Evolution of Generalized Space Curve as a Function of Its Local Geometry

Nassar H. Abdel-All¹, Samah G. Mohamed¹, Mariam T. Al-Dossary²

¹Mathematics Department, Faculty of Science, Assiut University, Assiut, Egypt
²Mathematics Department, Girls College of Science, University of Dammam, Dammam, KSA
Email: samah_gaber2000@yahoo.com

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Abstract
Kinematics of moving generalized curves in a $n$-dimensional Euclidean space is formulated in terms of intrinsic geometries. The evolution equations of the orthonormal frame and higher curvatures are obtained. The integrability conditions for the evolutions are given. Finally, applications in $R^2$ are given and plotted.

Keywords
Inelastic Curve, Curve Flow, Evolution of Curves

1. Introduction
The flow of a curve is called inelastic if the arclength of this curve is preserved. Inelastic curve flows have an importance in many applications such as engineering, computer vision [1] [2], computer animation [3] and even structural mechanics [4]. Physically, inelastic curve flows give rise to motion which no strain energy is induced. There exist such motions in many physical applications. G. S. Chirikjian and J. W. Burdick [5] studied applications of inelastic curve flows. M. Gage, R. S. Hamilton [6] and M. A. Grayson [7] investigated shrinking of closed plane curves to a circle via the heat equation. Also, D. Y. Kwon and F. C. Park [8] [9] derived the evolution equation for an inelastic plane and space curve. Latifi et al. [10] studied inextensible flows of curves in Minkowski 3-space.

The connection between integrable systems and differential geometry of curves and surfaces has been important topic of intense research [11] [12]. Goldstein and Petrich [13] showed that the celebrated mKdV equation naturally arises from inextensible motion of curves in Euclidean geometry. Nakayama, Segur and Wadati [14] set up a correspondence between the mKdV hierarchy and inextensible motions of plane curves in Euclidean geometry. Integrable systems satisfied by the curvatures of curves under inextensible motions in projective
geometries are identified in [15]. Inextensible flows of curves in Galilean space are investigated in [16].

In this paper, we shall present a general formulation of evolving generalized curves in \( \mathbb{R}^n \). The outline of this paper is as follows: In Section 2, we give the local differential geometry of curves in \( \mathbb{R}^n \). In Sections 3 and 4, we describe the motion of generalized curves in \( \mathbb{R}^n \). In Section 5, the integrability conditions for the considered model are obtained. In Section 6, we specialized the motion of curves \( \mathbb{R}^n \) to motion of plane curves (curves in \( \mathbb{R}^2 \)). Finally, Section 7 is devoted to conclusion.

2. Geometric Preliminaries

A generalized curve in a \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) can be regarded as a Riemannian submanifold of dimension 1 in \( \mathbb{R}^n \) [17].

**Definition 1** A differentiable manifold of dimension 1 immersed in \( \mathbb{R}^n \) is a topological Hausdorff space \( M \) with a differentiable structure \( (I_{\beta},\phi_{\beta}) \) with dimension one, where \( I_{\beta} \) is an open interval in \( \mathbb{R} \) and \( \phi_{\beta} \) is a diffeomorphism mapping:

\[
\phi_{\beta} : I_{\beta} \to \mathbb{R}^n,
\]

and \( \beta \) belongs to some index set \( \Lambda \).

**Definition 2** A Generalized curve \( C \) in \( \mathbb{R}^n \) is an image of a diffeomorphism \( \phi : I \to \mathbb{R}^n \), where \( I \) is an open interval of \( \mathbb{R} \). The representation of \( C \) in \( \mathbb{R}^n \) is given by

\[
u(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \in \mathbb{R}^n,
\]

where \( u \) is called the parameter of the curve \( C \).

The representation (1.1) is called the regular parametric representation of \( C \) in \( \mathbb{R}^n \), when

\[
\frac{d\phi}{du} \neq 0, \quad u \in I.
\]

Also \( \phi \) represents an immersion in \( \mathbb{R}^n \) if \( \frac{d\phi}{du} \bigg|_{u} = 1 \), the parameter \( u \), in this case, is called the arclength parameter and is denoted by \( s \), \( \phi(s) \) is called arclength parametrization.

