Lattices Associated with a Finite Vector Space

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Abstract

Let $F_q^n$ be a $n$-dimensional row vector space over a finite field $F_q$. For $1 \leq d \leq n-1$, let $W_0$ be a $d$-dimensional subspace of $F_q^n$. $L(n,d)$ denotes the set of all the spaces which are the subspaces of $F_q^n$ and not the subspaces of $W_0$ except $\{0\}$. We define the partial order on $L(n,d)$ by ordinary inclusion (resp. reverse inclusion), and then $L(n,d)$ is a poset, denoted by $L_0(n,d)$ (resp. $L_R(n,d)$). In this paper we show that both $L_0(n,d)$ and $L_R(n,d)$ are finite atomic lattices. Further, we discuss the geometricity of $L_0(n,d)$ and $L_R(n,d)$, and obtain their characteristic polynomials.

Keywords
Vector Space; Geometric Lattice; Characteristic Polynomial

1. Introduction

Let $P$ be a poset. For $a, b \in P$, we say $a$ covers $b$, denoted by $b <\cdot a$, if $b < a$ and there doesn’t exist $c \in P$ such that $b < c < a$. If $P$ has the minimum (resp. maximum) element, then we denote it by 0 (resp. 1) and say that $P$ is a poset with 0 (resp. 1). Let $P$ be a finite poset with 0. By a rank function on $P$, we mean a function $r$ from $P$ to the set of all the integers such that $r(0) = 0$ and $r(a) = r(b) + 1$ whenever $b \leq a$. Observe the rank function is unique if it exists. $P$ is said to be ranked whenever $P$ has a rank function.

Let $P$ be a finite ranked poset with 0 and 1. The polynomial $\chi(P, x) = \sum_{a \in P} \mu(0, a)x^{r(a)}$ is called the characteristic polynomial of $P$, where $\mu$ is the Möbius function on $P$ and $r$ is the rank function of $P$. A poset $P$ is said to be a lattice if both $a \lor b = \sup\{a, b\}$ and $a \land b = \inf\{a, b\}$ exist for any two elements $a, b \in P$. $a \lor b$ and $a \land b$ are called the join and meet of $a$ and $b$, respectively. Let $P$ be a finite lattice with 0. By an
atom in $P$, we mean an element in $P$ covering 0. We say $P$ is atomic if any element in $P \setminus \{0\}$ is the join of atoms. A finite atomic lattice $P$ is said to be a geometric lattice if $P$ admits a rank function $r$ satisfying $r(a \land b) + r(a \lor b) \leq r(a) + r(b), \forall a, b \in P$. Notations and terminologies about posets and lattices will be adopted from books [1] [2].

The special lattices of rough algebras were discussed in [3]. The lattices generated by orbits of subspaces under finite (singular) classical groups were discussed in [4] [5]. Wang et al. [6]-[8] constructed some sublattices of the lattices in [4]. The subspaces of a $d$-bounded distance-regular have similar properties to those of a vector space. Gao et al. [9]-[11] constructed some lattices and posets by subspaces in a $d$-bounded distance-regular graph. In this paper, we continue this research, and construct some new sublattices of the lattices in [4], discussing their geometricity and computing their characteristic polynomials.

Let $F_q$ be a finite field with $q$ elements, where $q$ is a prime power. For a positive integer $n$, let $F_q^n$ be the $n$-dimensional row vector space over $F_q$. Let $1 \leq d \leq n-1$. For a fixed $d$-dimensional subspace $W_0$ of $F_q^n$, let $L(n,d) = \{ P | P$ is a subspace of $F_q^n$ and is not of $W_0 \cup \{0\} \}$. If we define the partial order on $L(n,d)$ by ordinary inclusion (resp. reverse inclusion), then $L(n,d)$ is a poset, denoted by $L_0(n,d)$ (resp. $L_\text{r}(n,d)$). In the present paper we show that both $L_0(n,d)$ and $L_\text{r}(n,d)$ are finite atomic lattices, discuss their geometricity and compute their characteristic polynomials.

