A Note on the Almost Sure Central Limit Theorem in the Joint Version for the Maxima and Partial Sums of Certain Stationary Gaussian Sequences*

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Abstract

Considering a sequence of standardized stationary Gaussian random variables, a universal result in the almost sure central limit theorem for maxima and partial sum is established. Our result generalizes and improves that on the almost sure central limit theory previously obtained by Marcin Dudzinski [1]. Our result reaches the optimal form.

Keywords

Almost Sure Central Limit Theorem, Stationary Gaussian Sequence, Slowly Varying Functions at Infinity

1. Introduction

In this paper, we let \( \left( X, X_n \right)_{n \in \mathbb{N}} \) be a standardized stationary Gaussian sequence, also let

\[
M_n := \max_{i \in \mathbb{N}} X_i, \quad M_{m,n} := \max_{m+1 \leq i \leq m+n} X_i, \quad S_n := \sum_{i=1}^{n} X_i, \quad \sigma_n := \sqrt{\text{Var}(S_n)}, \quad D_n = \sum_{k=1}^{n} d_k, \quad \{d_k\}
\]

be some sequence of weights, \( I(\cdot) \) and \( \Phi(\cdot) \) denote the indicator function and the standard normal distribution function, respectively.

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The ASCLT has been first introduced independently by Brosamler [2] and Schatte [3] for partial sum, since then the concept has already started to receive applications in many fields. For example, Fahrner and Sadtmuller [4] and Cheng et al. [5] extended this almost sure central limit theorem for partial sums to the case of maxima of i.i.d. random variables. Under some conditions, they proved as follows:

$$\lim_{n \to \infty} \frac{1}{\ln n} \sum_{k=1}^{\infty} \frac{1}{k} \mathbb{I}(a_k(S_k - b_k) \leq x) = G(x) \quad \text{a.s.},$$

for all \( x \in \mathbb{R} \), where \( a_k > 0 \) and \( b_k \in \mathbb{R} \) satisfy \( a_k(S_k - b_k) \xrightarrow{d} G \), where \( G(\cdot) \) is some non-degenerate distribution function. Afterwards, Marcin Dudzinski [5] showed the ASCLT in its two-dimensional version, i.e.

$$\lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \mathbb{I}\left(a_k(M_k - b_k) \leq x, \frac{S_k}{\sigma_k} \leq y\right) = e^{-\tau} \Phi(y) \quad \text{a.s. for } \forall x, y \in \mathbb{R}.$$  \hspace{1cm} (1.1)

In this paper, we extend the weight \( d_k = \frac{1}{k} \) to the weight \( d_k = \frac{\exp\left((\ln k)^{\beta}\right)}{k} \), \( 0 \leq \beta < \frac{1}{2} \).

Our purpose is to prove that if \((X, X_n)_{n \in \mathbb{N}}\) is a standardized stationary Gaussian sequence, the covariance function \( r(t) \) fulfills

$$r(t) = \frac{L(t)}{t^{\alpha}}, \quad \alpha > 0 \quad \text{and} \quad t = 1, 2, \ldots,$$

where \( L(t) \) is a positive slowly varying function at infinity. Moreover if the numerical sequence \( \{u_n\} \) satisfies the following relation

$$\lim_{n \to \infty} n(1 - \Phi(u_n)) = \tau \quad \text{for some } 0 \leq \tau < \infty,$$

then we have

$$\lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \mathbb{I}\left(a_k(M_k - b_k) \leq x, \frac{S_k}{\sigma_k} \leq y\right) = e^{-\tau} \Phi(y) \quad \text{a.s. for any } x, y \in \mathbb{R},$$

where \( D_n = \sum_{k=1}^{n} d_k \) and \( d_k = \frac{\exp\left((\ln k)^{\beta}\right)}{k} \), \( 0 \leq \beta < \frac{1}{2} \).

In the following, denote by \( a_n \sim b_n \) if \( \frac{a_n}{b_n} \to 1 \) as \( n \to \infty \), by \( a_n \ll b_n \) if there exists a constant \( c > 0 \) such that \( a_n \leq cb_n \) for sufficiently large \( n \). The \( c \) stands for a constant, which may vary from one line to another.

2. Main Results

**Theorem 2.1.** Assume that \((X, X_n)_{n \in \mathbb{N}}\) be a standardized stationary Gaussian sequence, the covariance function \( r(t) \) satisfies (1.2) for some \( \alpha > 0 \). If the numerical sequence \( \{u_n\} \) satisfies (1.3), then (1.4) holds with \( d_k = \frac{\exp\left((\ln k)^{\beta}\right)}{k} \), \( 0 \leq \beta < \frac{1}{2} \).