**Frenet Frame**

A Frenet frame is a moving reference frame of \( n \) orthonormal vectors \( e_i(s) \) which are used to describe the curve locally at each point \( \phi(s) \). It is the main tool in differential geometric treatment of curves as it is far easier and more natural to describe the local properties (e.g. curvature and torsion) in terms of local reference system than using a global one like the Euclidean coordinates.

Give a curve \( \phi \) in \( \mathbb{R}^n \) which is regular of order \( n \). The Frenet frame for the curve is the set of orthonormal vectors (Frenet vectors) \( \mathfrak{F} = \{e_1(s), e_2(s), \ldots, e_n(s)\} \), and they are constructed from the derivatives of

\[
\{\phi(s), \phi'(s), \phi''(s), \ldots, \phi^{(n)}(s)\},
\]

which are linearly independent vectors, \( \left(\phi^{(n)}(s) = \frac{d^n}{ds^n}\right) \).

Using the Gram-Schmidt orthogonalization algorithm which convert linearly independent vectors

\[
\{\phi'(s), \phi''(s), \ldots, \phi^{(n)}(s)\}
\]

into the orthonormal one \( \{e_1(s), e_2(s), \ldots, e_n(s)\} \) as follows:

\[
e_i(s) = \phi'(s), e_1(s) = \frac{W_2}{W_1}, e_2(s) = \frac{W_3}{W_1}, i = 2, \ldots, n,
\]

where

\[
W_2(s) = \phi''(s) - \{\phi'(s), e_1(s)\} e_1(s), \ldots, W_l(s) = \phi^{(l)}(s) - \sum_{k=1}^{l-1} \{\phi^{(k)}(s), e_k(s)\} e_k(s).
\]

By this way, one obtain an orthonormal \( n \)-tuple of vectors at \( \phi(s) \), called the Frenet \( n \)-frame associated with the generalized curve at the point \( \phi(s) \), with \( e_i(s) \) is of class \( C^{k-i} \) if \( \phi(s) \in C^k \). The derivatives of the frenet \( n \)-frame at \( \phi(s) \) satisfy the following Frenet formulas:
\[ e'_j = k_{e_{j+2}} \] for \( j = 1, 2, \ldots, n \), where \( k_n = 0 \). These equations can be written in a matrix form:

\[
\begin{bmatrix}
0 & k_1 & 0 & 0 & \cdots & 0 & 0 \\
-k_1 & 0 & k_2 & 0 & \cdots & 0 & 0 \\
0 & -k_2 & 0 & k_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -k_{n-2} & 0 & k_{n-1} \\
0 & 0 & 0 & \cdots & 0 & -k_n & 0
\end{bmatrix}
\]

\[ \mathbf{Q} = (e_1, e_2, \ldots, e_n) \],

\[ \mathbf{E} = \begin{bmatrix} \mathbf{E}_1 \mathbf{E}_2 \vdots \mathbf{E}_n \end{bmatrix} \]

where \( \mathbf{E}_i \) are higher curvature functions or Euclidean curvatures of the curve. The \( m \)-th Euclidean curvature \( k_m \) gives the speed of rotation of the osculating \( m \)-plane around the osculating \( (m-1) \)-plane.

3. Dynamics of Curves in \( R^n \)

Consider a smooth curve in \( n \)-dimension space. Assume that \( u \) is the parameter along the curve in \( R^n \). Let \( r(u, t) \) denotes the position vector of a point on the curve at time \( t \). The metric on the curve is:

\[
g(u, t) = \frac{\partial r}{\partial u} \cdot \frac{\partial r}{\partial u}
\]

The arclength along the curve is given by:

\[
s(u, t) = \int_0^u \sqrt{g(\sigma, t)} d\sigma, \quad \frac{\partial}{\partial s} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u},
\]

we use \( \{u, t\} \) as coordinates of a point on the curve. At \( r(u, t) \), consider the orthonormal frame \( \mathcal{I} = \{e_1, e_2, \ldots, e_n\} \) such that \( e_1 \) is the tangent vector and \( e_2, e_3, \ldots, e_n \) denote the normal vectors at any point on the curve.

