

The Generalized Search for a Randomly Moving Target

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Abstract

A target is assumed to move randomly on one of two disjoint lines L_1 and L_2 according to a stochastic process $\{S(t), t \in \mathbb{R}^+\}$. We have two searchers start looking for the lost target from some points on the two lines separately. Each of the searchers moves continuously along his line in both directions of his starting point. When the target is valuable as a person lost on one of disjoint roads, or is serious as a car filled with explosives which moves randomly in one of disjoint roads, in these cases the search effort must be unrestricted and then we can use more than one searcher. In this paper we show the existence of a search plan such that the expected value of the first meeting time between the target and one of the two searchers is minimum.

Keywords

Stochastic Process; Expected Value; Linear Search; Optimal Search Plan

1. Introduction

The search for lost targets that are either stationary or randomly moving has recently applications, such as: searching for lost persons on roads, the search for a petroleum or gas underground, and so on (see, Abd-Elmoneim [1], Ohsumi [2], El-Rayes and Abd-Elmoneim [3], and Washburn [4]).

When the target to be found is stationary or moves randomly on the real line, this problem is of interest because it may arise in many real world situations (see El-Rayes *et al.* [5] and Balkhi [6]). Search problems with stationary target on line are well studied (see El-Rayes and Abd-Elmoneim [7], El-Rayes *et al.* [8], Balkhi [6] [9], Rousseeuw [10], Abd-Elmoneim and Abu-Gabl [11], and Stone [12]). In the case of randomly moving target on the line and the searcher starts search from the origin, a deal of work has been done for deriving conditions for optimal search path which minimizes the effort of finding the target (see El-Rayes *et al.* [5], Fristedt

and Heath [13]). If the lost target is a valuable target as a person lost on one of disjoint roads, or is serious as a car filled with explosives which moves randomly in one of disjoint roads, then the effort of the search (the cost of search) must be unrestricted, in these cases using more than one searcher (see Abd El-Moneim *et al.* [14]). The search problem for a randomly moving target on one of two disjoint lines will be considered, in previous studies (see Abd El-Moneim and Abu-Gabl [15]), using a searcher for each line where each searcher starts looking for the lost target from the origin of his line, but Abd El-Moneim *et al.* [14] used searchers starting searching for the target from the origin that is the intersection point of these lines. Each of the searchers moves continuously along his line in both directions of the starting point, in both cases the target motion is a Brownian motion. In this article, a target is assumed to move randomly on one of two disjoint lines L_1 and L_2 according to a stochastic process $\{S(t), t \in \mathfrak{R}^+\}$, where \mathfrak{R}^+ is the set of real numbers. This stochastic process satisfies the following conditions:

(i) Let $c \geq \mu$, where μ is the drift of the process and c is a constant. Then for any $t > 0$ and for some $\varepsilon > 0$, $P\{S(t) > ct\} \leq \frac{1}{2}\varepsilon^t$,

(ii) Let $\mu \neq 0, z_1 \geq z_2$ and $t \geq \max(z_1/\mu, z_2/\mu)$, then $P\{z_2 \leq S(t) \leq z_1\}$ is non-increasing with t ,

(iii) Let T be a stopping time for $S(t)$, then

$$E|S(T)| \leq \sigma\sqrt{E(T)} + |\mu|E(T),$$

where E stands for the expectation value and σ^2 is the variance and

(iv) Let

$$S(n) = \sum_{i=1}^n Z_i, n \geq 1,$$

where $\{Z_i\}$ is a sequence of independent identically distributed random variables (i. i. d. r. v.) and

$$q(j, j+1) = \sum_{n=0}^{\infty} P[-(j+1) < S(n) < -j],$$

then $q(j, j+1)$ satisfies the renewal theorem.

Two searcher start looking for the target from some point ϕ_0 for the first line L_1 and $\bar{\phi}_0$ for the second line L_2 . We assume that the speeds of the searchers are v_1 and v_2 . Assume that T is the set of real numbers and T^+ is the non-negative part of T . For any $t \in T^+$, let $S(t)$ be a random variable with $S(0) = 0$, and assume that Z_0 is the initial position of the target to be a random variable and independent of $S(t)$, $t > 0$. A search plan $(\Phi, \bar{\Phi})$ with speed v_1, v_2 is defined such that $\phi: T^+ \rightarrow T$ and $\bar{\phi}: T^+ \rightarrow T$, respectively, such that

$$|\phi(t_1) - \phi(t_2)| \leq v_1|t_1 - t_2|$$

$$|\bar{\phi}(t_1) - \bar{\phi}(t_2)| \leq v_2|t_1 - t_2|, \forall t_1, t_2 \in T^+, \phi \in \Phi, \bar{\phi} \in \bar{\Phi}$$

the first meeting time is a random variable valued in T^+ which is defined

$$\tau_{\hat{\phi}} = \inf \{t: \phi(t) = X_0 + S(t) \text{ or } \bar{\phi}(t) = Y_0 + S(t)\}$$

where $Z_0 = X_0$ if the target moves on the first line and $Z_0 = Y_0$ if the target moves on the second line. Let the search plan of two searchers be represented by $\hat{\Phi} = \{\phi, \bar{\phi}\}$, where $\hat{\Phi}$ is the set of all search plans. The problem is to find search plan $\hat{\phi} = \{\phi, \bar{\phi}\}, \phi \in \Phi, \bar{\phi} \in \bar{\Phi}$, such that $E\tau_{\hat{\phi}} < \infty$ in this case we call $\hat{\Phi}$ is a finite search plan and if $E\tau_{\hat{\phi}^*} < E\tau_{\hat{\phi}} \forall \hat{\phi} \in \hat{\Phi}$, where E terms to expectation value, then we call $\hat{\phi}^*$ is optimal.

