A Family of Generalized Stirling Numbers of the First Kind

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Abstract

A modified approach via differential operator is given to derive a new family of generalized Stirling numbers of the first kind. This approach gives us an extension of the techniques given by El-Desouky [1] and Gould [2]. Some new combinatorial identities and many relations between different types of Stirling numbers are found. Furthermore, some interesting special cases of the generalized Stirling numbers of the first kind are deduced. Also, a connection between these numbers and the generalized harmonic numbers is derived. Finally, some applications in coherent states and matrix representation of some results obtained are given.

Keywords
Stirling Numbers, Comtet Numbers, Creation, Annihilation, Differential Operator, Maple Program

1. Introduction

Gould [2] proved that

\[(e^x D)^n = e^{xn} \sum_{k=0}^{n} \binom{n}{k} x^k s(n,k) D^k = e^{xn} \sum_{k=0}^{n} s(n,k) D^k,\quad D = \frac{d}{dx},\]

where \(s(n,k)\) and \(s(n,k)\) are the usual Stirling numbers and the singles Stirling numbers of the first kind, respectively, defined by

\[(x)_n = \sum_{k=0}^{n} s(n,k) x^k, s(n,0) = \delta_{n,0} \text{ and } s(n,k) = 0 \text{ for } k > n.\]

\[(x)_n = \sum_{k=0}^{n} s(n,k) x^k, s(n,0) = \delta_{n,0} \text{ and } s_i(n,k) = 0 \text{ for } k > n.\]

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where \((x)_n = x(x-1)\cdots(x-n+1)\) and \(\langle x \rangle = x(x+1)\cdots(x+n-1)\).

These numbers satisfy the recurrence relations
\[
s(n+1,k) = s(n,k-1) - ns(n,k),
\]
\[
s_i(n+1,k) = s_i(n,k-1) + ns_i(n,k).
\]

EL-Desouky [1] defined the generalized Stirling numbers of the first kind \(s(k;\bar{\alpha},\bar{\beta})\), called \((\bar{\alpha},\bar{\beta})\)-Stirling numbers of the first kind by
\[
e^{\bar{\alpha}x}D^\alpha \cdots e^{\bar{\beta}x}D^\beta = e^{\left(\sum_{l=1}^{\infty} \frac{x^l}{l!}\right)\bar{s}_{n+l-k-i}} s(k;\bar{\alpha},\bar{\beta}) D^k,
\]
where \(\bar{s}(k;\bar{\alpha},\bar{\beta}) = 0\) for \(k < s_i\) or \(k > \sum_{j=1}^{n} s_j\) and \(s(0;\bar{\alpha},\bar{\beta}) = 1\), where \(\bar{\alpha} := (r_1, r_2, \cdots, r_n)\) is a sequence of real numbers and \(\bar{\beta} := (s_1, s_2, \cdots, s_n)\) is a sequence of nonnegative integers.

Equation (6) is equivalent to
\[
e^{\bar{\alpha}x} a^\alpha \cdots e^{\bar{\beta}x} a^\beta = e^{\left(\sum_{l=1}^{\infty} \frac{x^l}{l!}\right)\bar{s}_{n+l-k-i}} s(k;\bar{\alpha},\bar{\beta}) a^k,
\]
where \(a^\alpha\) and \(a^\beta\) are boson creation and annihilation operators, respectively, and satisfy the commutation relation \([a^\alpha, a^\beta] = 1\).

The numbers \(s(k;\bar{\alpha},\bar{\beta})\) satisfy the recurrence relation
\[
s(k;\bar{\alpha},\bar{\beta}) = \sum_{i=0}^{n} \binom{n}{i} s_{n+i-k} \sum_{j=1}^{i} \binom{i}{j} s_j s_{n+i-k-j},
\]
with the notations \(\bar{\alpha} \oplus r_{n+1} = (r_1, r_2, \cdots, r_{n+1})\) and \(\bar{\beta} \oplus s_{n+1} = (s_1, s_2, \cdots, s_{n+1})\).

The numbers \(s(k;\bar{\alpha},\bar{\beta})\) have the explicit formula
\[
s(k;\bar{\alpha},\bar{\beta}) = \sum_{\sigma_k = 0}^{\infty} \prod_{j=0}^{n} \left( \binom{n}{i} \sum_{j=1}^{i} \binom{i}{j} s_j s_{n+i-k-j} \right)^{\sigma_k},
\]
where \(\sigma_k = \sum_{j=0}^{n} j i_j\), with \(i_0 = 0\) and \(\sum_{j=0}^{n} j s_j\).

Moreover EL-Desouky [1] derived many special cases and some applications. For the proofs and more details, see [1].

The generalized falling factorial of \(x\) associated with the sequence \(\bar{\alpha} := (\alpha_0, \alpha_1, \cdots, \alpha_n)\) of order \(n\), where \(\alpha_0, \alpha_1, \cdots, \alpha_n\) are real numbers, is defined by \((x,\bar{\alpha})_n = \prod_{l=0}^{n-1} (x - \alpha_l)(x,\bar{\alpha})_0 = 1\).