Dynamics of the curve in \( R^n \) (motion of a point on the curve) can be specified by the form:

\[
\dot{r} = \frac{dr}{dt} = \sum_{j=1}^n v_j e_j,
\]

where \( v_j \) are the velocities along the frame \( e_j \). Consider a local motion that is the velocities \( v_j \) depend only on the local values of the curvatures \( \{k_1, k_2, \ldots, k_{n+1}\} \).

**Lemma 1** The evolution equation for the metric \( g \) is given by:

\[
\dot{g} = 2g \left( \frac{\partial v_1}{\partial s} - k_1 v_2 \right).
\]

**Proof 1** Take the \( t \) derivative of (1.5) and \( s \) derivative of (1.7), and since \( \frac{\partial}{\partial u}, \frac{\partial}{\partial t} \) commute, then we have:

\[
\dot{g} = \frac{\partial g}{\partial t} = 2 \left( \frac{\partial r}{\partial u}, \frac{\partial \partial r}{\partial u} \right) = 2g \left( \frac{\partial r}{\partial s}, \frac{\partial \partial r}{\partial s} \right) = 2g \left( e_1, \sum_{j=1}^n \frac{\partial v_j}{\partial s} e_j + \sum_{j=1}^n v_j \frac{\partial e_j}{\partial s} \right).
\]

Using (1.2), then we have

\[
\dot{g} = 2g \left( e_1, \lambda e_1 + \sum_{j=2}^n A_j e_j \right),
\]
where,

\[
\lambda = \left( \frac{\partial v_i}{\partial s} - k_i v_i \right),
\]

\[
A_j = v_{j,i} + k_{j,i} v_{j,i+1} - k_i v_{j+1},
\]

\( j = 2, 3, \ldots, n, \quad k_0 = k_n = 0. \)  \hspace{1cm} (1.9)

Then

\[ \dot{g} = 2g \lambda. \]

Hence the lemma holds.

**Lemma 2** *For a simple closed curve, the evolution of the length of the curve is given by:*

\[ \frac{\partial L}{\partial t} = \int_0^L \left( \frac{\partial v_i}{\partial s} - k_i v_i \right) d\sigma, \quad u \in [0, L]. \]

**Proof 2** *From the definition of the length \( L \), we have*

\[ \frac{\partial L}{\partial t} = \frac{\partial s}{\partial t} = \int_0^L \frac{\partial}{\partial t} (\sqrt{g}) d\sigma = \int_0^L \frac{\dot{g}}{2\sqrt{g}} d\sigma. \]  \hspace{1cm} (1.10)

Substitute from (1.8) into (1.10), then the lemma holds.

### 4. Main Results

**Definition 3** *An inelastic curve is a curve whose length is preserved, i.e., it doesn’t evolve in time.*

\[ \frac{\partial s}{\partial t} = 0, \quad i.e., \quad g_i = \dot{g} = 0. \]  \hspace{1cm} (1.11)

The necessary and sufficient conditions for inelastic flow are then given by the following theorem:

**Theorem 1** *The flow of the curve is inelastic if and only if \( \frac{\partial v_i}{\partial s} = k_i v_i. \)

**Proof 3** \( \Rightarrow \) *Assume that the curve is inelastic.*

From (1.6), the variation of the arclength is

\[ \dot{s} = \frac{\partial s}{\partial t} = \int_0^L \frac{g_i}{2\sqrt{g}} d\sigma. \]  \hspace{1cm} (1.12)

Substitute from (1.8) into (1.12), then

\[ \dot{s} = \int_0^L \left( \frac{\partial v_i}{\partial s} - k_i v_i \right) d\sigma. \]

Since the curve is inelastic, so \( \dot{s} = 0 \), hence

\[ \frac{\partial v_i}{\partial s} = k_i v_i. \]

\( \Leftarrow \) *Assume that \( \frac{\partial v_i}{\partial s} = k_i v_i. \) Substituting from this equation into (1.8), so \( g_i = 0 \), then \( \dot{s} = 0 \), this means that the arclength of the curve is preserved, hence the curve is inelastic.