2. The Search Plans

Let ζ_1, ζ_2 be positive integers and v_1 and v_2 are rational numbers such that:

1) $v_1 > |\mu|$ and $v_2 > |\mu|$,

2) $\zeta_1, \zeta_2 > 1$, such that $C_1 = v_1 \frac{\zeta_1 - 1}{\zeta_1 + 1}, C_2 = v_2 \frac{\zeta_2 - 1}{\zeta_2 + 1}$,

Now we shall define sequences $\{G_i\}_{i \geq 0}, \{r_i\}_{i \geq 0}, \{H_i\}_{i \geq 0}$ for the searcher on the first line L_1 ,

$\{\bar{G}_i\}_{i \geq 0}, \{\bar{r}_i\}_{i \geq 0}, \{\bar{H}_i\}_{i \geq 0}$ for the searcher on the second line L_2 and search plans with speed 1 as follows.

$$G_i = \zeta_1^i - 1, r_i = (-1)^{i+1} C_1 \left[G_i + 1 + (-1)^{i+1} \right], H_i = r_i + \phi_0, i \geq 0$$

$$\bar{G}_i = \zeta_2^i - 1, \bar{r}_i = (-1)^{i+1} C_2 \left[\bar{G}_i + 1 + (-1)^{i+1} \right], \bar{H}_i = \bar{r}_i + \bar{\phi}_0, i \geq 0$$

Let O be a finite set of numbers, such that

$$O = \{1, 3, 5, \dots, m\} \text{ for } j \in O, i \in I^+$$

we have

$$\dots < H_2 < \phi_0 < H_1 < H_3 < \dots < H_{j-2} < H_j < 0 < H_{j+2} < \dots$$

for the first searcher, and

$$\dots < \bar{H}_2 < \bar{\phi}_0 < \bar{H}_1 < \bar{H}_3 < \dots < \bar{H}_{j-2} < \bar{H}_j < 0 < \bar{H}_{j+2} < \dots$$

for the second searcher

for any $t \in \mathbb{R}^+$, if $G_{2i-1} \leq t < G_{2i}$ then $\phi(t) = \phi_0 - H_{2i-1} - [t - G_{2i-1}]$

and if $\bar{G}_{2i-1} \leq t < \bar{G}_{2i}$, $1 \leq i \leq \frac{j+1}{2}$, then $\bar{\phi}(t) = \bar{\phi}_0 - \bar{H}_{2i-1} - [t - \bar{G}_{2i-1}]$,

if $G_i \leq t < G_{i+1}$, then $\phi(t) = H_i + (-1)^i [t - G_i]$

and if $\bar{G}_i \leq t < \bar{G}_{i+1}$, $i \geq j + 1$, then $\bar{\phi}(t) = \bar{H}_i + (-1)^i [t - \bar{G}_i]$, and

if $G_{2i} \leq t < G_{2i+1}$, then $\phi(t) = H_{2i} - \phi_0 + [t - G_{2i}]$

and $\bar{G}_{2i} \leq t < \bar{G}_{2i+1}$, $1 \leq i \leq \frac{j-1}{2}$, then $\bar{\phi}(t) = \bar{H}_{2i} - \bar{\phi}_0 + [t - \bar{G}_{2i}]$

We use the following notations where $k_1(t), \bar{k}_1(t), k_2(t)$ and $\bar{k}_2(t)$ are positive functions, $\varphi_1 = s(t) - k_1(t)$, $\tilde{\varphi}_1 = s(t) + k_2(t)$ on the first line and $\varphi_2 = s(t) - \bar{k}_1(t)$, $\tilde{\varphi}_2 = s(t) + k_1(t)$ on the second line.

Lemma 2.1. if $0 < a, b < 1$ then $ab < a + b$

Theorem 2.1. If $(\phi, \bar{\phi}) \in \hat{\Phi}$ is a search plan, and let γ_1, γ_2 are measurable induced by the initial position of the target on the first and second line respectively, then $E(\tau_{\bar{\phi}})$ is finite if:

$$\left. \begin{aligned} & \int_{-\infty}^{\phi_0} \left[\sum_{i=1}^{j/2} \zeta_1^{2i} P(\varphi_1(G_{2i}) < -x) + \sum_{i=\frac{j}{2}+1}^{\infty} \zeta_1^{2i} P(\tilde{\varphi}_1(G_{2i}) \leq -x) \gamma_1 \right] (dx), \\ & \int_{\phi_0}^{\infty} \sum_{i=1}^{\infty} \zeta_1^{2i+1} P(\varphi_1(G_{2i+1}) > -x) \gamma_1 (dx), \\ & \int_{-\infty}^{\bar{\phi}_0} \sum_{i=1}^{j/2} \zeta_2^{2i} P(\varphi_2(\bar{G}_{2i}) < -y) + \sum_{i=\frac{j}{2}+1}^{\infty} \zeta_2^{2i} P(\tilde{\varphi}_2(\bar{G}_{2i}) \leq -y) \gamma_2 (dy), \\ & \int_{\bar{\phi}_0}^{\infty} \sum_{i=1}^{\infty} \zeta_2^{2i+1} P(\varphi_2(\bar{G}_{2i+1}) > -y) \gamma_2 (dy). \end{aligned} \right\} \text{ If } \phi_0, \bar{\phi}_0 > 0,$$

$$\left. \begin{aligned} & \int_{\phi_0}^{\infty} \left[\sum_{i=(j+1)/2}^{\infty} \zeta_1^{2i+1} P(\varphi_2(G_{2i+1}) > -x) + \sum_{i=1}^{(j-1)/2} \zeta_1^{2i} P(\tilde{\varphi}_1(G_{2i+1}) \leq -x) \right] \gamma_1 (dx) \\ & \int_{-\infty}^{\phi_0} \sum_{i=1}^{\infty} \zeta_1^{2i} P(\tilde{\varphi}_1(G_{2i}) < -x) \gamma_1 (dx), \\ & \int_{\bar{\phi}_0}^{\infty} \left[\sum_{i=(j+1)/2}^{\infty} \zeta_2^{2i+1} P(\varphi_2(\bar{G}_{2i+1}) > -y) + \sum_{i=1}^{(j-1)/2} \zeta_2^{2i} P(\tilde{\varphi}_2(\bar{G}_{2i+1}) \leq -y) \right] \gamma_2 (dy) \\ & \int_{-\infty}^{\bar{\phi}_0} \sum_{i=1}^{\infty} \zeta_2^{2i} P(\tilde{\varphi}_2(\bar{G}_{2i}) < -y) \gamma_2 (dy). \end{aligned} \right\} \text{ If } \phi_0, \bar{\phi}_0 < 0,$$