Let \( D = \{D_n\}_{n \in \mathbb{N}} \) be a positive non-decreasing sequence with \( \lim_{n \to \infty} D_n = \infty \). We say that \( \{x_k\}_{k \in \mathbb{N}} \) is \( D \)-summable to a finite limit \( x \) if

$$\lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k x_k = x,$$

where \( d_k = D_k - D_{k-1} \).
Remark 2.1. By a classical theorem of Harly (see Chandrasekharan and Minakshisundaram [6]), if two sequences \( D \) and \( D^* \) satisfy \( D_n = O(D_n) \), then \( D \)-summability implies the \( D^* \)-summability, i.e., if a sequence \( \{x_n\}_{n \in \mathbb{N}} \) is \( D \)-summable to \( x \), then it is also \( D^* \)-summable to \( x \). These results show that if (2.1) holds with the weight \( \{d_k\}_{k \in \mathbb{N}} \), then for \( 0 \leq d'_k \leq d_k \), \( D^* \to \infty \), (2.1) also holds with the weight \( \{d'_k\}_{k \in \mathbb{N}} \). So, if ASCLT holds with \( \{d_k\}_{k \in \mathbb{N}} \), then for \( 0 \leq d'_k \leq d_k \), \( D^* \to \infty \), ASCLT also holds with \( \{d'_k\}_{k \in \mathbb{N}} \). So, by this, if we use larger weights, we should expect to get stronger results. Theorem 2.1 remains valid if we extend the weights from \( \frac{1}{k}d_k \) to \( \exp\left(\frac{\ln k}{k}\right) \), \( 0 < \beta < \frac{1}{2} \). When \( \beta = 0 \), the weights \( \frac{1}{k}d_k \).

Lemma 2.2. Under the same assumptions as in Theorem 2.1, if the numerical sequence \( \{u_n\} \) fulfills (3), then there exists some \( \gamma > 0 \) and for any \( x, y \in \mathbb{R} \) and \( m < n \),

\[
E\left[I\left(M_n \leq u_n, \frac{S_n}{\sigma_n} \leq y\right) - I\left(M_m \leq u_m, \frac{S_m}{\sigma_m} \leq y\right)\right] \leq \left(\frac{m}{n}\right)^\gamma ,
\]

(2.2)

\[
\text{Cov}\left[I\left(M_n \leq u_n, \frac{S_n}{\sigma_n} \leq y\right) - I\left(M_m \leq u_m, \frac{S_m}{\sigma_m} \leq y\right)\right] \leq \left(\frac{m}{n}\right)^\gamma ,
\]

(2.3)

where \( u_n = \frac{x}{a_n} + b_n, u_m = \frac{x}{a_m} + b_m \).

3. Proofs

3.1. Proof of Theorem 2.1

Under the assumptions of Theorem 1 on \( X_i, X_2, \ldots, r(i) \) and \( \{u_n\} \), by Theorem 4.3.3 in Leadbetter et al. [7], we have \( \lim_{n \to \infty} P(M_n \leq u_n) = e^{-T} \) for \( T \) which is defined in (1.3). Let \( y \) be a real number, for each \( n \geq 1 \), \( Y_n \) denotes a standard normal variable, which is independent of \( (X_1, \ldots, X_n) \) and has the same distribution as \( \frac{S_n}{\sigma_n} \). From the proof of Lemma 2.2, we get that \( P\left(M_n \leq u_n, \frac{S_n}{\sigma_n} \leq y\right) - P\left(M_n \leq u_n\right)P\left(Y_n \leq y\right) \leq \frac{1}{n} \to 0 \), as \( n \to \infty \). Thus we have

\[
\lim_{n \to \infty} P\left(M_n \leq u_n, \frac{S_n}{\sigma_n} \leq y\right) = \lim_{n \to \infty} P\left(M_n \leq u_n\right)P\left(Y_n \leq y\right) = e^{-T} \Phi(y),
\]

by Toeplitz Lemma, we obtain

\[
\lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^n d_kP\left(M_k \leq u_k, \frac{S_k}{\sigma_k} \leq y\right) = e^{-T} \Phi(y)
\]

(3.1)

where \( D_n = \sum_{k=1}^n d_k \) and \( d_k = \frac{\exp\left(\frac{\ln k}{k}\right)}{k} \), \( 0 < \beta < \frac{1}{2} \).