**Theorem 2** *Consider an elastic curve \( r(u, t) \). For the curve flow \( \frac{dr}{dt} = \sum_{j=1}^n y_j e_j \), then*

1) *The evolution for the frame \( E = (e_1, e_2, \cdots, e_n)' \), can be given in a matrix form:

\[ E_t = M \cdot E, \]  \hspace{1cm} (1.13)

where \( M \) is the evolution matrix and it takes the form:
\[ M = \begin{pmatrix}
0 & M_{12} & M_{13} & \cdots & M_{1n} \\
-M_{12} & 0 & M_{23} & \cdots & M_{2n} \\
-M_{13} & -M_{23} & 0 & \cdots & M_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-M_{(n-1)2} & -M_{2(n-1)} & -M_{3(n-1)} & \cdots & M_{(n-1)n} \\
-M_{1n} & -M_{2n} & -M_{3n} & \cdots & 0 
\end{pmatrix}, \]

where the elements of the matrix \( M \) are given explicitly by:

\[ M_{1j} = A_j = v_{j,t} + k_{j-1}v_{j-1} - k_jv_{j+1}, \]

\[ j = 2, 3, \ldots, n. \]

\[ M_{\alpha\beta} = \frac{1}{k_{\mu-\alpha}} \left( A_{(\alpha-1)\mu,\alpha} + k_{\mu-1}A_{(\alpha-1)(\mu-1)} - k_\mu A_{(\alpha-1)(\mu+1)} + k_{\alpha-2}A_{(\alpha-2)\mu} \right), \]  

(1.14)

\[ \alpha = 2, \ldots, n-1, \]

\[ \alpha < \mu \leq n, \quad k_0 = k_n = 0. \]

2) The evolution equations for the curvatures take the form:

\[ k_{1,t} = M_{1,2s} - k_1\lambda - k_2M_{1,3}, \]

\[ k_{\alpha,t} = M_{\alpha,\mu s} - k_\alpha\lambda + k_{\alpha-1}M_{\alpha,\mu} - k_{\alpha+1}M_{\alpha(\mu+1)} - M_{\alpha(\mu-1)}, \]  

(1.15)

**Proof 4** Consider the elastic curve \( r(u, t) \) i.e., \( g \neq 0 \). Take the \( u \) derivative of (1.7), then we have:

\[ r_u = \sqrt{g}r'_u = \sqrt{g} \left( \lambda e_t + \sum_{j=2}^{n} A_j e_j \right), \]  

(1.16)

Since \( r_u = \sqrt{g}r'_u = \sqrt{g}e_t \), take the \( t \) derivative of this equation, then we have

\[ r_{tt} = \sqrt{g} \left( \frac{g_t}{2g} e_t + e_{tt} \right). \]  

(1.17)

Since

\[ r_{tt} = r_{uu}. \]  

(1.18)

Substitute from (1.9), (1.16) and (1.17) into (1.18), then we have

\[ e_{tt} = \sum_{j=2}^{n} A_j e_j. \]  

(1.19)

Take the \( u \) derivative of (1.19), then we have:

\[ e_{1,uu} = \sqrt{g} \left( (-k_1A_2)e_t + (A_{2,t} - k_2A_1)e_t + \sum_{j=3}^{n} (A_{j,t} + k_{j-1}A_{j-1} - k_jA_{j+1})e_j \right). \]  

(1.20)

Since \( e_{1,uu} = \sqrt{g}e_{1,tt} = \sqrt{g} (k_1e_2) \), take the \( t \) derivative of this equation, then we have

\[ e_{1,ut} = \sqrt{g} \left( \frac{g_t}{2g} k_1 + k_{1,t} \right) e_2 + k_1e_{2,tt}. \]  

(1.21)

Since

\[ e_{1,ut} = e_{1,uu}. \]  

(1.22)