$$\left. \begin{aligned} & \int_{-\infty}^{\phi_0} \left[\sum_{i=1}^{j/2} \zeta_1^{2i} P(\varphi_1(G_{2i}) < -x) + \sum_{i=\frac{j}{2}+1}^{\infty} \zeta_1^{2i} P(\tilde{\varphi}_1(G_{2i}) < -x) \right] \gamma_1(dx), \\ & \int_{\phi_0}^{\infty} \sum_{i=1}^{\infty} \zeta_1^{2i+1} P(\varphi_1(G_{2i+1}) > -x) \gamma_1(dx), \\ & \int_{\phi_0}^{\infty} \left[\sum_{i=(j+1)/2}^{\infty} \zeta_2^{2i+1} P(\varphi_2(\bar{G}_{2i+1}) > -y) + \sum_{i=1}^{(j-1)/2} \zeta_2^{2i+1} P(\tilde{\varphi}_2(\bar{G}_{2i+1}) > -y) \right] \gamma_2(dy), \\ & \int_{-\infty}^{\bar{\phi}_0} \sum_{i=0}^{\infty} \zeta_2^{2i} P(\tilde{\varphi}_2(\bar{G}_{2i}) < -y) \gamma_2(dy) \end{aligned} \right\} \text{If } \phi_0 > 0, \bar{\phi}_0 < 0,$$

And

$$\left. \begin{aligned} & \int_{\phi_0}^{\infty} \left[\sum_{i=(j+1)/2}^{\infty} \zeta_1^{2i+1} P(\varphi_1(G_{2i+1}) > -x) + \sum_{i=1}^{(j-1)/2} \zeta_1^{2i+1} P(\tilde{\varphi}_1(G_{2i+1}) < -x) \right] \gamma_1(dx), \\ & \int_{-\infty}^{\phi_0} \sum_{i=1}^{\infty} \zeta_1^{2i} P(\tilde{\varphi}_1(G_{2i}) < -x) \gamma_1(dx) \\ & \int_{-\infty}^{\bar{\phi}_0} \left[\sum_{i=1}^{j/2} \zeta_2^{2i} P(\varphi_2(\bar{G}_{2i}) < -y) + \sum_{i=\frac{j}{2}+1}^{\infty} \zeta_2^{2i} P(\tilde{\varphi}_2(\bar{G}_{2i}) < -y) \right] \gamma_2(dy), \\ & \int_{\bar{\phi}_0}^{\infty} \sum_{i=1}^{\infty} \zeta_2^{2i+1} P(\varphi_2(\bar{G}_{2i+1}) > -y) \gamma_2(dy), \end{aligned} \right\} \text{If } \phi_0 < 0, \bar{\phi}_0 > 0$$

are finite

Proof: The continuity of $S(t)$, $\phi(t)$ and $\bar{\phi}(t)$ imply that if $X_0 > \phi_0$ then $X_0 + S(t)$ is greater than $\phi(t)$ on the first line until the first meeting, also if $X_0 < \phi_0$ then $X_0 + S(t)$ is smaller than $\phi(t)$ on the first line until the first meeting the same for the second line by replacing X_0 by Y_0 and ϕ_0 by $\bar{\phi}_0$. Hence for any $i \geq 0$.

$$P(\tau_{\phi} > t) = P(\tau_{\phi} > t \text{ and } \tau_{\bar{\phi}} > t) = P(\tau_{\phi} > t) \cdot P(\tau_{\bar{\phi}} > t)$$

$$P(\tau_{\phi} > G_{2i+1}) \leq \int_{-\infty}^{\phi_0} P(X_0 + s(G_{2i}) < H_{2i}/X_0 = x) \gamma_1(dx) + \int_{\phi_0}^{\infty} P(X_0 + s(G_{2i+1}) > H_{2i+1}/X_0 = x) \gamma_1(dx)$$

we get

$$P(\tau_{\phi} > G_{2i+1}) \leq \int_{-\infty}^{\phi_0} P(\tilde{\varphi}_1(G_{2i}) < -x) \gamma_1(dx) + \int_{\phi_0}^{\infty} P(\varphi_1(G_{2i+1}) > -x) \gamma_1(dx)$$

Also

$$P(\tau_{\bar{\phi}} > G_{2i+1}) \leq \int_{-\infty}^{\bar{\phi}_0} P(\tilde{\varphi}_2(\bar{G}_{2i}) < -y) \gamma_2(dy) + \int_{\bar{\phi}_0}^{\infty} P(\varphi_2(\bar{G}_{2i+1}) > -y) \gamma_2(dy)$$

$$P(\tau_{\phi} > G_{2i}) \leq \int_{-\infty}^{\phi_0} P(\tilde{\varphi}_1(G_{2i}) < -x) \gamma_1(dx) + \int_{\phi_0}^{\infty} P(\varphi_1(G_{2i-1}) > -x) \gamma_2(dx)$$

and

$$P(\tau_{\bar{\phi}} > \bar{G}_{2i}) \leq \int_{-\infty}^{\bar{\phi}_0} P(\tilde{\varphi}_2(\bar{G}_{2i}) < -y) \gamma_2(dy) + \int_{\bar{\phi}_0}^{\infty} P(\varphi_2(\bar{G}_{2i-1}) > -y) \gamma_1(dy)$$