To prove Theorem 2.1 for \( 0 < \beta < \frac{1}{2} \). We should prove the following:

\[
\frac{1}{D_n} \sum_{k=1}^n d_k \left[ I\left(M_k \leq u_k, \frac{S_k}{\sigma_k} \leq y\right) - P\left(M_k \leq u_k, \frac{S_k}{\sigma_k} \leq y\right) \right] \to 0 \quad \text{a.s.}
\]

(3.2)

In order to prove (3.2), it suffices to show the following holds (see Lemma 3.1 in Csaki and Gonchigdanzan
for some \( \varepsilon > 0 \)

\[
\text{Var} \left( \sum_{k=1}^{n} d_k I \left( M_k \leq u_k, \frac{S_k}{\sigma_k} \leq y \right) \right) \approx D_n^2 \left( \ln D_n \right)^{(1+\varepsilon)} \tag{3.3}
\]

Set \( \xi_k = I \left( M_k \leq u_k, \frac{S_k}{\sigma_k} \leq y \right) - I \left( M_k \leq u_k, \frac{S_k}{\sigma_k} \leq y \right) \), by Lemma 2.2, we have

\[
\left| E(\xi_i, \xi_j) \right| = \left| \text{Cov} \left( I \left( M_k \leq u_k, \frac{S_k}{\sigma_k} \leq y \right), I \left( M_l \leq u_l, \frac{S_l}{\sigma_l} \leq y \right) \right) \right| \\
\leq \left| \text{Cov} \left( I \left( M_k \leq u_k, \frac{S_k}{\sigma_k} \leq y \right), I \left( M_l \leq u_l, \frac{S_l}{\sigma_l} \leq y \right) \right) - I \left( M_k \leq u_k, \frac{S_k}{\sigma_k} \leq y \right) \right| \\
+ \left| \text{Cov} \left( I \left( M_k \leq u_k, \frac{S_k}{\sigma_k} \leq y \right), I \left( M_l \leq u_l, \frac{S_l}{\sigma_l} \leq y \right) \right) \right| \\
\ll \left| E \left( I \left( M_k \leq u_k, \frac{S_k}{\sigma_k} \leq y \right) \right) - I \left( M_k \leq u_k, \frac{S_k}{\sigma_k} \leq y \right) \right| \\
+ \left| \text{Cov} \left( I \left( M_k \leq u_k, \frac{S_k}{\sigma_k} \leq y \right), I \left( M_l \leq u_l, \frac{S_l}{\sigma_l} \leq y \right) \right) \right| \\
\ll \left( \frac{k}{l} \right) \quad \text{for} \ 1 \leq k < l.
\]

We know that

\[
\text{Var} \left( \sum_{k=1}^{n} d_k I \left( M_k \leq u_k, \frac{S_k}{\sigma_k} \leq y \right) \right) \leq E \left( \sum_{k=1}^{n} d_k \xi_k \right)^2 \leq 2 \sum_{1 \leq k \leq n} d_k d_i \left| E(\xi_k, \xi_i) \right| \\
\ll \sum_{1 \leq k \leq n} d_k d_i \left( \frac{k}{l} \right)^2 + \sum_{1 \leq k \leq n} d_k d_i = T_{n1} + T_{n2}
\]

For \( T_{n1} \), we have the following estimate:

\[
T_{n1} \leq \sum_{1 \leq k \leq n} d_k d_i \left( \frac{k}{l} \right)^2 \ll \sum_{1 \leq k \leq n} d_k d_i \frac{1}{(\ln D_n)^{1-\beta}} \ll \frac{D_n^2}{(\ln D_n)^{(1+\beta)}}
\]

By the elementary calculation, it is easy to see that \( D_n \sim \frac{1}{\beta}(\ln n)^{1-\beta} \exp\left(\left(\ln n\right)^{\beta}\right) \), \( \ln D_n \sim (\ln n)^{\beta} \), \( \ln \ln D_n \sim \ln \ln n \).

For \( 0 < \beta < \frac{1}{2} \), we have \( \varepsilon := \frac{1-2\beta}{2\beta} > 0 \) and \( \frac{1}{2\beta} = 1 + \varepsilon \). Then

\[
T_{n2} \ll \sum_{1 \leq k \leq n} d_k d_i \ll \sum_{1 \leq k \leq n} d_k \sum_{1 \leq k \leq n} \frac{\exp\left(\left(\ln n\right)^{\beta}\right)}{l} \ll \frac{D_n}{(\ln D_n)^{\frac{1-\beta}{\beta}}} \sum_{k=1}^{n} d_k \ln D_n \ll \frac{D_n^2 \ln \ln D_n}{(\ln D_n)^{1-\beta}}
\]

Then (9) and thus (8) follows from above estimates. By using (7), we complete the proof of Theorem 2.1.
3.2. Proof of Lemma 2.2

We first consider that (1.2) holds with some $0 < \alpha < 1$. Let $1 \leq i \leq n$, we have

$$0 < \text{Cov}\left( X_i, \frac{S_n}{\sigma_n} \right) = \frac{1}{\sigma_n} \text{Cov}\left( X_i, \sum_{j=1}^{n} X_j \right) = \frac{1}{\sigma_n} \left( 1 + \sum_{j=1}^{i-1} r(i-j) + \sum_{j=i+1}^{n} r(j-i) \right).$$

$$\leq \frac{2}{\sigma_n} + \frac{2}{\sigma_n} \sum_{j=1}^{n} \frac{L(t)}{t^\alpha}$$

for some $0 < \alpha < 1$

By the application of the Karamata’s theorem (see Mielniczuk [9]), we obtain

$$\sigma_n \sim \left( \frac{2}{(1-\alpha)(2-\alpha)} \right)^{\frac{1}{2}} L(n)^{\frac{1}{2} - \frac{1}{2}} n^{-\frac{\alpha}{2}}.$$  (4.2)