Substitute from (1.20) and (1.21) into (1.22), then we have
\[ k_{1,j} = A_{2,j} - k_i \lambda - k_j A_j, \]
\[ e_{2,j} = -A_j e_j + \sum_{j=3}^{n} B_j e_j, \]
\[ B_j = \frac{1}{k_i} \left( A_{j,i} + k_{j-1} A_{j,i-1} - k_j A_{j,i} \right), \quad j = 3, \ldots, n. \]

Since \( e_{2,u} = \sqrt{g} e_{2,v} = \sqrt{g} \left( -k_i e_i + k_2 e_j \right) \), take the \( t \) derivative of this equation, then we have
\[ e_{2,u} = \sqrt{g} \left( -\left( k_2 \lambda + k_j e_j \right) e_i - \left( k_i A_i + k_2 A_j + k_j A_j \right) e_j + k_2 e_{j,j} - k_i \sum_{j=4}^{n} A_i e_j \right). \]
(1.23)

Take the \( u \) derivative of (1.23), then we have
\[ e_{2,u} = \sqrt{g} \left( -A_j e_i + B_j e_j \left( B_{3,j} - k_3 B_j \right) e_j + \left( k_2 A_2 + k_3 B_3 - k_4 B_4 \right) e_4 + \cdots \right. \]
\[ + \left. \left( B_{n-1,j} + k_{n-2} B_{n-2} - k_{n-1} B_{n-1} \right) e_{n-1} + \left( B_{n+1,j} + k_{n} B_{n} \right) e_n \right). \]
(1.24)

Since
\[ e_{2,u} = e_{2,u}. \]
(1.25)

Substitute from (1.24) and (1.25) into (1.26), and use the first equation of (1.23), then we have
\[ k_{2,j} = B_{3,j} - k_j \lambda - k_j B_j + k_j A_j, \]
\[ e_{3,j} = -A_j e_i - B_j e_j \sum_{j=4}^{n} C_j e_j, \]
(1.26)

Take the \( u \) derivative of the second equation of (1.27), then we have
\[ e_{3,u} = \sqrt{g} \left( \left( -A_{3,j} + k_j B_j \right) e_i - \left( B_{3,j} + k_j A_j \right) e_j - \left( k_2 B_2 + k_3 C_3 \right) e_3 + \left( C_{4,j} - k_4 C_4 \right) e_4 + \left( C_{5,j} + k_5 C_5 - k_6 C_6 \right) e_5 + \cdots \right. \]
\[ + \left. \left( C_{n-1,j} + k_{n-2} C_{n-2} - k_{n-1} C_{n-1} \right) e_{n-1} + \left( C_{n+1,j} + k_{n} C_n \right) e_n \right). \]
(1.27)

Since \( e_{3,u} = \sqrt{g} e_{3,v} = \sqrt{g} \left( -k_2 e_j + k_j e_4 \right) \), take the \( t \) derivative of this equation and use the first equation of (1.27), then we have
\[ e_{3,u} = \sqrt{g} \left( k_2 A_2 e_i + \left( -B_{3,j} + k_2 B_2 \right) e_j - \left( k_2 B_2 - k_4 B_4 \right) e_4 + \left( k_j \lambda + k_2 B_4 + k_3 A_3 \right) e_5 + \left( k_3 e_3 - k_4 B_j \right) e_j - k_i \sum_{j=3}^{n} B_j e_j \right) \]
(1.28)

Since
\[ e_{3,u} = e_{3,u}. \]
(1.29)

Substitute from (1.28) and (1.29) into (1.30), then we have
\[ k_{3,j} = C_{4,j} + k_2 B_2 - k_4 \lambda - k_4 C_4, \]
\[ e_{4,j} = -A_j e_i - B_j e_j - C_j e_j + \sum_{j=5}^{n} D_j e_j, \]
(1.30)