Hence

$$E(\tau_{\phi}) \leq \int_0^{\infty} P(\tau_{\phi} > t) dt$$

from Lemma 2.1 then we get:

$$\begin{aligned}
E(\tau_\phi) &\leq \int_0^\infty [P(\tau_\phi > t) + P(\tau_{\bar{\phi}} > t)] dt \leq \sum_{i=0}^\infty \left[\int_{G_i}^{G_{i+1}} P(\tau_\phi > G_i) + \int_{\bar{G}_i}^{\bar{G}_{i+1}} P(\tau_{\bar{\phi}} > \bar{G}_i) \right] dt \\
&\leq \sum_{i=0}^\infty \left[\int_{G_i}^{G_{i+1}} P(\tau_\phi > t) + \int_{\bar{G}_i}^{\bar{G}_{i+1}} P(\tau_{\bar{\phi}} > t) \right] dt \leq \sum_{i=0}^\infty [(G_{i+1} - G_i)P(\tau_\phi > G_i) + (\bar{G}_{i+1} - \bar{G}_i)P(\tau_{\bar{\phi}} > \bar{G}_i)] \\
&= \lambda_1 \sum_{i=0}^\infty (\zeta_1^{i+1} - \zeta_1^i) P(\tau_\phi > G_i) + \lambda_2 \sum_{i=0}^\infty (\zeta_2^{i+1} - \zeta_2^i) P(\tau_{\bar{\phi}} > \bar{G}_i) \\
&= \lambda_1 (\zeta_1 - 1) \sum_{i=0}^\infty \zeta_1^i P(\tau_\phi > G_i) + \lambda_2 (\zeta_2 - 1) \sum_{i=0}^\infty \zeta_2^i P(\tau_{\bar{\phi}} > \bar{G}_i) \\
&= \lambda_1 (\zeta_1 - 1) [P(\tau_\phi > 0) + \zeta_1 P(\tau_\phi > G_1) + \zeta_1^2 P(\tau_\phi > G_2) + \zeta_1^3 P(\tau_\phi > G_3) + \dots] \\
&\quad + \lambda_2 (\zeta_2 - 1) [P(\tau_{\bar{\phi}} > 0) + \zeta_2 P(\tau_{\bar{\phi}} > \bar{G}_1) + \zeta_2^2 P(\tau_{\bar{\phi}} > \bar{G}_2) + \zeta_2^3 P(\tau_{\bar{\phi}} > \bar{G}_3) + \dots]
\end{aligned}$$

If $\phi_0, \bar{\phi}_0 < 0$, then

$$\begin{aligned}
E(\tau_\phi) &\leq \lambda_1 (\zeta_1 - 1) \left[P(\tau_\phi > 0) + \zeta_1 P(\tau_\phi > G_1) + \zeta_1^2 \left\{ \int_{\phi_0}^\infty P(\tilde{\phi}_1(G_1) > -x) \gamma_1(dx) \right. \right. \\
&\quad \left. \left. + \int_{-\infty}^{\phi_0} P(\tilde{\phi}_1(G_2) < -x) \gamma_1(dx) \right\} + \zeta_1^3 \left\{ \int_{\phi_0}^\infty P(\tilde{\phi}_1(G_3) > -x) \gamma_1(dx) + \int_{-\infty}^{\phi_0} P(\tilde{\phi}_1(G_2) < -x) \gamma_1(dx) \right\} \right. \\
&\quad \left. + \zeta_1^4 \left\{ \int_{\phi_0}^\infty P(\tilde{\phi}_1(G_3) > -x) \gamma_1(dx) + \int_{-\infty}^{\phi_0} P(\tilde{\phi}_1(G_4) < -x) \gamma_1(dx) \right\} + \dots + \zeta_1^j \left\{ \int_{\phi_0}^\infty P(\tilde{\phi}_1(G_j) > -x) \gamma_1(dx) \right. \right. \\
&\quad \left. \left. + \int_{-\infty}^{\phi_0} P(\tilde{\phi}_1(G_{j-1}) < -x) \gamma_1(dx) \right\} + \zeta_1^{j+1} \left\{ \int_{\phi_0}^\infty P(\tilde{\phi}_1(G_{j+1}) > -x) \gamma_1(dx) + \int_{-\infty}^{\phi_0} P(\tilde{\phi}_1(G_j) < -x) \gamma_1(dx) \right\} \right. \\
&\quad \left. + \zeta_1^{j+2} \left\{ \int_{\phi_0}^\infty P(\phi_1(G_{j+2}) > -x) \gamma_1(dx) + \int_{-\infty}^{\phi_0} P(\tilde{\phi}_1(G_{j+1}) < -x) \gamma_1(dx) \right\} \right. \\
&\quad \left. + \zeta_1^{j+3} \left\{ \int_{\phi_0}^\infty P(\phi_1(G_{j+3}) > -x) \gamma_1(dx) + \int_{-\infty}^{\phi_0} P(\tilde{\phi}_1(G_{j+2}) < -x) \gamma_1(dx) \right\} + \dots \right] \\
&\quad + \lambda_2 (\zeta_2 - 1) \left[P(\tau_{\bar{\phi}} > 0) + \zeta_2 P(\tau_{\bar{\phi}} > \bar{G}_1) + \zeta_2^2 \left\{ \int_{\bar{\phi}_0}^\infty P(\tilde{\phi}_2(\bar{G}_1) > -y) \gamma_2(dy) \right. \right. \\
&\quad \left. \left. + \int_{-\infty}^{\bar{\phi}_0} P(\tilde{\phi}_2(\bar{G}_2) < -y) \gamma_2(dy) \right\} + \zeta_2^3 \left\{ \int_{\bar{\phi}_0}^\infty P(\tilde{\phi}_2(\bar{G}_3) > -y) \gamma_2(dy) + \int_{-\infty}^{\bar{\phi}_0} P(\tilde{\phi}_2(\bar{G}_2) < -y) \gamma_2(dy) \right\} + \dots \right. \\
&\quad \left. + \zeta_2^j \left\{ \int_{\bar{\phi}_0}^\infty P(\tilde{\phi}_2(\bar{G}_j) > -y) \gamma_2(dy) + \int_{-\infty}^{\bar{\phi}_0} P(\tilde{\phi}_2(\bar{G}_{j-1}) < -y) \gamma_2(dy) \right\} + \zeta_2^{j+1} \left\{ \int_{\bar{\phi}_0}^\infty P(\tilde{\phi}_2(\bar{G}_{j+1}) > -y) \gamma_2(dy) \right. \right. \\
&\quad \left. \left. + \int_{-\infty}^{\bar{\phi}_0} P(\tilde{\phi}_2(\bar{G}_j) < -y) \gamma_2(dy) \right\} + \zeta_2^{j+2} \left\{ \int_{\bar{\phi}_0}^\infty P(\phi_2(\bar{G}_{j+2}) > -y) \gamma_2(dy) + \int_{-\infty}^{\bar{\phi}_0} P(\tilde{\phi}_2(\bar{G}_{j+1}) < -y) \gamma_2(dy) \right\} \right. \\
&\quad \left. + \zeta_2^{j+3} \left\{ \int_{\bar{\phi}_0}^\infty P(\phi_2(\bar{G}_{j+3}) > -y) \gamma_2(dy) + \int_{-\infty}^{\bar{\phi}_0} P(\tilde{\phi}_2(\bar{G}_{j+2}) < -y) \gamma_2(dy) \right\} + \dots \right]
\end{aligned}$$