Comtet [3] [4] and [5] defined \(s_{\bar{\sigma}}(n,k)\) the generalized Stirling numbers of the first kind, which are called Comtet numbers, by
\[
(x,\bar{\alpha})_n = \sum_{k=0}^{n} s_{\bar{\sigma}}(n,k)x^k.
\]
These numbers satisfy the recurrence relation
\[
s_{\bar{\sigma}}(n,k) = s_{\bar{\sigma}}(n-1,k-1) - \alpha_{n-k} s_{\bar{\sigma}}(n-1,k).
\]

El-Desouky and Cakic [6] defined \(s_{\bar{\sigma}}(n,k;\bar{s})\), the generalized Comtet numbers by
\[
\prod_{j=0}^{n-1} \left( x^{\alpha_j} \delta x^{\alpha_j} \right)^{\sigma_j} = \prod_{j=0}^{n-1} \left( \delta - \alpha_j \right)^{\sigma_j} = \sum_{k=0}^{n-1} s_{\bar{\sigma}}(n,k;\bar{s}) \delta^k,
\]
where \(|\bar{s}| = \sum_{j=0}^{n-1} s_j (n,k;\bar{s}) = 0\) for \(k > |\bar{s}|\), \(s_{\bar{\sigma}}(n,0;\bar{s}) = (-1)^{|\bar{s}|}\prod_{j=0}^{n-1} \left( \alpha_j \right)^{r_j}\) and \(\bar{s} := (s_0, s_1, \cdots, s_n)\).

For more details on generalized Stirling numbers via differential operators, see [7]-[10] and [11].
The paper is organized as follows:

In Section 2, using the differential operator \((e^{x^2}D^x e^{x^3}) \cdots (e^{x^2}D^y e^{x^3})(e^{y^2}D^x e^{y^3})\) we define a new family of generalized Stirling numbers of the first kind, denoted by \(s(k; \tilde{r}, \tilde{s})\). A recurrence relation and an explicit formula of these numbers are derived. In Section 3, some interesting special cases are discussed. Moreover some new combinatorial identities and a connection between \(s(n,k,r,s)\) and the generalized harmonic numbers \(O^{(i)}\) are given. In Section 4, some applications in coherent states and matrix representation of some results obtained are given. Section 5 is devoted to the conclusion, which handles the main results derived throughout this work. Finally, a computer program is written using Maple and executed for calculating the generalized Stirling numbers of the first kind and some special cases, see Appendix.

2. Main Results

Let \(\tilde{r} := (r_1, r_2, \cdots, r_n)\) be a sequence of real numbers and \(\tilde{s} := (s_1, s_2, \cdots, s_n)\) be a sequence of nonnegative integers.

**Definition 2.1**

The generalized Stirling numbers \(s(k; \tilde{r}, \tilde{s})\) are defined by

\[
\left( e^{x^2}D^x e^{x^3} \right) \cdots \left( e^{x^2}D^y e^{x^3}(e^{y^2}D^x e^{y^3}) \right) = e^{(\Sigma \sigma_i r_i)} \sum_{k=0}^{\beta_0} s(k; \tilde{r}, \tilde{s}) D^k,
\]

where \(\beta_0 = \sum_{j=1}^{n} s_j\), \(s(k; \tilde{r}, \tilde{s}) = 0\) for \(k > \beta_0\) and \(s(0; \tilde{r}, \tilde{s}) = 1\).

Equation (13) is equivalent to

\[
\left( e^{a^2}e^{x^3} \right) \cdots \left( e^{a^2}e^{y^3}(e^{y^2}e^{x^3}) \right) = e^{(\Sigma \sigma_i r_i)} \sum_{k=0}^{\beta_0} s(k; \tilde{r}, \tilde{s}) a^k.
\]

**Theorem 2.1**

The numbers \(s(k; \tilde{r}, \tilde{s})\) satisfy the recurrence relation

\[
s(k; \tilde{r} \oplus r_{n+1}, \tilde{s} \oplus s_{n+1}) = \sum_{i=0}^{s_{n+1}} \binom{s_{n+1}}{i} \left( \sum_{j=1}^{n} r_j + r_{n+1} \right)^{s_{n+1}-i} s(k-i; \tilde{r}, \tilde{s}),
\]

with the notations \(\tilde{r} \oplus r_{n+1} := (r_1, r_2, \cdots, r_{n+1})\) and \(\tilde{s} \oplus s_{n+1} := (s_1, s_2, \cdots, s_{n+1})\).

**Proof**

\[
e^{(\Sigma \sigma_i r_i)} \sum_{k=0}^{\beta_0} s(k; \tilde{r}, \tilde{s}) D^k = \left( e^{\sigma_1 x} D^x e^{\sigma_2 x^3} \right) \left( e^{(\Sigma \sigma_i r_i)} \sum_{k=0}^{\beta_0} s(k; \tilde{r}, \tilde{s}) D^k \right)
\]

\[
= e^{\sigma_1 x} D^x \left( e^{(\Sigma \sigma_i r_i)} \sum_{k=0}^{\beta_0} s(k; \tilde{r}, \tilde{s}) D^k \right)
\]

\[
= e^{(\Sigma \sigma_i r_i)} \sum_{k=0}^{\beta_0} \left( \sum_{j=1}^{n} r_j + r_{n+1} \right)^{s_{n+1}-i} s(k-i; \tilde{r}, \tilde{s}) D^k.
\]

Equating the coefficients of \(D^k\) on both sides yields (15).