By using the mathematical induction, we can extend the previous results to \( n \)-dimensional space as follows:
\[ E_i = M \cdot E, \]
where \( E = (e_1, e_2, \ldots, e_n)^T \), and
\[
M = \begin{pmatrix}
0 & A_2 & A_3 & A_4 & \cdots & A_n \\
-A_2 & 0 & B_3 & B_4 & \cdots & B_n \\
-A_3 & -B_3 & 0 & C_4 & \cdots & C_n \\
-A_4 & -B_4 & -C_4 & 0 & \cdots & D_n \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-A_n & -B_n & -C_n & -D_n & \cdots & 0
\end{pmatrix}.
\]
(1.32)

We can rewrite (1.9) and the third equation of (1.23), (1.27) and (1.31) as follows:
\[
A_j = M_{1j}, \quad 1 < j \leq n,
\]
\[
B_j = M_{1j}, \quad 2 < j \leq n,
\]
\[
C_j = M_{1j}, \quad 3 < j \leq n,
\]
\[
D_j = M_{1j}, \quad 4 < j \leq n.
\]
(1.33)

Hence
\[
M_{1j} = A_j = v_{j,1} + k_{j,j-1}v_{j-1} - k_{j}v_{j,1}, \quad j = 2, 3, \ldots, n
\]

\[
M_{\mu\nu} = \frac{1}{k_{\mu+\nu}} \left( M_{(\mu-1)\nu} + k_{\mu-1}M_{(\mu-2)(\nu-1)} - k_{\nu}M_{(\mu-3)\nu} + k_{\mu-2}M_{(\mu-4)\nu} \right),
\]
(1.34)

\( \alpha = 2, \ldots, n-1, \quad \mu = 3, \ldots, n, \quad \alpha < \mu, \quad k_0 = k_n = 0. \)

So we can rewrite (1.32) in the following form:
\[
M = \begin{pmatrix}
0 & M_{12} & M_{13} & \cdots & M_{1n} \\
-M_{12} & 0 & M_{23} & \cdots & M_{2n} \\
-M_{13} & -M_{23} & 0 & \cdots & M_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-M_{1n-1} & -M_{2n-1} & -M_{3n-1} & \cdots & 0
\end{pmatrix}
\]
(1.35)

By using the mathematical induction, we can extend the results in the first equation of (1.23), (1.27) and (1.31) as follows:
\[
k_{1,s} = M_{12,s} - k_{1}\lambda - k_{2}M_{13},
\]
\[
k_{a,j} = M_{a\nu,s} - k_{a}\lambda + k_{a-1}M_{(a-1)\nu} - k_{a-2}M_{(a-2)\nu},
\]
(1.36)

Hence the theorem holds.

**Lemma 3** If the curve flow (1.7) is inelastic, then the evolution equations for curvatures (1.36) take the form:
\[
k_{1,s} = M_{12,s} - k_{2}M_{13},
\]
\[
k_{a,j} = M_{a\nu,s} + k_{a-1}M_{(a-1)\nu} - k_{a+1}M_{(a+1)\nu},
\]
(1.37)

**Proof 5** If the curve flow (1.7) is inelastic, then
\[
g_{s} = 0, \text{ i.e., } \lambda = 0.
\]
(1.38)

Then, substitute from (1.38) into (1.36), then the lemma holds.

### 5. Integrability Conditions

**Theorem 3** The curve \( r(u,t) \) is inelastic curve \( (g_{s} = 0) \) if and only if the integrability condition (sometimes
called the zero curvature condition is given by:

\[ Q_t - M_s + [Q, M] = 0, \]  

(1.39)

where \([Q, M] = Q \cdot M - M \cdot Q\) is the Lie bracket.