Then

$$\begin{aligned}
 E(\tau_{\hat{\phi}}) &\leq \lambda_1 (\zeta_1 - 1) \left[g_1 + (\zeta_1 + 1) \zeta_1^2 \int_{-\infty}^{\phi_0} P(\tilde{\varphi}_1(G_2) < -x) \gamma_1(dx) \right. \\
 &\quad + (\zeta_1 + 1) \zeta_1^4 \int_{-\infty}^{\phi_0} P(\tilde{\varphi}_1(G_4) < -x) \gamma_1(dx) + \dots + (\zeta_1 + 1) \zeta_1^{j+1} \int_{-\infty}^{\phi_0} P(\tilde{\varphi}_1(G_{j+1}) < -x) \gamma_1(dx) + \dots \\
 &\quad + (\zeta_1 + 1) \zeta_1^3 \int_{\phi_0}^{\infty} P(\tilde{\varphi}_1(G_3) > -x) \gamma_1(dx) + (\zeta_1 + 1) \zeta_1^5 \int_{\phi_0}^{\infty} P(\tilde{\varphi}_1(G_5) > -x) \gamma_1(dx) + \dots \\
 &\quad \left. + (\zeta_1 + 1) \zeta_1^j \int_{\phi_0}^{\infty} P(\tilde{\varphi}_1(G_j) > -x) \gamma_1(dx) + \dots + (\zeta_1 + 1) \zeta_1^{j+2} \int_{\phi_0}^{\infty} P(\tilde{\varphi}_1(G_{j+2}) > -x) \gamma_1(dx) + \dots \right] \\
 &\quad + \lambda_2 (\zeta_2 - 1) \left[\bar{g}_2 + (\zeta_2 + 1) \zeta_2^2 \int_{-\infty}^{\bar{\phi}_0} P(\tilde{\varphi}_2(\bar{G}_2) < -y) \gamma_2(dy) + (\zeta_2 + 1) \zeta_2^4 \int_{-\infty}^{\bar{\phi}_0} P(\tilde{\varphi}_2(\bar{G}_4) < -y) \gamma_2(dy) + \dots \right. \\
 &\quad + (\zeta_2 + 1) \zeta_2^{j+2} \int_{-\infty}^{\bar{\phi}_0} P(\tilde{\varphi}_2(\bar{G}_{j+2}) < -y) \gamma_2(dy) + \dots + (\zeta_2 + 1) \zeta_2^3 \int_{\bar{\phi}_0}^{\infty} P(\tilde{\varphi}_2(\bar{G}_3) > -y) \gamma_2(dy) \\
 &\quad + (\zeta_2 + 1) \zeta_2^5 \int_{\bar{\phi}_0}^{\infty} P(\tilde{\varphi}_2(\bar{G}_5) > -y) \gamma_2(dy) + \dots + (\zeta_2 + 1) \zeta_2^j \int_{\bar{\phi}_0}^{\infty} P(\tilde{\varphi}_2(\bar{G}_j) > -y) \gamma_2(dy) + \dots \\
 &\quad \left. + (\zeta_2 + 1) \zeta_2^{j+2} \int_{\bar{\phi}_0}^{\infty} P(\tilde{\varphi}_2(\bar{G}_{j+2}) > -y) \gamma_2(dy) + \dots \right] \\
 E(\tau_{\hat{\phi}}) &\leq \lambda_1 (\zeta_1 - 1) \left[g_1 (\zeta_1 + 1) \left\{ \int_{-\infty}^{\phi_0} m_1(x) \gamma_1(dx) + \int_{\phi_0}^{\infty} q_1(x) \gamma_1(dx) + \int_{\phi_0}^{\infty} \tilde{q}_1(x) \gamma_1(dx) \right\} \right] \\
 &\quad + \lambda_2 (\zeta_2 - 1) \left[\bar{g}_1 (\zeta_2 + 1) \left\{ \int_{-\infty}^{\bar{\phi}_0} \bar{m}_1(y) \gamma_2(dy) + \int_{\bar{\phi}_0}^{\infty} \bar{q}_1(y) \gamma_2(dy) + \int_{\bar{\phi}_0}^{\infty} \bar{\tilde{q}}_1(y) \gamma_2(dy) \right\} \right],
 \end{aligned}$$