**Theorem 2.2**

The numbers \(s(k; \tilde{r}, \tilde{s})\) have the explicit formula

\[
s(k; \tilde{r}, \tilde{s}) = \sum_{\sigma_\alpha = \beta_0-k, j \geq 0} \prod_{i=1}^{n} \frac{\sigma_\alpha}{i} \left( \sum_{j=0}^{s_i} r_j + r_{n+1} \right), \quad \text{where} \quad \sigma_\alpha = \sum_{j=1}^{n} i_j, r_0 = 0.
\]

(16)
Proof

\[ e^{\beta^s} D^h e^{\alpha^s} = e^{\beta^s} e^{\alpha^s} (D + r_j I)^h = e^{\beta^s} \sum_{k=0}^{\frac{h}{s}} \binom{h}{k} (r_j)^k D^{h-k}, \]

\[ (e^{\beta^s} D^h e^{\alpha^s}) (e^{\beta^s} D^h e^{\alpha^s}) = e^{2\beta^s} D^h e^{2\alpha^s} (e^{\beta^s} D^h e^{\alpha^s}) = e^{2\beta^s} D^h e^{2\beta^s} \sum_{k=0}^{\frac{h}{s}} \binom{h}{k} (r_j)^k D^{h-k} \]

\[ = e^{\beta^s} (2r_j + r_j^2) D^{h-s} + \sum_{k=0}^{\frac{h}{s}} \binom{h}{k} (r_j)^k D^{h-k} \]

\[ = e^{2\beta^s} (2r_j + r_j^2) D^{h-s} + \sum_{k=0}^{\frac{h}{s}} \binom{h}{k} (r_j)^k D^{h-k} \]

thus, by iteration, we get

\[ (e^{\beta^s} D^h e^{\alpha^s}) \cdots (e^{\beta^s} D^h e^{\alpha^s}) (e^{\beta^s} D^h e^{\alpha^s}) = e^{\beta^s} (2r_j + r_j^2)^{h-s} + \sum_{k=0}^{\frac{h}{s}} \binom{h}{k} (r_j)^k D^{h-k} \]

(17)

Comparing (13) and (18) yields (16).

3. Special Cases

Setting \( r_j = r \) and \( s_i = s, i = 1, \ldots, n \) in (13), we have the following definition.

**Definition 3.1**
For any real number \( r \) and nonnegative integer \( s \), let the numbers \( s(n, k; r, s) \), be defined by

\[ (e^{\beta^s} D^h e^{\alpha^s})^n = e^{(2\alpha^s)x} \sum_{k=0}^{\frac{h}{s}} s(n, k; r, s) D^k, \]

(19)

where \( s(0, 0; r, s) = 1 \) and \( s(n, k; r, s) = 0 \) for \( k > ns \).

Equation (19) is equivalent to

\[ (e^{\alpha^s} D^h e^{\alpha^s})^n = e^{(2\alpha^s)x} \sum_{k=0}^{\frac{h}{s}} s(n, k; r, s) D^k, \]

(20)

**Corollary 3.1**
The numbers \( s(n, k; r, s) \) satisfy the recurrence relation

\[ s(n+1, k; r, s) = \sum_{i=0}^{k} \binom{k}{i} ((2n+1)r)^{i-k} s(n, k-i; r, s). \]

(21)

**Proof**
The proof follows directly from Equation (15) by setting \( r_j = r \) and \( s_i = s, i = 1, 2, \ldots, n \).

**Corollary 3.2**
The numbers \( s(n, k; r, s) \) have the explicit formula

\[ s(n, k; r, s) = r^{x-s} \sum_{i=0}^{k} \binom{k}{i} (2j-1)^{i-j}. \]

(22)

**Proof**
By substituting \( r_j = r \) and \( s_i = s, i = 1, 2, \ldots, n \) in Equation (17), yields
\((e^nDe^n)^n = e^{(2n)a} \sum_{k=0}^{n} \left( \prod_{j=1}^{s} \left( \prod_{i=1}^{s} \left( 2j-1 \right)^{i} \right)^{r_{\sigma_{j}}} \right) D^{n-\sigma_{a}},\)

then setting \(ns - \sigma_{a} = k\) we have

\((e^nDe^n)^n = e^{(2n)a} \sum_{k=0}^{n} \sum_{\sigma_{a} = n-k,j \geq 0} \left( \prod_{j=1}^{s} \left( \prod_{i=1}^{s} \left( 2j-1 \right)^{i} \right)^{r_{\sigma_{j}}} \right) D^{k},\)

(23)

hence comparing Equations (19) and (23) we obtain Equation (22).

Furthermore we handle the following special cases.

i) If \(r = 1\), then we have

**Definition 3.2**

\((e^nDe^n)^n = e^{(2n)a} \sum_{k=0}^{n} s(n,k;1,s) D^{k},\)

(24)

where \(s(0,0,1,s) = 1\) and \(s(n,k;1,s) = 0\) for \(k > ns\).

**Corollary 3.3**

The numbers \(s(n,k;1,s)\) satisfy the recurrence relation

\(s(n+1,k;1,s) = \sum_{i=1}^{s} (2n+1)^{r_{i}} s(n,k-i;1,s)\).