**Proof 6** Consider the Frenet frame \(E = (e_1, e_2, \ldots, e_n)\) that satisfy (1.4) and (1.35). Since

\[ E_s = \sqrt{g} E_t = \sqrt{g} Q \cdot E. \]  

(1.40)

Take the \( t \) derivative of (1.40) and use (1.13), then we have

\[ E_{st} = \sqrt{g} \left( \frac{g_t}{2g} Q + Q_t + Q \cdot M \right) \cdot E. \]  

(1.41)

Differentiating (1.13) with respect to \( u \) and use (1.40), then

\[ E_{u} = \sqrt{g} (M_s + M \cdot Q) \cdot E. \]  

(1.42)

From (1.41) and (1.42), then

\[ E_{st} - E_{u} = \sqrt{g} \left( \frac{g_t}{2g} Q_t - M_s + [Q,M] \right) \cdot E. \]  

(1.43)

(\( \Rightarrow \)) **First**, If the curve is inelastic, so \((g_t = 0)\) and \( v \) and \( t \) commute, then \( E_{st} = E_{u} \), hence

\[ Q_t - M_s + [Q,M] = 0. \]  

(1.44)

From (1.4) and (1.35), we have

\[
[Q,M] = \begin{bmatrix}
0 & k_2 M_{13} & M_{13,s} & \cdots & M_{1n,s} \\
-k_2 M_{13} & 0 & -k_1 M_{13} + k_3 M_{24} & \cdots & M_{2n,s} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-M_{1n-1,s} & -M_{2n-1,s} & -M_{3n-1,s} & \cdots & -k_{n-2} M_{n-2,n} \\
-M_{ln,s} & -M_{2n,s} & -M_{3n,s} & \cdots & 0
\end{bmatrix}. 
\]  

(1.45)

Differentiating (1.4) with respect to \( t \) and (1.35) with respect to \( s \) and use (1.36), then we have

\[
M_s - Q_t = \begin{bmatrix}
0 & -k_2 M_{13} + \lambda k_1 & M_{13,s} & \cdots & M_{1n,s} \\
-k_2 M_{13} + \lambda k_1 & 0 & -\lambda k_2 - k_3 M_{13} + k_4 M_{24} & \cdots & M_{2n,s} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-M_{1n-1,s} & -M_{2n-1,s} & -M_{3n-1,s} & \cdots & -\lambda k_{n-1} - k_{n-2} M_{n-2,n} \\
-M_{ln,s} & -M_{2n,s} & -M_{3n,s} & \cdots & 0
\end{bmatrix}. 
\]  

(1.46)

Substitute from (1.45) and (1.46) into (1.44), then we have

\[
\begin{bmatrix}
0 & \lambda k_1 & 0 & \cdots & 0 \\
\lambda k_1 & 0 & -\lambda k_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -\lambda k_{n-1} \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix}
\]  

(1.47)

Hence

\[ \lambda k_1 = 0, \lambda k_2 = 0, \ldots, \lambda k_{n-1} = 0. \]

Since \( k_m \neq 0 \), for \( m = 1, 2, \cdots, n-1 \). Then \( \lambda = 0 \). Hence \( g_t = 0 \), \( g = \) constant \( i.e.\), the arclength is preserved. Hence the curve is inelastic.
Theorem 4 In $n$-dimensional Euclidean space, consider inelastic curve $r(u,t)$. If the matrices $Q$ and $M$ are abelian, then the elements in the evolution matrix $M$ take the form:

$$M_{(\alpha-1)\mu} = 0, \quad \alpha = 2,3,\ldots,n-1, \mu = \alpha + 1.$$ 

Proof 7 Since the matrices $Q$ and $M$ are abelian, so $[Q,M] = 0$, then the integrability condition (1.39) takes the form:

$$M_{s} - Q_{t} = 0.$$ 

Since the curve is inelastic, so $\lambda = 0$, then

$$M_{s} - Q_{t} = \begin{pmatrix} 0 & k_{2}M_{13} & \cdots & M_{1n,s} \\ -k_{2}M_{13} & 0 & \cdots & M_{2n,s} \\ \vdots & \vdots & \ddots & \vdots \\ -M_{(n-1),s} & -M_{2(n-1),s} & \cdots & 0 \end{pmatrix}.$$ 

(1.49)

Substitute from (1.49) into (1.48), then for $n = 10$, we have

$$M_{13} = M_{15} = M_{37} = M_{29} = 0, M_{24} = M_{46} = M_{88} = M_{8(10)} = 0,$$

By using the mathematical induction, we can extend the previous results to $n$-dimensional space, then we have

$$M_{(\alpha-1)\mu} = 0, \quad \alpha = 2,3,\ldots,n-1, \mu = \alpha + 1.$$ 

6. Applications

Here we give some applications for time evolution equations for plane curve. We are in a position to derive time evolution of geometrical quantities. For $n = 2$, and from (1.13), we have motion of the Frenet frame of the curve in the plane.