From (4.1) and (4.2), for some $0 < \alpha < 1$, we have

$$\text{Cov}\left( X_i, \frac{S_n}{\sigma_n} \right) \leq \frac{1}{n^{\alpha/2}} = \frac{1}{n^{\alpha/2}}$$

for any $\alpha > 0$ and some $0 < \alpha < 1$. Hence,

$$0 < \sup_{1 \leq i \leq n} \text{Cov}\left( X_i, \frac{S_n}{\sigma_n} \right) \leq \frac{1}{n^{\alpha/2}} \to 0,$$

as $n \to \infty$, for some $0 < \beta < \frac{1}{2}$. (4.3)

so there exist $\lambda$, $n_0$ such that $0 < \sup_{1 \leq i \leq n} \text{Cov}\left( X_i, \frac{S_n}{\sigma_n} \right) < \lambda < 1$, for any $n > n_0$.  (4.4)

By (1.2), $\lim_{t \to \infty} \frac{L(t)}{t^\alpha} = 0$, thus there exist $\delta$, $n_1$ such that

$$0 < \sup_{t \in [0, n_1]} r(t) = \delta < 1.$$  (4.5)

Let $y$ be an arbitrary real number and $m < n$. Suppose that $Y_n$ is a random variable, which has the same distribution as $\frac{S_n}{\sigma_n}$, but $Y_n$ is independent of $(X_1, \ldots, X_n)$, then we have

$$E \left| \left( M_{u_n, S_n} \leq u_n, \frac{S_n}{\sigma_n} \leq y \right) - I \left( M_{u_n, S_n} \leq u_n, \frac{S_n}{\sigma_n} \leq y \right) \right|$$

$$= P \left( M_{u_n, S_n} \leq u_n, \frac{S_n}{\sigma_n} \leq y \right) - P \left( M_{u_n, S_n} \leq u_n, \frac{S_n}{\sigma_n} \leq y \right)$$

$$\leq P \left( M_{u_n, S_n} \leq u_n, \frac{S_n}{\sigma_n} \leq y \right) - P \left( M_{u_n, S_n} \leq u_n \right) P \left( Y_n \leq y \right)$$

$$+ P \left( M_{u_n, S_n} \leq u_n, \frac{S_n}{\sigma_n} \leq y \right) - P \left( M_{u_n, S_n} \leq u_n \right) P \left( Y_n \leq y \right)$$

$$\leq A_1 + A_2 + A_3$$
By Theorem 4.2.1 in Leadbetter et al. (1983) and (4.3), (4.4) and (4.5), we get that
\[ A_1 + A_2 \ll \sum_{i=1}^{n} \text{Cov} \left( X_i, \frac{S_n}{\sigma_n} \right) \exp \left( -\frac{u_i^2}{2(1+\lambda)} \right) \ll n^{1/n^\beta} \exp \left( -\frac{u_i^2}{2(1+\lambda)} \right). \] (4.6)

As we know \( \{u_n\} \) fulfills (1.3), which implies that
\[ \exp \left( -\frac{u_i^2}{2} \right) \sim \frac{\sqrt{2\pi} u_i}{n}, \quad u_n \sim (2\ln n)^{\frac{1}{2}}, \quad \exp \left( -\frac{u_n^2}{2(1+\lambda)} \right) \ll \frac{(\ln n)^{\frac{1}{2(1+\lambda)}}}{n^{1+\lambda}}, \]

combine this and (4.6), we have
\[ A_1 + A_2 \ll n^{1/n^\beta} \left( \frac{\ln n}{n^{1+\lambda}} \right)^{\frac{1}{2(1+\lambda)}} = \frac{\left( \frac{\ln n}{n^{1+\lambda}} \right)^{\frac{1}{2(1+\lambda)}}}{n^{1+\lambda-1}}. \]

We set \( \beta + \frac{1}{1+\lambda} - 1 > 0 \), thus
\[ A_1 + A_2 \ll n^{1/n^\beta} = \left( \frac{m}{n} \right)^{\gamma} \] for some \( 0 < \gamma < \beta + \frac{1}{1+\lambda} - 1 \). (4.7)

Next, we estimate \( A_3 \).
\[ A_3 = \left| P \left( M_{m,n} \leq u_n \right) - P \left( M_n \leq u_n \right) \right| \]
\[ \leq \left| P \left( M_{m,n} \leq u_n \right) - \prod_{i=1}^{n} \Phi(u_n) \right| + \left| P \left( M_n \leq u_n \right) - \prod_{i=1}^{n} \Phi(u_n) \right| \]
\[ + \left| \prod_{i=1}^{n} \Phi(u_n) - \prod_{i=1}^{n} \Phi(u_n) \right| \]
\[ = B_1 + B_2 + B_3. \] (4.8)