where

$$\begin{aligned}
 g_1 &= P(\tau_{\hat{\phi}} > 0) + \zeta_1 P(\tau_{\hat{\phi}} > G_1) + \zeta_1^2 \int_{\phi_0}^{\infty} P(\tilde{\varphi}_1(G_1) > -x) \gamma_1(dx) \\
 \bar{g}_1 &= P(\tau_{\bar{\phi}} > 0) + \zeta_2 P(\tau_{\bar{\phi}} > \bar{G}_1) + \zeta_2^2 \int_{\bar{\phi}_0}^{\infty} P(\tilde{\varphi}_2(\bar{G}_1) > -y) \gamma_2(dy)
 \end{aligned}$$

$$m_1(x) = \zeta_1^2 P(\tilde{\varphi}_1(G_{2i}) < -x),$$

$$\bar{m}_1(y) = \zeta_2^2 P(\tilde{\varphi}_2(\bar{G}_{2i}) < -y),$$

$$q_1(x) = \sum_{j=1}^{(j-1)/2} \zeta_1^{2i+1} P[\tilde{\varphi}_1(G_{2j+1}) > -x]$$

$$\bar{q}_1(y) = \sum_{i=1}^{(j-1)/2} \zeta_1^{2i+1} P[\tilde{\varphi}_2(\bar{G}_{2i+1}) > -y]$$

$$\tilde{q}_1(x) = \sum_{i=(j+1)/2}^{\infty} \zeta_2^{2i+1} P[\varphi_1(G_{2i+1}) > -x]$$

and

$$\bar{\tilde{q}}_1(y) = \sum_{i=(j+1)/2}^{\infty} \zeta_2^{2i+1} P[\varphi_2(\bar{G}_{2i+1}) > -y]$$

the other cases can be proved by similar way

Lemma 2.2. Let $a_n \geq 0$ for $n \geq 0$, and $a_{n+1} \leq a_n$, $\{d_n\}$ $n \geq 0$ be a strictly increasing sequence of integers with $d_0 = 0$ then for any $k \leq 0$,

$$\sum_{n=k}^{\infty} [d_{n+1} - d_n] a_{d_{n+1}} \leq \sum_{n=d_k}^{\infty} a_n \leq \sum_{n=k}^{\infty} [d_{n+1} - d_n] a_{d_n} \quad (\text{see EL-Rayes et al. [5]})$$

Theorem 2.2. The chosen search plan satisfies

$$\begin{aligned} \tilde{q}_2(x) &\leq L(|x|), \bar{q}_2(y) \leq L(|y|) \\ q_2(x) &\leq L'(|x|), \bar{q}_2(y) \leq L'(|y|), \phi_0, \bar{\phi}_0 > 0 \\ m_1(x) &\leq L''(|x|), \bar{m}_1(y) \leq L''(|y|) \\ \tilde{q}_1(x) &\leq L'''(|x|), \bar{q}_1(y) \leq L'''(|y|), \phi_0, \bar{\phi}_0 < 0 \end{aligned}$$

where $L(|x|), L'(|x|), L''(|x|), L'''(|x|), L(y), L'(|y|), L''(|y|), L'''(|y|)$ are linear functions.

Proof. We shall prove the theorem for $\tilde{q}_2(x)$ and $\bar{q}_2(y)$, since the other cases can be proved by a similar way.

$$\tilde{q}_2(x) = \sum_{i=1}^{\infty} \zeta_1^{2i+1} P[\varphi(G_{2i+1}) > -x], \quad \phi_0 > 0,$$

$$\bar{q}_2(y) = \sum_{i=1}^{\infty} \zeta_2^{2i+1} P[\varphi(\bar{G}_{2i+1}) > -y], \quad \bar{\phi}_0 > 0$$

(i) if $x \geq \phi_0$

$$\tilde{q}_2(x) = \tilde{q}_2(\phi_0) + \sum_{i=1}^{\infty} \zeta_1^{2i+1} P[-x < \varphi(G_{2i+1}) \leq -\phi_0]$$

and $y \geq \bar{\phi}_0$

$$\bar{q}_2(y) = \bar{q}_2(\bar{\phi}_0) + \sum_{i=1}^{\infty} \zeta_2^{2i+1} P[-y < \varphi(\bar{G}_{2i+1}) \leq -\bar{\phi}_0]$$

(ii) if $0 \leq x \leq \phi_0$

$$\tilde{q}_2(x) = \tilde{q}_2(0) + \sum_{i=1}^{\infty} \zeta_1^{2i+1} P[-x < \varphi(G_{2i+1}) \leq 0]$$

and $0 \leq y \leq \bar{\phi}_0$

$$\bar{q}_2(y) = \bar{q}_2(0) + \sum_{i=1}^{\infty} \zeta_2^{2i+1} P[-y < \varphi(\bar{G}_{2i+1}) \leq 0]$$

(iii) if $x \leq 0$

$$\tilde{q}_2(x) = \tilde{q}_2(0) - \sum_{i=1}^{\infty} \zeta_1^{2i+1} P[0 < \varphi(G_{2i+1}) \leq -x]$$

and $y \leq 0$

$$\bar{q}_2(y) = \bar{q}_2(0) - \sum_{i=1}^{\infty} \zeta_2^{2i+1} P[0 < \varphi(\bar{G}_{2i+1}) \leq -y]$$

we have $x \geq \phi_0$ and $y \geq \bar{\phi}_0$ in (i)

$$\tilde{q}_2(x) \leq \tilde{q}_2(0) \quad \text{and} \quad \bar{q}_2(y) \leq \bar{q}_2(0)$$

$$\tilde{q}_2(x) = \tilde{q}_2(\phi_0) + \sum_{i=1}^{\infty} \zeta_1^{2i+1} P[-x < \varphi(G_{2i+1}) \leq -\phi_0]$$

$$\bar{q}_2(y) = \bar{q}_2(\bar{\phi}_0) + \sum_{i=1}^{\infty} \zeta_2^{2i+1} P[-y < \varphi(\bar{G}_{2i+1}) \leq -\bar{\phi}_0]$$

but,

$$\tilde{q}_2(x) = \tilde{q}_2(0) + \sum_{i=1}^{\infty} \zeta_1^{2i+1} P[-x < \varphi(G_{2i+1}) < 0]$$

