2)- (8.2) it follows that $h_{xyz}^2 = 2h_{xyz}$. The rest is obvious from the table. QED.

4. Complex and Hypercomplex Numbers in Differential Geometry

For the manifold $M$ we consider the products $M^2 = M \times M = \{ (x,y) \mid x, y \in M \}$, $M^4 = M^2 \times M^2 = \{ (x,y; u,v) \mid x, y, u, v \in M \}$ and the diagonals $\Delta(M^2) = \{ (x,x) \mid x \in M^2 \}$, $\Delta(M^4) = \{ (x,x; x,x) \in M^4 \}$. It is obvious that the manifold $\Delta(M^2)$ and $\Delta(M^4)$ are diffeomorphic to $M$ ($\Delta(M^2) \cong \Delta(M^4) \cong M$).

**Theorem 5** [1]. Let $(M, \nabla)$ be a manifold with a connection $\nabla$ and $\pi : TM \to M$ be the canonical projection. Then there exists such a neighborhood $N_0$ of the null section $O_0$ in TM that the mapping

$$\varphi : \pi \times \exp : X \to \left( \pi(X), \exp_{\pi(X)} X \right)$$

is the diffeomorphic of $N_0$ on a neighborhood $N_\Delta$ of the diagonal $\Delta(M^2)$.

Further, $\nabla$ is a Riemannian connection of the Riemannian metric $g$. Combining the Theorems 3 and 5 we have obtained the following.

**Theorem 6.** The diffeomorphism $\varphi$ induces the Kaehlerian structure $(\tilde{J}_2, \tilde{g})$ on the neighborhood $N_\Delta$ of the diagonal $\Delta(M^2)$ and $\Delta(M^2) \cong M$ is a totally geodesic submanifold of the Kaehlerian manifold $(N_\Delta, \tilde{J}_2, \tilde{g})$. 

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Remark. Generally speaking, the complex structure of the Kaehlerian manifold \((N, \overline{J}, \overline{g})\) is not compatible with the product structure of \(M^4\). It means that if \(z_j, l=1,n\) are the complex coordinates of a point \((x, y) \in N\), then, generally speaking, we can not find such real coordinates \(x_j, y_j, l=1,n\) of the points \((x, y) \in M\) respectively that \(z_j = x_j + iy_j\) where \(i^2 = -1\).

Combining the Theorems 2, 3, 4, 5 and 6 we have obtained the following.

Theorem 7. There exists the hyper Kaehlerian structure \((\overline{N}, \overline{J}, \overline{J}_y, \overline{g})\) on a neighborhood \(N\) of the diagonal \(M\) and \(M \cong \Delta(M^4)\) is a totally geodesic submanifold of the hyper Kaehlerian manifold \((N, \overline{J}, \overline{J}_y, \overline{g})\).

Remark. Generally speaking, the hypercomplex structure of the hyper Kaehlerian manifold \((\overline{N}, \overline{J}, \overline{J}_y, \overline{g})\) is not compatible with the product structure of \(M^4\). It means that if \(q_j, l=1,n\) are the hypercomplex coordinates of a point \((x; y; u; v) \in \overline{N}\), then, generally speaking we can not find such real coordinates \(x_j, y_j, u_j, v_j, l=1,n\) of the points \((x; y; u; v) \in M\) respectively that \(q_j = x_j + iy_j + ju_j + kv_j\) where \(i^2 = j^2 = k^2 = -1\), \(ij = –ji = k\).

5. A Local Construction of Kaehlerian and Riemannian Metrics

1°. We consider a Riemannian manifold \((M, g)\) as a totally geodesic subanifold of the Kaehlerian manifold \(Tb\left(M, \frac{\varepsilon(p)}{2}; J = J_1, g\right)\) (see Theorem 3) then \(\overline{g}|_\mu = g\).

Let \(x_1, \ldots, x_n\) be coordinates in some coordinate neighborhood \(U \subset M\) and \(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\) be the corresponding vector fields. We can choose a neighborhood \(\overline{U} = U \times D = \bigcup_{p \in U} D(p; \varepsilon) \subset Tb\left(M, \frac{\varepsilon(p)}{2}\right)\) where \(\varepsilon \leq \frac{\varepsilon(p)}{2}\) for every point \(p \in U\). It is clear from 3°, 1 that \(U \times D\) is a Riemannian product with respect the metric \(\overline{g}\). For every point \(x \in \overline{U}\) where \(\pi(x) = p\) we denote \(Y_j = J \frac{\partial}{\partial x_j}, j=1,n\) and the vector fields \(Y_j\) define the coordinates \(y_1, \ldots, y_n\) on \(D(p; \varepsilon)\) hence \(Y_j = \frac{\partial}{\partial y_j}\) is tangent to \(D(p; \varepsilon)\) for \(j = 1,n\).