2. The Lattice $L_0(n,d)$

In this section we prove that the lattice $L_0(n,d)$ is a finite geometric lattice, and compute its characteristic polynomial. We begin with a useful proposition.

**Proposition 2.1.** ([12], Lemma 9.3.2 and [13], Corollaries 1.8 and 1.9) For $0 \leq k \leq m \leq n$, the following hold:

1) The number of $k$-dimensional subspaces contained in a given $m$-dimensional subspace of $F_q^n$ is
\[
\binom{m}{k}_q = \prod_{i=m-k+1}^{m} (q^i - 1) / \prod_{i=1}^{k} (q^i - 1).
\]

2) The number of $m$-dimensional subspaces containing a given $k$-dimensional subspace of $F_q^n$ is
\[
\binom{n-k}{m-k}_q.
\]

3) Let $P$ be a fixed $m$-dimensional subspace of $F_q^n$. Then the number of $k$-dimensional subspaces $Q$ of $F_q^n$ satisfying $\dim(P \cap Q) = t$ is
\[
q^{(m-t)(k-t)} \binom{n-m}{k-t} \binom{m}{t}_q.
\]

**Theorem 2.2.** $L_0(n,d)$ is a geometric lattice.

**Proof.** For any two elements $P, Q \in L_0(n,d)$,
\[
P \lor Q = P + Q, P \land Q = \begin{cases} \{P \cap Q\} & \text{if } P \cap Q \neq \emptyset; \\ \{0\} & \text{otherwise.} \end{cases}
\]

Therefore $L_0(n,d)$ is a finite lattice. Note that $\{0\}$ is the unique minimum element. Let $P(n,d;j)$ be the set of all the $j$-dimensional subspaces of $L_0(n,d)$, where $1 \leq j \leq n$. Then $P(n,d;1)$ is the set of all the atoms in $L_0(n,d)$. In order to prove $L_0(n,d)$ is atomic, it suffices to show that every element of $P(n,d;j)$ is a join of some atoms. The result is trivial for $j = 1$. Suppose that the result is true for $j = l > 1$. Let $U \in P(n,d;l+1)$. By Proposition 2.1 and $\dim(W_e \cap U) \leq l$, the number of $l$-dimensional subspaces of $L_0(n,d)$ contained in $U$ at least is
\[
\left(\frac{l+1}{l}_q\right) - 1 = \frac{q^{l+1} - 1}{q-1} \geq 2.
\]

Therefore there exist two different $l$-dimensional subspaces $U', U'' \subseteq U$ of $L_0(n,d)$ such that $U = U' \lor U''$. 


By induction \( U \) is a join of some atoms. Hence \( L_0(n,d) \) is a finite atomic lattice. For any \( U \in L_0(n,d) \), define \( r_0(U) = \dim U \). It is routine to check that \( r_0 \) is the rank function on \( L_0(n,d) \). For any \( U, V \in L_0(n,d) \), we have

\[
r_0(U \lor V) + r_0(U \land V) = \dim(U + V) + \dim(U \land V) \\
\leq \dim(U + V) + \dim(U \cap V) \\
= \dim U + \dim V = r_0(U) + r_0(V).
\]

Hence \( L_0(n,d) \) is a geometric lattice. \( \square \)

**Lemma 2.3.** For any \( P, Q \in L_0(n,d) \), suppose that \( \dim P = t \), \( \dim Q = t + s \) and \( \dim(W \cap Q) = m \). Then the Möbius function of \( L_0(n,d) \) is

\[
\mu(P,Q) = \begin{cases} 
(-1)^{l} q^{\frac{s}{2}} & \text{if } \{0\} \neq P \leq Q \text{ or } P = Q = \{0\}; \\
\sum_{j=1}^{n} (-1)^{l-1} \left( \begin{bmatrix} s \\ l \end{bmatrix} - \begin{bmatrix} m \\ l \end{bmatrix} \right) q^{\frac{t-j}{2}} & \text{if } \{0\} = P < Q; \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof:** The Möbius function of \( L_0(n,d) \) is

\[
\mu(P,Q) = \begin{cases} 
(-1)^{l} q^{\frac{s}{2}} & \text{if } \{0\} \neq P \leq Q \text{ or } P = Q = \{0\}; \\
\sum_{j=1}^{n} -\mu(U,Q) & \text{if } \{0\} = P < Q; \\
0 & \text{otherwise.}
\end{cases}
\]