$$\bar{q}_2(y) = \bar{q}_2(0) + \sum_{i=1}^{\infty} \zeta_2^{2i+1} P[-y < \varphi(\bar{G}_{2i+1}) < 0]$$

from [1]

$$\tilde{q}_2(x) = \sum_{i=1}^{\infty} \zeta_1^{2i+1} P[\varphi(G_{2i+1}) > 0] \leq \sum_{i=1}^{\infty} \zeta_1^{2i+1} \varepsilon^{G_{2i+1}} \leq \frac{\zeta_1^3}{\zeta_1^2 - 1}$$

$$\bar{q}_2(y) = \sum_{i=1}^{\infty} \zeta_2^{2i+1} P[\varphi(\bar{G}_{2i+1}) > 0] \leq \sum_{i=1}^{\infty} \zeta_2^{2i+1} \varepsilon^{\bar{G}_{2i+1}} \leq \frac{\zeta_2^3}{\zeta_2^2 - 1}, \quad 0 < \varepsilon < 1$$

we define the following

1) $d_n = G_{2n+1} = (\zeta_1^{2n+1} - 1)$

$\bar{d}_n = \bar{G}_{2n+1} = (\zeta_2^{2n+1} - 1)$

2) $k(n) = \varphi(n) = \sum_{i=1}^n W_i$, where $\{W_i\}_{i \geq 0}$ is sequence of (i. i. d. r. v.)

$\bar{k}(n) = \bar{\varphi}(n) = \sum_{i=1}^n \bar{W}_i$ where $\{\bar{W}_i\}_{i \geq 0}$ is sequence of (i. i. d. r. v.)

3) $a(n) = P[-x < k(n) \leq 0] = \sum_{j=0}^{\lfloor x \rfloor} P[-(j+1) < k(n) \leq -j]$

$\bar{a}(n) = P[-y < \bar{k}(n) \leq 0] = \sum_{j=0}^{\lfloor y \rfloor} P[-(j+1) < \bar{k}(n) \leq -j]$

4) n_1 is an integer such that $d_{n_1} = b_1|x| + b_2$; n_2 is an integer such that $\bar{d}_{n_2} = \bar{b}_1|x| + \bar{b}_2$

5) $U(j, j+1) = \sum_{n=0}^{\infty} t_2 P[-(j+1) < k(n) \leq -j]$

$\bar{U}(j, j+1) = \sum_{n=0}^{\infty} \bar{t}_2 P[-(j+1) < \bar{k}(n) \leq -j]$

6) $\alpha_1 = \zeta_1^2 / (\zeta_1^2 - 1)$

$\alpha_2 = \zeta_2^2 / (\zeta_2^2 - 1)$

If $n > d_n$ and \bar{d}_n then $a_{(n)}$, $\bar{a}_{(n)}$ are non-increasing, see [15], and we can apply Lemma 2.2 in suitable steps.

$$\begin{aligned} \tilde{q}_2(x) - \tilde{q}_2(0) &= \sum_{i=1}^{\infty} \zeta_1^{2i+1} P[-x < \varphi(G_{2i+1}) \leq -0] = \sum_{n=1}^{n_1} \zeta_1^{2n+1} a(d_n) + \sum_{n=n_1+1}^{\infty} \zeta_1^{2n+1} a(d_n) \\ &\leq \sum_{n=1}^{n_1} \zeta_1^{2n+1} + \alpha_1 \sum_{n=n_1+1}^{\infty} (d_n - d_{n-1}) a(d_n) \\ &\leq \sum_{n=1}^{n_1} \zeta_1^{2n+1} + \alpha_1 \sum_{n=d_{n_1}}^{\infty} a(n) \\ &\leq \sum_{n=1}^{n_1} \zeta_1^{2n+1} + \alpha_1 \sum_{j=0}^{\lfloor x \rfloor / 2} U(j, j+1) \end{aligned}$$

and

$$\begin{aligned}
 \bar{q}_2(y) - \bar{q}_2(0) &= \sum_{i=1}^{\infty} \zeta_2^{2i+1} P[-y < \varphi(\bar{G}_{2i+1}) \leq 0] = \sum_{n=1}^{n_2} \zeta_2^{2n+1} a(\bar{d}_n) + \sum_{n=n_2+1}^{\infty} \zeta_2^{2n+1} \bar{a}(\bar{d}_n) \\
 &\leq \sum_{n=1}^{n_2} \zeta_2^{2n+1} + \alpha_2 \sum_{n=n_2+1}^{\infty} \zeta_2^{2n+1} \bar{a}(\bar{d}_n) \\
 &\leq \sum_{n=1}^{n_2} \zeta_2^{2n+1} + \alpha_2 \sum_{n=n_2+1}^{\infty} (\bar{d}_n - \bar{d}_{n-1}) \bar{a}(n) \\
 &\leq \sum_{n=1}^{n_2} \zeta_2^{2n+1} + \alpha_2 \sum_{n=d_{n_2}}^{\infty} \bar{a}(n) \\
 &\leq \sum_{n=1}^{n_2} \zeta_2^{2n+1} + \alpha_2 \sum_{j=0}^{|y|/2} \bar{U}(j, j+1)
 \end{aligned}$$

Since $U(j, j+1)$, $\bar{U}(j, j+1)$ satisfies the conditions of renewal theorem (see [15]) hence $U(j, j+1)$, $\bar{U}(j, j+1)$ is bounded for all j by a constant, so

$$\begin{aligned}
 \tilde{q}_2(x) &\leq \tilde{q}_2(\phi_0) + M_1 + M_2 |x| = L(|x|) \\
 \bar{q}_2(y) &\leq \bar{q}_2 L(\bar{\phi}_0) + \bar{M}_1 + \bar{M}_2 |y| = L(|y|)
 \end{aligned}$$

Theorem 2.3. If there exist a finite search plan $(\phi, \bar{\phi}) \in \hat{\Phi}$ then $E|Z_0|$ is finite.