(25)

**Proof:**

The proof follows directly from Equation (21) by setting \(r = 1\).

**Corollary 3.4**

The numbers \(s(n,k;1,s)\) have the explicit formula

\(s(n,k;1,s) = \sum_{\sigma_{a} = n-k,j \geq 0} \left[ \prod_{j=1}^{s} \left( \prod_{i=1}^{s} \left( 2j-1 \right)^{i} \right)^{r_{\sigma_{j}}} \right] \).

(26)

**Proof:**

The proof follows directly from Equation (22) by setting \(r = 1\).

ii) If \(s = 1\), then we have

**Definition 3.3**

The numbers \(s(n,k;r,1)\) are defined by

\((e^nDe^n)^n = e^{(2n)a} \sum_{k=0}^{n} s(n,k;r,1) D^{k},\)

(27)

where \(s(n,0;r,1) = \delta_{n,0}\) and \(s(n,k;r,1) = 0\) for \(k > n\).

**Corollary 3.5**

The numbers \(s(n,k;r,1)\) satisfy the triangular recurrence relation

\(s(n,k;r,1) = s(n-1,k-1;r,1) + (2n-1)r s(n-1,k;r,1)\).

(28)

**Proof:**

The proof follows easily from (22) by setting \(s = 1\).

**Corollary 3.6**

The numbers \(s(n,k;r,1)\) have the following explicit formula

\(s(n,k;r,1) = r^{n-k} \sum_{\sigma_{a} = n-k,j \in [0,1]} \prod_{j=1}^{s} (2j-1)^{\sigma_{j}}\).

(29)

**Proof:**

The proof follows from (22) by setting \(s = 1\).

Also, using the recurrence relation (28) we can find the following explicit formula.
Theorem 3.1
The numbers \(s(n,k;r,1)\) have the following explicit expression
\[
s(n,k;r,1) = \sum_{i_1+\cdots+i_{n-k+1},j_1\in[0,1]} \left( \frac{i_1 + r}{1 - i_1} \right) \left( \frac{i_2 + 3r}{1 - i_2} \right) \cdots \left( \frac{i_{n-k} + (2n-1)r}{1 - i_{n-k}} \right) \quad (30)
\]

Proof
For \(k = 0\), \(s(n,0;r,1) = (r)(3r)(5r)\cdots((2n-1)r) = r^n \left(1.2.3\cdots(2n-1)\right)\).

For \(i_n \in \{0,1\}\), we get
\[
s(n,k;r,1) = \sum_{i_1+\cdots+i_{n-k+1},j_1\in[0,1]} \left( \frac{i_1 + r}{1 - i_1} \right) \left( \frac{i_2 + 3r}{1 - i_2} \right) \cdots \left( \frac{i_{n-k} + (2n-1)r}{1 - i_{n-k}} \right) \cdot (2n-1)r
\]

That is the same recurrence relation (28) for the numbers \(s(n,k;r,1)\). This completes the proof.

iii) If \(r = 1\) and \(s = 1\), then we have

Definition 3.4
The numbers \(s(n,k)r,1)\) are defined by
\[
\left( e^r De^n \right)^n = e^{2\pi i n} \sum_{k=0}^{\infty} s(n,k)D^k, \quad (31)
\]

where \(s(n,0) = \delta_{n,0}\) and \(s(n,k) = 0\) for \(k > n\).

Equation (31) is equivalent to
\[
\left( e^{sr} De^n \right)^n = e^{2\pi i n} \sum_{k=0}^{\infty} s(n,k)a^k. \quad (32)
\]

Corollary 3.7
The numbers \(s(n,k)\) satisfy the triangular recurrence relation
\[
s(n+1,k) = s(n,k-1) + (2n+1)s(n,k). \quad (33)
\]

Proof
The proof follows by setting \(r = 1\) in Equation (28).

Corollary 3.8
The numbers \(s(n,k)\) have the explicit formula
\[
s(n,k) = \sum_{\sigma_\geq-k, j_\geq0, j \in[0,1]} \prod_{j=1}^{n} (2j-1)^{\sigma_j}. \quad (34)
\]

Proof
The proof follows by setting \(r = 1\) in Equation (29).

Moreover \(s(n,k)\) have the following explicit formula.

Corollary 3.9
The numbers \(s(n,k)\) have the following explicit expression
\[
s(n,k) = \sum_{i_1+\cdots+i_{n-k+1},j_1\in[0,1]} \left( \frac{i_1 + 1}{1 - i_1} \right) \left( \frac{i_2 + 3}{1 - i_2} \right) \cdots \left( \frac{i_{n-k} + (2n-1)}{1 - i_{n-k}} \right) \quad (35)
\]