Let \( \delta \) is defined as in (4.5), set \( \delta < \frac{\alpha}{2 - \alpha} \) such that \( \frac{1}{1+\delta} + \alpha - 2 > 0 \). By Theorem 4.2.1 in Leadbetter et al. and (1.3), we have
\[ B_1 + B_2 \ll n \exp \left( -\frac{u_i^2}{1+\delta} \right) \sum_{t=1}^{n} r(t) \ll \frac{(\ln n)^{1/\delta}}{n^{1+\delta-1}} \sum_{t=1}^{n} \frac{L(t)}{r^\alpha} \ll \frac{(\ln n)^{1/\delta}}{n^{1+\delta-2}} L(n), \]

since \( L(n) \) and \( \ln n \) are slowly varying functions at infinity, we have \( \left( \ln n \right)^{\frac{1}{1+\delta}} L(n) \ll n^\varepsilon \) for any \( \varepsilon > 0 \), then
\[ B_1 + B_2 \ll \frac{1}{n^\varepsilon} = \left( \frac{m}{n} \right)^\gamma \] for some \( \gamma > 0 \). (4.9)

Meanwhile, following from the elementary inequality \( x^m - x^\alpha \leq \frac{m}{n}, \quad 0 \leq x \leq 1 \). We get
\[ B_2 \leq \frac{m}{n} \ll \left( \frac{m}{n} \right)^\gamma \] for \( m < n \). (4.10)

By (4.8), (4.9) and (4.10), if (1.2) holds with \( 0 < \alpha < 1 \), then (2.2) holds.

Provided (1.2) also holds with some \( \alpha \geq 1 \), since \( r(.) \) is positive, we get \( \sigma_n = \sqrt{\text{Var} \left( S_n \right)} \geq n^\frac{1}{2} \), this imply
0 < Cov \left( X_i, \frac{S_m}{\sigma_m} \right) \ll \frac{2}{n^2} \sum_{i=1}^n L(t) t^\alpha. \quad \text{As } \sum_{i=1}^n \frac{L(t)}{t^\alpha} \text{ is a slowly varying function infinity, we get } \sum_{i=1}^n \frac{L(t)}{t^\alpha} \ll n^\varepsilon \text{ for any } \varepsilon > 0. \text{ We obtain that (4.3) and (4.4). By using Theorem 4.2.1 in Leadbetter et al. and by (4.6)-(4.10), let } A_i - A_j \text{ be replaced by } C_i - C_j \text{ in (4.3), we also get } C_i + C_2 \ll \left( \frac{m}{n} \right) , \quad \text{let } D_i - D_j \text{ be replaced by } B_i - B_j \text{ in (4.8), we also obtain that } D_i + D_j \ll n \exp \left( \frac{u^2}{1+\delta} \right) \sum_{i=1}^n r(t) \ll \frac{(\ln n)^{\varepsilon \delta}}{n^{1+\varepsilon \delta}} n^\varepsilon \ll \left( \frac{m}{n} \right) \gamma \text{ for some } \gamma > 0, \text{ like (4.11), } D_i \ll \frac{m}{n} \ll \left( \frac{m}{n} \right) \gamma, \quad \text{we get } C_i \ll \left( \frac{m}{n} \right) \gamma, \text{ at last, we obtain when (1.2) holds with } \alpha \geq 1, \text{ then (2.2) also holds.}

Next, we prove (2.3). First we consider the situation of some } 0 < \alpha < 1. \text{ Let } i \geq m + 1. \text{ Since } L(t) \text{ is a positive slowly varying function, then we have } \left( L(m) \right)^{1/2} \ll m^\varepsilon \text{ for any } \varepsilon > 0, \frac{L(t)}{t^\alpha} \text{ is monotonic decreasing, so } \sum_{i=1}^n \frac{L(t)}{t^\alpha} \ll \sum_{i=1}^n \frac{L(t)}{t^\alpha}, \text{ by (1.2) and (4.2), we obtain }

\begin{align*}
0 < Cov \left( X_i, \frac{S_m}{\sigma_m} \right) & \ll \frac{1}{\sigma_m} \sum_{i=1}^n \frac{L(t)}{t^\alpha} \ll \frac{1}{L(m)^{1/2}} \sum_{i=1}^n \frac{L(t)}{t^\alpha} \\
& \ll \frac{1}{L(m)^{1/2}} \sum_{i=1}^n \frac{L(t)}{t^\alpha} \ll \frac{L(m)^{1/2}}{L(m)^{1/2}} \frac{m^{-\alpha}}{m^2} \ll \frac{m^\varepsilon}{m^2} \ll \frac{1}{m^\eta} \quad \text{for some } 0 < \eta < \frac{1}{2}
\end{align*}