So, \(\overline{U}\) is an coordinate neighborhood of the Kaehlerian manifold \(\left(Tb\left(M, \frac{\varepsilon(p)}{2}\right), J, \overline{g}\right)\), with complex coordinates \(z_j = x_j + iy_j, j=1,n, i^2 = -1\), and the vector fields \(\frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j}\right), \alpha, \beta = 1,n\). It is known [3] that the Kaehlerian metric \(\overline{g}\) has on \(\overline{U}\) the following decomposition

\[
ds^2 = 2 \sum_{\alpha, \beta} \overline{g}_{\alpha, \beta} dx^\alpha dy^\beta, \quad \overline{g}_{\alpha, \beta} = \frac{\partial^2 u}{\partial z_\alpha \partial z_\beta},
\]

where \(u\) is a real-valued function on \(\overline{U}\).

We have

\[
\frac{\partial^2 u}{\partial z_\alpha \partial z_\beta} = \frac{1}{4} \left(\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} - \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} \right) + \frac{1}{4} \left(\frac{\partial^2 u}{\partial y_\alpha \partial x_\beta} + \frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} \right) = 0,
\]

\[
\frac{\partial^2 u}{\partial z_\alpha \partial z_\beta} = \frac{1}{4} \left(\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} - \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} \right) + \frac{1}{4} \left(\frac{\partial^2 u}{\partial y_\alpha \partial x_\beta} + \frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} \right) = 0.
\]
It follows that
\[
\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} = \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta}, \quad \frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} = -\frac{\partial^2 u}{\partial y_\alpha \partial x_\beta}.
\]

Further, we obtain
\[
\overline{g}_{\alpha\beta}^x = \frac{\partial^2 u}{\partial z_\alpha \partial z_\beta} = \frac{1}{4} \left( \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} + i \left( \frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} - \frac{\partial^2 u}{\partial y_\alpha \partial x_\beta} \right) \right) = \frac{1}{2} \left( \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + i \frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} \right),
\]
\[
\overline{g}_{\alpha\beta}^y = \frac{\partial^2 u}{\partial z_\alpha \partial z_\beta} = \frac{1}{4} \left( \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} - i \left( \frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} - \frac{\partial^2 u}{\partial y_\alpha \partial x_\beta} \right) \right) = \frac{1}{2} \left( \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} - i \frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} \right).
\]

Finally, we get
\[
\overline{g} \left( \frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta} \right) = \frac{1}{2} \text{Reg} \left( \frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta} \right) = \frac{1}{2} \text{Reg} \left( \frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta} \right) = \text{Re} \left( \overline{g}_{\alpha\beta}^x + \overline{g}_{\alpha\beta}^y + \overline{g}_{\alpha\beta}^x + \overline{g}_{\alpha\beta}^y \right) = \frac{\partial^2 u}{\partial x_\alpha \partial y_\beta}.
\]

We can consider the restriction of \( \overline{g} \) and the function \( u \) on the neighborhood \( U \). So, we have obtained.

**Theorem 8.** Let \((M, g)\) be a Riemannian manifold and \( x_1, \cdots, x_n \) be coordinates is some coordinate neighborhood \( U \subset M \). There exists a smooth function \( u: U \to \mathbb{R} \) that \( g_{ij} = g \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \frac{\partial^2 u}{\partial x_i \partial x_j} \) on \( U \).

2°. Let \((M, J, g)\) be a Kaehlerian manifold \( x_1, \cdots, x_n, \, y_1, \cdots, y_n \), be coordinates is some coordinate neighborhood \( U \subset M \), where \( \frac{\partial}{\partial y_\alpha} = J_\frac{\partial}{\partial x_\alpha}, \, \alpha = 1, n \). We consider a function \( u: U \to \mathbb{R} \) from Theorem 5. Then, we have the following conditions on this function.

\[
\frac{\partial^2 u}{\partial x_\alpha \partial y_\beta} = g \left( \frac{\partial}{\partial x_\alpha}, J_\frac{\partial}{\partial x_\beta} \right) = -g \left( J_\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta} \right) = -\frac{\partial^2 u}{\partial y_\alpha \partial y_\beta};
\]
\[
\frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} = g \left( J_\frac{\partial}{\partial x_\alpha}, J_\frac{\partial}{\partial x_\beta} \right) = g \left( \frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta} \right) = \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta}, \, \alpha, \beta = 1, n.
\]

**6. Conclusion**

We consider such mappings in the category of Riemannian manifolds that metrics are invariant with respect to them. It follows that only totally geodesic submanifolds are “naturally good”. Theorems 6 and 7 allow considering any Riemannian manifold as a totally geodesic submanifold of a Kaehlerian (hyper Kaehlerian) one \( i.e. \) to apply the results of Kaehlerian (hyper Kaehlerian) geometry to Riemannian metrics. We remark that Whittles embeddings are not suitable in this context.

**References**


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