Proof
The proof follows by setting \(r = 1\) in (30).
From Equations (29) and (30) (also from Equations (34) and (35)) we have the combinatorial identities

\[
\sum_{i_1+\cdots+i_n=k, j_1+\cdots+j_n=l} \left( \begin{array}{c} l_1 + r_1 \\ 1 - i_1 \\ 1 - i_2 \\ \vdots \\ 1 - i_n \end{array} \right) \left( \begin{array}{c} l_2 + 3r_2 \\ 1 - i_1 \\ 1 - i_2 \\ \vdots \\ 1 - i_n \end{array} \right) \cdots \left( \begin{array}{c} l_n + (2n-1)r_n \\ 1 - i_1 \\ 1 - i_2 \\ \vdots \\ 1 - i_n \end{array} \right) = r^{n-k} \sum_{a_n=0}^{n} \prod_{j=1}^{n} (2j-1)^{a_j}. \tag{36}
\]

\[
\sum_{i_1+\cdots+i_n=k, j_1+\cdots+j_n=l} \left( \begin{array}{c} l_1 + 1 \\ 1 - i_1 \\ 1 - i_2 \\ \vdots \\ 1 - i_n \end{array} \right) \left( \begin{array}{c} l_2 + 3 \\ 1 - i_1 \\ 1 - i_2 \\ \vdots \\ 1 - i_n \end{array} \right) \cdots \left( \begin{array}{c} l_n + (2n-1) \\ 1 - i_1 \\ 1 - i_2 \\ \vdots \\ 1 - i_n \end{array} \right) = r^{n-k} \sum_{a_n=0}^{n} \prod_{j=1}^{n} (2j-1)^{a_j}. \tag{37}
\]

From Equations (29) and (34) we obtain that

\[
\mathbf{s}(n,k; r, 1) = r^{n-k} \mathbf{s}(n,k). \tag{38}
\]

**Remark 3.1**

Operating with both sides of Equation (13) on the exponential function \( e^{x} \), we get

\[
(l + r_1)^x (l + 2r_1 + r_2)^x \cdots (l + 2r_1 + \cdots + 2r_{n-1} + r_n)^x = \sum_{k=0}^{\infty} \mathbf{s}(k; \bar{r}, \bar{s}) t^k.
\]

Therefore, since a nonzero polynomial can have only a finite set of zeros, we have

\[
\prod_{j=0}^{n-1} \left( 2 \sum_{m=0}^{j} r_m + r_{j+1} + x \right)^{\frac{x}{j+1}} = \sum_{k=0}^{\infty} \mathbf{s}(k; \bar{r}, \bar{s}) x^k, r_0 = 0. \tag{39}
\]

If \( x = 1 \), we obtain

\[
\sum_{k=0}^{\infty} \mathbf{s}(k; \bar{r}, \bar{s}) = \prod_{j=0}^{n-1} \left( 2 \sum_{m=0}^{j} r_m + r_{j+1} \right)^{\frac{1}{j+1}}. \tag{40}
\]

**Remark 3.2**

From relation (39), by replacing \( s_j \) with \( s_{j-1} \), and relation (18) we conclude that

\[
\mathbf{s}(k; \bar{r}, \bar{s}) = \mathbf{s}(n,k; \bar{s}), \text{ where } \alpha_i = -\left( 2 \sum_{m=0}^{j} r_m + r_{j+1} \right), i = 0,1,\cdots, n-1. \tag{41}
\]

This gives us a connection between \( \mathbf{s}(k; \bar{r}, \bar{s}) \) and \( \mathbf{s}_n(n,k; \bar{s}) \), the generalized Comtet numbers, see [6]. Setting \( r_i = r \) and \( s_i = s \), we get

\[
\prod_{j=0}^{n-1} \left( 2j + 1 + x \right)^{\frac{1}{j+1}} = \sum_{k=0}^{\infty} \mathbf{s}(n,k; r, s) x^k, \tag{42}
\]

hence, we have

\[
\mathbf{s}(n,k; r, s) = \mathbf{s}_n(n,k; s), \text{ where } \alpha_i = -(2i+1)r, i = 0,1,\cdots, n-1, \text{ see [6].}
\]

If \( x = 1 \), then

\[
\sum_{k=0}^{\infty} \mathbf{s}(n,k; r, s) = \prod_{j=0}^{n-1} \left( 2j + 1 + 1 \right)^{\frac{1}{j+1}}, n \geq 1. \tag{43}
\]

Next we discuss the following special cases of (42) and (43):

i) If \( r = 1 \), then

\[
\prod_{j=0}^{n-1} \left( 2j + 1 + x \right)^{\frac{1}{j+1}} = \sum_{k=0}^{\infty} \mathbf{s}(n,k; 1, s) x^k, \tag{44}
\]

hence we have

\[
\mathbf{s}(n,k; 1, s) = \mathbf{s}_n(n,k), \text{ the generalized Comtet numbers, where } \alpha_i = -(2i+1), i = 0,1,\cdots, n-1 \text{, see [6].}
\]

ii) If \( s = 1 \), then we have
\[
\prod_{j=0}^{n-1}((2j+1)r+x)=\sum_{k=0}^{n} s(n,k;r,1)x^k, \\
\sum_{k=0}^{n} s(n,k;r,1) = \prod_{j=0}^{n-1}((2j+1)r+1), n \geq 1.
\] (45)

hence we obtain \( s(n,k;r,s) = s_{\xi}(n,k) \), Comet numbers, where \( \alpha_i = -(2i+1)r, i = 0,1,\ldots,n-1 \), see [3] and [4].