Lemma 4 In 2-dimensional Euclidean space, consider an elastic curve $r(u,t)$. The time evolution equation for the frame $E = (e_{1},e_{2})^{T}$ is given by

$$\begin{pmatrix} e_{1} \\ e_{2} \end{pmatrix}_{t} = \begin{pmatrix} 0 & M_{12} \\ -M_{12} & 0 \end{pmatrix} \begin{pmatrix} e_{1} \\ e_{2} \end{pmatrix}$$

where,

$$M_{12} = v_{2,s} + k_{1}v_{1}.$$ 

Lemma 5 The time evolution equation for the curvature of the curve in $R^{2}$ is given explicitly by

$$k_{1,s} = v_{2,ss}^{2} + v_{1}^{2}k_{1},$$

(1.50)

This equation represents a quasilinear parabolic partial differential equation (PDE). This result coincide with [18].

Example 1 If

$$v_{1} = k_{1}^{2}, \quad v_{2} = k_{1,ss}.$$ 

Then (1.50) takes the form:

$$k_{1,s} = k_{1,ssss} + k_{1,ss}^{2}k_{1} + k_{1}^{2}k_{1,s}.$$ 

(1.51)

The solution of the PDE (1.51) is

$$k_{1}(s,t) = \sqrt{10} \tanh \left( c_{1} - \frac{4s + t}{4\sqrt{2}} \right).$$
where \( c_1 \) is constant. The curvature \( k_1(s,t) \) of the curve is plotted as a function of \( s \) and \( t \) (Figure 1(a)), and for different values of \( t \), the curvature \( k_1(s) \) is plotted (Figure 1(b)).

**Example 2**
If
\[
21121,1,vkvv=−
\]
Then (1.50) takes the form:
\[
v_1 = k_1^2, \quad v_2 = -k_{1,s}.
\]
The solution of the PDE (1.52) is
\[
k_1(s,t) = \frac{1}{\sqrt{c}} \left( c_1 e^{\sqrt{c}(s-\lambda t)} - c_2 e^{-\sqrt{c}(s-\lambda t)} \right) + c_3,
\]
where \( c_1, c_2, c_3 \) and \( \lambda \) are constants. The curvature \( k_1(s,t) \) of the curve is plotted as a function of \( s \) and \( t \) (Figure 2(a)), and for different values of \( t \), the curvature \( k_1(s) \) is plotted (Figure 2(b)).

**Example 3**
If
\[
21111,1,1,1,vkvkkk=+
\]
Then (1.50) takes the form:
\[
v_1 = k_1^2, \quad v_2 = k_{1,s}.
\]
The solution of the PDE (1.53) is
\[
k_1(s,t) = \sqrt{3} \text{sech}\left(c_2 + c_3s + c_1t\right),
\]
where \( c_1 \) and \( c_2 \) are constants. The curvature \( k_1(s,t) \) of the curve is plotted as a function of \( s \) and \( t \) (Figure 3(a)), and for different values of \( t \), the curvature \( k_1(s) \) is plotted (Figure 3(b)).

![Figure 1](image1.png)
**Figure 1.** The curvature of the curve for \( c_1 = 0.01, s \in [-1,1], t \in [0,1]. \)

![Figure 2](image2.png)
**Figure 2.** The curvature of the curve for \( \lambda = 1, c_1 = 0.04, c_2 = 0.4, c_3 = 0.9, s \in [-0.5,0.5], t \in [0,1]. \)
7. Conclusion

In this paper, we have discussed the motion of curves in $n$-dimension Euclidean space. We derived the evolution equations of the orthonormal frame and evolution equations for the higher curvatures. We get the integrability conditions for the evolutions. Moreover, we give some examples of motions of elastic curves in the plane.

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References


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