then there exist numbers } \mu, m_\eta, \text{ such that } 0 < \sup \limits_{j \geq m + 1} Cov \left( X_i, \frac{S_m}{\sigma_m} \right) < \mu < 1 \text{ for all } m > m_\eta. \text{ By the Normal Comparison Lemma, we obtain }

\begin{align*}
& \left| Cov \left( I \left( M_m \leq u_n, \frac{S_m}{\sigma_m} \leq y \right), I \left( M_m \leq u_n, \frac{S_m}{\sigma_m} \leq y \right) \right) \right| \\
& = \sup_{y \geq m + 1} Cov \left( I \left( M_m \leq u_n, \frac{S_m}{\sigma_m} \leq y \right), I \left( M_m \leq u_n, \frac{S_m}{\sigma_m} \leq y \right) \right) \\
& \ll \sum_{i=1}^n \sum_{j=1}^n r(j-i) \exp \left( \frac{u_n^2 + u_n^2}{2(1+r(j-i))} \right) + \sum_{i=1}^n Cov \left( X_i, \frac{S_m}{\sigma_m} \right) \exp \left( -\frac{u_n^2}{2 \left( 1 + Cov \left( X_i, \frac{S_m}{\sigma_m} \right) \right)} \right) + \sum_{i=1}^n Cov \left( X_i, \frac{S_m}{\sigma_m} \right) \exp \left( -\frac{u_n^2}{2 \left( 1 + Cov \left( X_i, \frac{S_m}{\sigma_m} \right) \right)} \right) \\
& \ll E_1 + E_2 + E_3 + E_4
\end{align*}
By (4.5), set $\frac{\alpha}{2-\alpha} < 1$, then $\frac{2}{1+\delta} + \alpha - 2 > 0$, we have

$$E_1 = \sum_{i=1}^{\infty} \sum_{j=m+1}^{\infty} r(j-i) \exp \left( -\frac{u^2_i + u^2_j}{2(1+r(j-i))} \right) \ll \exp \left( -\frac{u^2_i + u^2_j}{2(1+\delta)} \right) \sum_{i=1}^{m} \sum_{j=m+1}^{\infty} r(t)$$

$$\ll m \exp \left( -\frac{u^2_i + u^2_j}{2(1+\delta)} \right) \sum_{r=m+1}^{\infty} L(t) \ll m \exp \left( -\frac{u^2_i + u^2_j}{2(1+\delta)} \right) \sum_{j=1}^{m} \frac{L(t)}{t^\delta}$$

$$\ll \left( \frac{m}{n} \right)^{\frac{1}{1+\delta}} + \frac{1}{n^{1+\delta}} L(n) \ll \left( \frac{m}{n} \right)^{\gamma} \ll \left( \frac{m}{n} \right)^{\gamma}$$

for some $0 < \gamma < 1$

By (4.4), set $\frac{\beta}{1-\beta} < 1$, then $-\beta + 1 - \frac{1}{1+\lambda} < 0$, we have

$$E_2 = \sum_{i=1}^{m} \text{Cov} \left( X_i, S_{\frac{m}{\sigma_m}} \right) \exp \left( -\frac{u^2_i}{2(1+\text{Cov} \left( X_i, S_{\frac{m}{\sigma_m}} \right))} \right)$$

$$\ll \exp \left( -\frac{u^2_i}{2(1+\mu)} \right) \sum_{i=1}^{m} \text{Cov} \left( X_i, S_{\frac{m}{\sigma_m}} \right) \ll \frac{m}{n^\beta} \left( \text{ln} m \right)^{\frac{1}{2+\beta}}$$

and

$$\sum_{i=m+1}^{\infty} \text{Cov} \left( X_i, S_{\frac{m}{\sigma_m}} \right) = \frac{1}{\sigma_m} \sum_{i=m+1}^{\infty} \sum_{j=m+1}^{\infty} r(i-j) = \frac{1}{\sigma_m} \sum_{i=1}^{m} \sum_{j=m+1}^{\infty} r(j-i)$$

$$= \frac{1}{\sigma_m} \sum_{i=1}^{m} \sum_{k=m+1}^{\infty} r(k) \ll \frac{1}{\sigma_m} \sum_{k=1}^{m} \sum_{k=m+1}^{\infty} r(k) \ll \frac{m}{\sigma_m} \sum_{k=1}^{m} \frac{L(k)}{k^\alpha}$$

$$E_3 \ll \exp \left( -\frac{u^2_i}{2(1+\mu)} \right) \sum_{i=m+1}^{\infty} \text{Cov} \left( X_i, S_{\frac{m}{\sigma_m}} \right)$$

$$\ll \exp \left( -\frac{u^2_i}{2(1+\mu)} \right) \frac{m}{\sigma_m} \sum_{k=1}^{m} \frac{L(k)}{k^\alpha}$$