For example if \( n = 3, r = 2 \) and \( s = 2 \) in (43) we have
\[
\sum_{k=0}^{6} s(3,k;2,2) = \prod_{j=0}^{2}(4j+3)^2.
\] (46)

Using Table 2,
L.H.S. of (46) = \( s(3,0;2,2) + s(3,1;2,2) + s(3,2;2,2) + s(3,3;2,2) + s(3,4;2,2) + s(3,5;2,2) + s(3,6;2,2) = 14400 + 22080 + 12784 + 508 + 36 + 1 = 53361. \)
R.H.S. of (46) = \( \prod_{j=0}^{2}(4j+3)^3 = (3^3)(7^3)(11^3) = 53361. \)
This confirms (46) and hence (43).
Another example if \( n = 2, r = 2 \) and \( s = 3 \) in (43) we have
\[
\sum_{k=0}^{6} s(2,k;2,3) = \prod_{j=0}^{1}(4j+3)^3.
\] (47)

Using Table 3,
L.H.S. of (47) = \( s(2,0;2,3) + s(2,1;2,3) + s(2,2;2,3) + s(2,3;2,3) + s(2,4;2,3) + s(2,5;2,3) + s(2,6;2,3) = 1728 + 3456 + 2736 + 1088 + 228 + 24 + 1 = 9261. \)
R.H.S. of (46) = \( \prod_{j=0}^{1}(4j+3)^3 = (3^3)(7^3) = 9261. \)
This confirms (43).
iii) If \( r = s = 1 \), then we get
\[
\prod_{j=0}^{n-1}((2j+1)+x) = \sum_{k=0}^{n} s(n,k)x^k, \\
\sum_{k=0}^{n} s(n,k) = \prod_{j=0}^{n-1}(j+1) = 2^n n!, n \geq 1.
\] (48)

hence we have \( s(n,k) = s_{\xi}(n,k) \), which is a special case of Comet numbers, where \( \alpha_i = -(2i+1), i = 0,1,\ldots,n-1 \), see [3] and [4] and Table 1.
Setting \( e^x = t \), we have \( D := \frac{d}{dx} = t(\frac{d}{dt}) = tD = \delta \), then substituting in (2.1) it becomes
\[
(\frac{d^n e^x}{dx^n} \cdot \frac{d^n e^x}{dt^n}) = e^{(\sum_{k=0}^{n} \delta_{k})} \sum_{k=0}^{n} s(k;\xi) \delta_{k}. \] (49)

Using, see [12],
\[
F(\delta)(x^a f(x)) = x^a F(\delta + \alpha) f(x),
\]
then Equation (49) yields
\[
(\delta_i + 2r_i + \cdots + 2r_{n-1} + r_{n})^n \cdots (\delta_i + 2r_i + r_i)^n (\delta_i + r_i)^n = \sum_{k=0}^{n} s(k;\xi) \delta_{k}. \] (50)

Comparing this equation with Equation (4.1) in [6], we get
\[
\xi s(k;\xi) = s_{\xi}(n,k;\xi),
\] (51)
Furthermore, using our notations, it is easy from Equation (4.4) in [6] and (41) to show that

\[ S(n,i;\bar{z},\bar{s}) = \sum_{k=0}^{n} s(n,k;\bar{F},\bar{s}) S(k,i),[s] = s_0 + s_1 + \cdots + s_{n-1}, \]

where \( \alpha_i = -\left(2\sum_{a=0}^{n} r_a + r_{i+1}\right), i = 0,1,\ldots,n-1 \) and \( S(n,k) \) are the Stirling numbers of the second kind.

Next, we find a connection between \( s(n,k;r,s) \) and the generalized harmonic numbers \( O_n^{(\ell)} \) which are defined by, see [13] and [14],

\[ O_n^{(\ell)} = \sum_{j=1}^{n} \frac{1}{(2j-1)^\ell}. \]

From (42), we have

\[ \sum_{k=0}^{n} s(n,k;r,s)x^k = \prod_{j=0}^{n-1} \left(1 + \frac{x}{2j+1}\right) = \prod_{j=0}^{n-1} \left(1 + \frac{x}{2j+1}\right)e^{\sum_{s=1}^{n} s\log(1 + x/(2j+1)r)} = \prod_{j=0}^{n-1} \left(1 + \frac{x}{2j+1}\right)e^{\sum_{s=1}^{n} \sum_{j=0}^{n-1} \frac{(-1)^{j+1}}{j+i} x^j / i!} \]

\[ = \prod_{j=0}^{n-1} \left(1 + \frac{x}{2j+1}\right)e^{\sum_{s=1}^{n} \sum_{j=0}^{n-1} \frac{O_n^{(j)} (-1)^{j+1}}{r^j i^j} x^j} \]

\[ = \prod_{j=0}^{n-1} \left(1 + \frac{x}{2j+1}\right)e^{\sum_{s=1}^{n} \sum_{j=0}^{n-1} \frac{O_n^{(j)} (-1)^{j+1}}{r^j i^j} x^j} \]

Equating the coefficients of \( x^k \) on both sides, we obtain

\[ s(n,k;r,s) = r^{n-k} \prod_{j=0}^{n-1} (2j+1) \sum_{l=0}^{n} \frac{s^n}{l!} \sum_{k_1+\cdots+k_l = k} (-1)^{k+1} \frac{O_n^{(k_1)} \cdots O_n^{(k_l)}}{l_1 \cdots l_i r^k}. \]