By Proposition 2.1, we have

\[
\sum_{\{0\} \subset U \subset Q} -\mu(U,Q) = \sum_{j=1}^{n} \left( \begin{bmatrix} s \\ l \end{bmatrix} - \begin{bmatrix} m \\ l \end{bmatrix} \right) q^{\frac{t-j}{2}}.
\]

Thus, the assertion follows. \( \square \)

**Theorem 2.4.** The characteristic polynomial of \( L_0(n,d) \) is

\[
\chi(L_0(n,d),x) = x^n + \sum_{j=1}^{n} \left( \begin{bmatrix} n \\ j \end{bmatrix} - \begin{bmatrix} d \\ j \end{bmatrix} \right) q^{\frac{n-j}{2}} \\
+ \sum_{j=1}^{n} \sum_{l=0}^{\min(d,j-1)} (-1)^{l-1} q^{\frac{d-j}{2}} \left( \begin{bmatrix} d \\ l \end{bmatrix} - \begin{bmatrix} n-d \\ j \end{bmatrix} \right) x^{n-j}.
\]

**Proof:** By Proposition 2.1 and Lemma 2.3, we have

\[
\chi(L_0(n,d),x) = \sum_{P \in L_0(n,d)} \mu(P) x^{\alpha_0(P)} \\
= x^n + \sum_{\{0\} \subset P \subset L_0(n,d)} \mu(P) x^{\alpha_0(P)} \\
= x^n + \sum_{j=1}^{n} \left( \begin{bmatrix} n \\ j \end{bmatrix} - \begin{bmatrix} d \\ j \end{bmatrix} \right) q^{\frac{n-j}{2}} \\
+ \sum_{j=1}^{n} \sum_{l=0}^{\min(d,j-1)} (-1)^{l-1} q^{\frac{d-j}{2}} \left( \begin{bmatrix} d \\ l \end{bmatrix} - \begin{bmatrix} n-d \\ j \end{bmatrix} \right) x^{n-j}.
\]
3. The Lattice \( L_R(n,d) \)

In this section we prove that the lattice \( L_R(n,d) \) is a finite atomic lattice, classify its geometricity and compute its characteristic polynomial.

**Theorem 3.1.** The following hold:
1) \( L_R(n,d) \) is a finite atomic lattice.
2) \( L_R(n,d) \) is geometric if and only if \( n = 2 \).

**Proof.**
1) For any two elements \( P, Q \in L_R(n,d) \), \( P \cap Q = P + Q \) and
\[
P \lor Q = \begin{cases} \{P \cap Q\} & \text{if } P \cap Q \not\subseteq W; \\ \{0\} & \text{otherwise.} \end{cases}
\]

Therefore \( L_R(n,d) \) is a finite lattice. Note that \( \{0\} \) is the unique minimum element. Let \( \{0\} \subseteq P \subseteq W_0 \) be the set of all the \( j \)-dimensional subspaces of \( L_R(n,d) \), where \( 0 \leq j \leq n-1 \). Then \( P(n,d;n-1) \) is the set of all the atoms in \( L_R(n,d) \). In order to prove \( L_R(n,d) \) is atomic, it suffices to show that every element of \( P(n,d); 0 \leq j \leq n-1 \) is a join of some atoms. The result is trivial for \( j = n-1 \). Suppose that the result is true for \( j = n-l \). Let \( U \subseteq P(n,d;n-l-1) \). By Proposition 2.1, the number of \( n-l \)-dimensional subspaces of \( L_R(n,d) \) containing \( U \) is equal to
\[
|n-l\rangle - |d-1\rangle |1\rangle = \frac{q^{d-l} - q^{d-1} - 1}{q-1} \geq 2.
\]

Then there exist two different \( (n-l) \)-dimensional subspaces \( U \subseteq U', U'' \subseteq L_R(n,d) \) such that \( U = U' \lor U'' \). By induction \( U \) is a join of some atoms. Therefore \( L_R(n,d) \) is a finite atomic lattice.