$$\ll \left( \frac{m}{n} \right)^{\frac{\alpha}{n^{1+\beta}}} \left( \text{ln} n \right)^{\frac{1}{2+\beta}} \ll \left( \frac{m}{n} \right)^{\gamma} \ll \left( \frac{m}{n} \right)^{\gamma}$$

for $0 < \gamma < \frac{\alpha}{2}$

$$E_4 \ll \text{Cov} \left( S_{\frac{m}{\sigma_m}}, S_{\frac{m+2m}{\sigma_m}} - S_{2m} \right) + E \left( \frac{S_{\frac{m}{\sigma_m}}}{\sigma_{\frac{m}}}, S_{\frac{m+2m}{\sigma_m}} - S_{2m} \right)$$

$$= F_1 + F_2 + F_3$$

(4.12)
Since \( L(t)/t^\alpha \) is monotonic decreasing, so for \( 1 \leq i \leq m \), \( r(j-i) \leq r(j-m) \), by (4.2), Karamata’s theorem and the property of the slowly varying function, we obtain

\[
F_i \leq \frac{1}{\sigma_n^2} \text{Cov} \left( \sum_{i=1}^{m} X_i, \sum_{j=2m+1}^{2m+n} X_j \right) = \frac{1}{\sigma_n^2} \sum_{i=1}^{m} \sum_{j=2m+1}^{2m+n} r(j-i)
\]

\[
\leq \frac{1}{\sigma_n^2} \sum_{i=1}^{m} \sum_{j=2m+1}^{2m+n} r(j-m) \leq m \frac{1}{\sigma_n^2} \sum_{k=1}^{\infty} r(k)
\]

\[
\leq \frac{m}{L(m)^{1/2} L(n)^{1/2} m^{1/2 - \alpha/n} } L(n)^{1/2} \left( \frac{m}{n} \right)^{\alpha/n} \left( \frac{L(n)}{L(m)} \right)^{1/2}.
\]

\[
\leq \left( \frac{m}{n} \right)^{\alpha/n} \left( \frac{n}{m} \right)^{\alpha/n} \leq \left( \frac{m}{n} \right)^{\alpha/n} \quad \text{for} \ 0 < \alpha < \frac{1}{4}.
\]

By \( \sigma_n = \sqrt{\text{Var}(S_n)} \), then we get \( E \left( S_m / \sigma_m \right)^2 = 1 \), by the stationary of the sequence \((X, X_n)_{n \in \mathbb{N}}\), we get \( S_{n+2m} - S_n = S_{2m} \), thus \( E(S_{n+2m} - S_n)^2 = E(S_{2m})^2 \) and by the fact that \( EX_j > 0 \) for all \( i, j \) and (4.2), we deduce that

\[
F_2 \leq E \left( \frac{S_m}{\sigma_m} \right) \left| E \left( \frac{S_{n+2m} - S_n}{\sigma_n} \right) \right| \leq E \left( \frac{S_m}{\sigma_m} \right)^2 E \left( \frac{S_{n+2m} - S_{2m}}{\sigma_n} \right)^2.
\]

\[
\leq \frac{\sigma_{2m}}{\sigma_n} \leq \left( \frac{m}{n} \right)^{\alpha/n} \left( \frac{L(m)}{L(n)} \right)^{1/2} \left( \frac{m}{n} \right)^{\alpha/n} \leq \left( \frac{m}{n} \right)^{\alpha/n} \quad \text{for some} \ \gamma < 1
\]

where the second inequality follows from Jensen inequality.

For \( F_3 \), we have

\[
F_3 = -E \left( \frac{S_m}{\sigma_m} \right) E \left( \frac{S_n}{\sigma_n} \right) \leq 0,
\]

so we prove (2.3) for some \( 0 < \alpha < 1 \).

Next, we prove (2.3) for some \( \alpha > 1 \).

\[
\left| \text{Cov} \left( I \left( M_m \leq u_n \frac{S_m}{\sigma_m} \leq y \right), I \left( M_{n,2m} \leq u_n \frac{S_n}{\sigma_n} \leq y \right) \right) \right| = G_1 + G_2 + G_3 + G_4,
\]

where \( G_1 - G_4 \) are defined as \( E_1 - E_4 \) in (4.11), but for (1.2) holds for \( \alpha > 1 \). Similarly as the proof of \( E_1 - E_4 \), it is easy to check that \( G_1 + G_2 + G_3 \leq \left( \frac{m}{n} \right)^\alpha \) for some \( \gamma > 0 \).