From (22) and (53), we have the combinatorial identity

\[ \sum_{s_{(i,j)} \geq 0} \left[ \prod_{j=1}^{n} \sum_{i=1}^{j} \left(2j-1\right)^i \right] = \prod_{j=0}^{n} \left(2j+1\right) \sum_{l=0}^{n} \frac{s^n}{l!} \sum_{k_1+\cdots+k_l = k} (-1)^{k+1} \frac{O_n^{(k_1)} \cdots O_n^{(k_l)}}{l_1 \cdots l_i r^k}. \]

hence, setting \( s = 1 \), we get the identity

\[ \sum_{s_{(i,j)} \geq 0} \left[ \prod_{j=1}^{n} (2j-1)^i \right] = \prod_{j=0}^{n} (2j+1) \sum_{l=0}^{n} \sum_{k_1+\cdots+k_l = k} (-1)^{k+1} \frac{O_n^{(k_1)} \cdots O_n^{(k_l)}}{l_1 \cdots l_i l!}. \]

4. Some Applications

4.1. Coherent State and Normal Ordering

Coherent states play an important role in quantum mechanics especially in optics. The normally ordered form of the boson operator in which all the creation operators \( a^+ \) stand to the left of the annihilation operators \( a \). Using the properties of coherent states we can define and represent the generalized polynomial \( \bar{P}_{rs}(x) \) and generalized...
number $\overline{P}_{r,s}$ as follows.

**Definition 4.1**

The generalized polynomial $\overline{P}_{r,s}(x)$ is defined by

$$\overline{P}_{r,s}(x) = \sum_{k=0}^{d_r} s(k;\overline{r},\overline{s}) x^k,$$

and the generalized number $\overline{P}_{r,s}$

$$\overline{P}_{r,s} = \overline{P}_{r,s}(1) = \sum_{k=0}^{d_r} s(k;\overline{r},\overline{s}).$$

For convenience we apply the convention

$$s(k;\overline{r},\overline{s}) = 0 \text{ for } k < 0 \text{ or } k > \beta_s.$$

Now we come back to normal ordering. Using the properties of coherent states, see [7], the coherent state matrix element of the boson string yields the generalized polynomial $\overline{P}_{r,s}(x)$

$$\left\langle z \left| e^{a^*a} e^{a^*a} \right| \cdots \left| e^{a^*a} e^{a^*a} \right| e^{a^*a} e^{a^*a} \right\rangle = \left\langle z \left| e^{(s)\sum_{k=0}^{n} e^{a^*a} \sum_{k=0}^{n} s(k;\overline{r},\overline{s}) z^k \right\rangle = e^{(s)\sum_{k=0}^{n} e^{a^*a} \sum_{k=0}^{n} s(k;\overline{r},\overline{s}) z^k}ight.$$  

$$= e^{(s)\sum_{k=0}^{n} \overline{P}_{r,s}(z)} = e^{(s)\sum_{k=0}^{n} \gamma_r \gamma_s z^k}, \text{ where } \beta_n = \sum_{j=1}^{n} s_j.$$

**Definition 4.2**

We define the polynomial $\overline{P}(n,x)$ as

$$\overline{P}(n,x) = \sum_{k=0}^{n} s(n,k) x^k,$$

and the numbers

$$\overline{P}(n) = \overline{P}(n,1) = \sum_{k=1}^{n} s(n,k).$$

For convenience we apply the conventions

$$s(n,0) = \delta_{n,0} \text{ and } s(n,k) = 0 \text{ for } k > n \text{ and } \overline{P}(0) = \overline{P}(0,x) = 1.$$

Similarly, using the properties of coherent states and (32) we have

$$\left\langle z \left| e^{a^*a} e^{a^*a} \right| \cdots \left| e^{a^*a} e^{a^*a} \right| e^{a^*a} e^{a^*a} \right\rangle = e^{(s)\sum_{k=0}^{n} \sum_{k=0}^{n} s(n,k) z^k} = e^{(s)\sum_{k=0}^{n} \sum_{k=0}^{n} s(n,k) z^k}$$  

$$= e^{(s)\overline{P}(n,z)} = e^{(s)\overline{P}(n,z)} = e^{(s)\overline{P}(n,z)}.$$

**4.2. Matrix Representation**

In this subsection we derive a matrix representation of some results obtained.

Let $s_r$ be $n \times n$ lower triangle matrix, where $s_r$ is the matrix whose entries are the numbers $s(n,k;r,1)$, i.e. $s_r = [s(i,j;r,1)]_{i,j=0}$. Furthermore let $N_r$ be an $n \times n$ lower triangle matrix defined by
\( N_j = [e^{2irx}(i,j;r,1)]_{i+j=0} \), \( M_i \) is a diagonal matrix whose entries of the main diagonal are \( e^{2irx}, i = 0,1,\ldots,n \), i.e. \( M_i = \text{diag}(e^{2rx}, e^{4rx}, \ldots, e^{2nx}) \), \( R_i = \left(\left(e^{rx}D^x\right)^0, \left(e^{rx}D^x\right)^1, \ldots, \left(e^{rx}D^x\right)^x\right)^T \) and \( D = (D^0, D^1, D^2, \ldots, D^x)^T \).