2) For any \( U \subseteq L_R(n,d) \), we define \( r(U) = n - \dim U \). It is routine to check that \( r(U) \) is the rank function on \( L_R(n,d) \). It is obvious that \( L_R(2,1) \) is a geometric lattice. Now assume that \( n \geq 3 \). Let \( P \) be a \( 1 \)-dimensional subspace of \( n^* \) and \( P \subseteq W_0 \). By Proposition 2.1, the number of \( 2 \)-dimensional subspaces of \( L_R(n,d) \) containing \( P \) is equal to
\[
|n-l\rangle - |d-1\rangle |1\rangle = \frac{q^{d-l} - q^{d-1} - 1}{q-1} \geq 2.
\]

Therefore, there exist two different \( 2 \)-dimensional subspaces \( P \subseteq P', P'' \subseteq L_R(n,d) \) such that \( P = P' \lor P'' \).

**Lemma 3.2.** For any \( P, Q \subseteq L_R(n,d) \), suppose that \( \dim P = t+s \), \( \dim Q = t \) and \( \dim(W_0 \cap P) = m \). Then the Möbius function of \( L_R(n,d) \) is
\[
\mu(P,Q) = \begin{cases} (-1)^t q^{\frac{s}{2}} & \text{if } P \leq Q \neq \{0\} \text{ or } P = Q = \{0\}; \\ \sum_{l=1}^s (-1)^{l-1} \left( \begin{array}{c} s \cr l \end{array} \right) q^{l-1} & \text{if } P < Q = \{0\}; \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** The Möbius function of \( L_R(n,d) \) is
\[
\mu(P,Q) = \sum_{P \subseteq U \subseteq \{0\}} -\mu(P,U) & \text{if } P \leq Q = \{0\} \text{ or } P = Q = \{0\}; \\ 0 & \text{otherwise.}
\]

Proposition 2.1 implies that
\[
\sum_{P \in \mathcal{U} < [0]} -\mu(P, U) = \sum_{j=1}^{s} (-1)^{s-j} \left( \begin{bmatrix} n \cr j \end{bmatrix} - \begin{bmatrix} d \cr j \end{bmatrix} \right) q^{\frac{n-j}{2}}.
\]

**Theorem 3.3.** The characteristic polynomial of \( L_{n}(n, d) \) is
\[
\chi(L_{n}(n, d), x) = x^n - 1 + \sum_{j=1}^{n} (-1)^{n-j} \left( \begin{bmatrix} n \cr j \end{bmatrix} - \begin{bmatrix} d \cr j \end{bmatrix} \right) q^{\frac{n-j}{2}} (x^j - 1).
\]

**Proof.** By Proposition 2.1, we have
\[
\chi(L_{n}(n, d), x) = \sum_{P \in \mathcal{U}^*} \mu(F_q^n, P) x^{\deg(P)}
\]
\[
= x^n + \sum_{P \in \mathcal{U}^*} \mu(F_q^n, P) x^{\text{dim}(P)}
\]
\[
= x^n + \sum_{j=1}^{n} (-1)^{n-j} \left( \begin{bmatrix} n \cr j \end{bmatrix} - \begin{bmatrix} d \cr j \end{bmatrix} \right) q^{\frac{n-j}{2}} x^j + \sum_{j=1}^{n} (-1)^{n-j} \left( \begin{bmatrix} n \cr j \end{bmatrix} - \begin{bmatrix} d \cr j \end{bmatrix} \right) q^{\frac{n-j}{2}} (x^j - 1).
\]

**References**