Define \( G_4 =: H_1 + H_2 + H_3 \) as in (4.12), by Karamata’s theorem, we get

\[
H_1 \leq \frac{1}{m^2 n^2} \sum_{i=1}^{m} \sum_{j=2m+1}^{2m+n} r(j-i) \leq \frac{1}{m^{2n+1}} \sum_{i=1}^{m} \sum_{j=2m+1}^{2m+n} r(t)
\]

\[
\leq \frac{m^{2n+1}}{m^2 n^2} \sum_{t=1}^{m} r(t) \leq \left( \frac{m}{n} \right)^{1/2} L(m) \leq \left( \frac{m}{n} \right)^{\gamma} \quad \text{for} \ 0 < \gamma < \frac{1}{2}
\]

By the definition of \( \sigma_n \) and Karamata’s theorem, it is easy to obtain that

\[
n \leq \sigma_n^2 = n + 2 \sum_{t=1}^{\infty} (n-t) r(t) = n + 2 \sum_{t=1}^{\infty} \frac{L(t)}{t^\alpha} - 2 \sum_{t=1}^{\infty} t r(t) \leq n + 2 \sum_{t=1}^{\infty} \frac{t L(t)}{t^\alpha} \leq n + 2 \sum_{t=1}^{\infty} \frac{L(t)}{t^\alpha} \leq n + c n,
\]
then we have \( \sigma_n = O\left(n^{\frac{1}{2}}\right) \). Similarly as the proof of \( F_2 \),

\[
H_2 \ll \sigma_{2n} \ll \left( \frac{m}{n} \right)^{\frac{1}{2}} \left( \frac{m}{n} \right)^{\gamma} \quad \text{for} \quad 0 < \gamma < \frac{1}{2}.
\]

by the stationary of the sequence \( \{X_n\}_{n \in \mathbb{N}} \), it is easy to see that \( H_3 = -E\left( \frac{S_m}{\sigma_n} \right)E\left( \frac{S_{n+2m} - S_{2m}}{\sigma_n} \right) = 0 \).

So we have proved (2.3) for the case of \( \alpha > 1 \).

Finally we should prove (2.3) for \( \alpha = 1 \).

\[
\bigg| \text{Cov}\left( \left\{ M_{m} \leq u_n, \frac{S_m}{\sigma_n} \leq y \right\}, \left\{ M_{m,n} \leq u_n, \frac{S_m}{\sigma_n} \leq y \right\} \right) \bigg| = I_1 + I_2 + I_3 + I_4,
\]

where we replaced \( E_1 - E_4 \) by \( I_1 - I_4 \) as in (4.11), but for (1.2) holds for some \( \alpha = 1 \).

Similarly, it is easy to check that \( I_1 + I_2 + I_3 \ll \left( \frac{m}{n} \right)^{\gamma} \) for some \( \gamma > 0 \). Define \( I_4 = I_1 + I_2 + I_3 \). Since \( r(t) \) is monotonic decreasing, define \( \hat{L}(n) = 1 + 2\sum_{i=1}^{n} r(t) \), by the application of the proposition (Potter’s TH) of the slowly varying function, for any \( 0 < \delta \ll \frac{1}{2} \), we have

\[
J_1 \ll \frac{m}{\sigma_n \sigma_m} \sum_{i=1}^{m\cdot n+1} r(t) \ll \frac{m}{\sigma_n \sigma_m} \sum_{i=1}^{m\cdot n+1} r(t) \ll \frac{m\hat{L}(n+m)}{(\hat{L}(m))^\frac{1}{2} (\hat{L}(n))^\frac{1}{2} m^2 n^2}.
\]

\[
= \left( \frac{m}{n} \right)^{\frac{1}{2}} \left( \frac{(n+m)}{m} \right)^{\frac{1}{2}} \left( \frac{(n+m)}{n} \right)^{\frac{1}{2}} \ll \left( \frac{m}{n} \right)^{\frac{1}{2}} \left( \frac{(n+m)}{m} \right)^{\frac{1}{2}} \left( \frac{(n+m)}{n} \right)^{\frac{1}{2}} \quad \text{for some} \quad 0 < \gamma < \frac{1}{2}.
\]

\[
J_2 \ll \sigma_{2m} \ll \left( \frac{m}{n} \right)^{\frac{1}{2}} \left( \frac{(n+m)}{m} \right)^{\frac{1}{2}} \ll \left( \frac{m}{n} \right)^{\frac{1}{2}} \left( \frac{2n}{m} \right)^{\frac{1}{2}} \ll \left( \frac{m}{n} \right)^{\gamma} \quad \text{for some} \quad 0 < \gamma < \frac{1}{2}.
\]

Similarly as \( F_2 = 0 \) when \( 0 < \alpha < 1 \), we have \( J_3 = 0 \), So we have \( I_4 \ll \left( \frac{m}{n} \right)^{\gamma} \). Finally, we get that

\[
I_1 + I_2 + I_3 + I_4 \ll \left( \frac{m}{n} \right)^{\gamma} \quad \text{for some} \quad \gamma > 0, \quad \text{when} \quad \alpha = 1.
\]

References


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