Equation (27), may be represented in a matrix form as
\[
R_i = N_i D = M_i s D, \tag{64}
\]
for example if \( n = 3 \) then
\[
\begin{bmatrix}
(e^{rx}D^x)^0 \\
(e^{rx}D^x)^1 \\
(e^{rx}D^x)^2 \\
(e^{rx}D^x)^3
\end{bmatrix}
= \begin{bmatrix}
e^{rx} & 0 & 0 & 0 \\
e^{2rx} & e^{2rx} & 0 & 0 \\
e^{3rx} & 2e^{4rx} & 4e^{4rx} & 0 \\
e^{4rx} & 23e^{6rx} & 9e^{6rx} & e^{6rx}
\end{bmatrix}
D^0
\begin{bmatrix}
e^{rx} & 0 & 0 & 0 \\
e^{2rx} & 0 & 0 & 0 \\
e^{3rx} & 3e^{4rx} & 4e^{4rx} & 0 \\
e^{4rx} & 15e^{6rx} & 23e^{6rx} & 9e^{6rx}
\end{bmatrix}
D^1
\begin{bmatrix}
e^{rx} & 0 & 0 & 0 \\
e^{2rx} & 0 & 0 & 0 \\
e^{3rx} & 3 & 4 & 1 \\
e^{4rx} & 15 & 23 & 9 & 1
\end{bmatrix}
D^2
\begin{bmatrix}
e^{rx} & 0 & 0 & 0 \\
e^{2rx} & 0 & 0 & 0 \\
e^{3rx} & 1 & 0 & 0 \\
e^{4rx} & 0 & 0 & 0
\end{bmatrix}
D^3
\tag{65}
\]
its inverse is given by
\[
D = N_i^{-1} R_i = s_i^{-1} M_i^{-1} R_i. \tag{66}
\]
Setting \( r = 1 \) in (64), we get
\[
R_i = N_i D = M_i s D, \tag{67}
\]
\[
\begin{bmatrix}
e^{rx} & 0 & 0 & 0 \\
e^{2rx} & e^{2rx} & 0 & 0 \\
e^{3rx} & 3e^{4rx} & 4e^{4rx} & 0 \\
e^{4rx} & 15e^{6rx} & 23e^{6rx} & 9e^{6rx}
\end{bmatrix}
D^0
\begin{bmatrix}
e^{rx} & 0 & 0 & 0 \\
e^{2rx} & 0 & 0 & 0 \\
e^{3rx} & 3 & 4 & 1 \\
e^{4rx} & 15 & 23 & 9 & 1
\end{bmatrix}
D^1
\begin{bmatrix}
e^{rx} & 0 & 0 & 0 \\
e^{2rx} & 0 & 0 & 0 \\
e^{3rx} & 1 & 0 & 0 \\
e^{4rx} & 0 & 0 & 0
\end{bmatrix}
D^2
\begin{bmatrix}
e^{rx} & 0 & 0 & 0 \\
e^{2rx} & 0 & 0 & 0 \\
e^{3rx} & 0 & 0 & 0 \\
e^{4rx} & 0 & 0 & 0
\end{bmatrix}
D^3
\tag{68}
\]
\[
D = N_i^{-1} R_i = s_i^{-1} M_i^{-1} R_i.
\]
For \( n = 3 \), we have
\[
\begin{bmatrix}
D^0 \\
D^1 \\
D^2 \\
D^3
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 0 \\
-1 & e^{-2rx} & 0 & 0 \\
1 & -4e^{-2rx} & e^{-4rx} & 0 \\
-1 & 13e^{-2rx} & -9e^{-4rx} & e^{-6rx}
\end{bmatrix}
\begin{bmatrix}
e^{rx} & 0 & 0 & 0 \\
e^{2rx} & e^{2rx} & 0 & 0 \\
e^{3rx} & 4e^{4rx} & e^{4rx} & 0 \\
e^{4rx} & 23e^{6rx} & 9e^{6rx} & e^{6rx}
\end{bmatrix}
\begin{bmatrix}
e^{rx} & 0 & 0 & 0 \\
e^{2rx} & 0 & 0 & 0 \\
e^{3rx} & 0 & 0 & 0 \\
e^{4rx} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
e^{rx} & 0 & 0 & 0 \\
e^{2rx} & 0 & 0 & 0 \\
e^{3rx} & 0 & 0 & 0 \\
e^{4rx} & 0 & 0 & 0
\end{bmatrix}
\tag{69}
\]

5. Conclusion
In this article we investigated a new family of generalized Stirling numbers of the first kind. Recurrence relations and an explicit formula of these numbers are derived. Moreover some interesting special cases and new
combinatorial identities are obtained. A connection between this family and the generalized harmonic numbers is given. Finally, some applications in coherent states and matrix representation of some results are obtained.

References


Appendix

Tables of $s(n, k; r, s)$ calculated using Maple, for some values of $n, k, r$ and $s$:

### Table 1. $0 \leq n, k \leq 4$, $r = s = 1$.

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### Table 2. $0 \leq n, k \leq 4$, $r = s = 2$.

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### Table 3. $0 \leq n, k \leq 4$, $r = 2$, and $s = 3$.

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Notice that the last column in all tables is just the sum of the entries of the corresponding row.