Proof. If $E\tau_{\hat{\phi}} < \infty$, then $P(\tau_{\hat{\phi}} < \infty) = 1$ or $p(\tau_{\bar{\phi}} < \infty) = 1$

then $P(\tau_{\hat{\phi}} < \infty) + (\tau_{\bar{\phi}} < \infty) = 1$, that is $Z_0 = \phi(\tau_{\hat{\phi}}) - S(\tau_{\bar{\phi}})$ with probability 1, so

$$|Z_0| = \phi(\tau_{\hat{\phi}}) + |S(\tau_{\bar{\phi}})| \leq \tau_{\hat{\phi}} + |S(\tau_{\bar{\phi}})|$$

Hence $E|Z_0| \leq E\tau_{\hat{\phi}} + E|S(\tau_{\bar{\phi}})|$, but $|S(\tau_{\bar{\phi}})| \leq \tau_{\bar{\phi}}$ then $E|S(\tau_{\bar{\phi}})| \leq E\tau_{\bar{\phi}}$ and $E|Z_0| < \infty$.

Remark A direct consequence of theorems 1, 2 and 3 in Section 2 is the existence of a finite search plan $\hat{\phi} \in \hat{\Phi}(t)$ if and only if $E|Z_0| \leq \infty$, and then the set of search plans, which defined in Section 2, satisfies the conditions of Theorem 1 if the expectation value of initial position of the lost target is finite.

3. Existence of an Optimal Path

Definition. Let $\hat{\phi}_n \in \hat{\Phi}(t)$ be a sequence of search plans, we say that $\hat{\phi}_n$ converges to $\hat{\phi}$ as n tends to ∞ if for any $t \in I^+ \rightarrow \hat{\phi}_n(t)$ converges to $\hat{\phi}(t)$ on every compact subset.

Theorem 3.1. Let for any $t \in \mathfrak{R}^+$, $S(t)$ be a process with continuous sample paths. The mapping $(\phi, \bar{\phi}) \rightarrow E(\tau_{\bar{\phi}}) \in \mathfrak{R}^+$ is lower semi-continuous on $\hat{\Phi}(t)$.

Proof. Let ξ be a sample point on L_1 corresponding to the sample path $\xi(t)$ of $X_o + S(t)$ and $\bar{\xi}$ be a sample point on L_2 corresponding to the sample path $\bar{\xi}(t)$ of $Y_o + S(t)$. Let $\{\phi_n\}_{n \geq 1}$ be a sequence of search paths which converges to $\phi \in \Phi_{V_1}(t)$ and $\{\bar{\phi}_n\}_{n \geq 1}$ converges to $\bar{\phi} \in \Phi_{V_2}(t)$. Given $t \in \mathfrak{R}^+$, we define for any $n \geq 1$

$$B_n(t) = \left\{ \xi : \min_{0 \leq x \leq t} |\xi(x) - \phi_n(x)| > 0 \right\},$$

$$B(t) = \left\{ \xi : \min_{0 \leq x \leq t} |\xi(x) - \phi(x)| > 0 \right\},$$

$$\bar{B}_n(t) = \left\{ \bar{\xi} : \min_{0 \leq y \leq t} |\bar{\xi}(y) - \bar{\phi}_n(y)| > 0 \right\}$$

$$\text{and } \bar{B}(t) = \left\{ \bar{\xi} : \min_{0 \leq y \leq t} |\bar{\xi}(y) - \bar{\phi}(y)| > 0 \right\}$$

Let $\xi \in B(t)$, $\bar{\xi} \in \bar{B}(t)$ and since $\{\phi_n\}_{n \geq 1}$ converges uniformly on $[0, t]$ to ϕ , then there exists an integer $n(\xi)$ such that for all $n \geq n(\xi)$ and for any $0 \leq x \leq t$

$$|\phi_n(x) - \phi(x)| < \varepsilon = 0.5 \min_{0 \leq x \leq t} |\xi(x) - \phi(x)|$$

and $|\xi(x) - \phi(x)| \leq |\xi(x) - \phi_n(x)| + |\phi(x) - \phi_n(x)|$,

then $|\xi(x) - \phi_n(x)| \geq |\xi(x) - \phi(x)| - |\phi(x) - \phi_n(x)| \geq 2\varepsilon - \varepsilon = \varepsilon > 0$.

Hence $\xi \in B_n(t)$ for all $n > n(\xi)$ and so $B(t) \subset \liminf_{n \rightarrow \infty} B_n(t)$,

$$\int_0^\infty P(B(t)) dt \leq \int_0^\infty P\left(\liminf_{n \rightarrow \infty} B_n(t)\right) dt \leq \int_0^\infty \liminf_{n \rightarrow \infty} P(B_n(t)) dt$$

by the same way we can get

$$\int_0^\infty P(\bar{B}(t)) dt \leq \int_0^\infty P\left[\liminf_{n \rightarrow \infty} \bar{B}_n(t)\right] dt \leq \int_0^\infty \liminf_{n \rightarrow \infty} P(\bar{B}_n(t)) dt$$

since the sample paths are continuous then

$$B_n(t) = \{\tau_{\phi_n} > t\}, \quad B(t) = \{\tau_\phi > t\}, \quad \bar{B}_n(t) = \{\tau_{\bar{\phi}_n} > t\} \quad \text{and} \quad B = \{\tau_\phi > t\}$$

we get $\int_0^\infty p(\tau_\phi > t) dt \leq \int_0^\infty \liminf_{n \rightarrow \infty} p(\tau_{\phi_n} > t) dt$ and $\int_0^\infty p(\tau_{\bar{\phi}} > t) dt \leq \int_0^\infty \liminf_{n \rightarrow \infty} p(\tau_{\bar{\phi}_n} > t) dt$, we obtain

$$\int_0^\infty [p(\tau_{\bar{\phi}} > t) + p(\tau_\phi > t)] dt \leq \int_0^\infty \liminf_{n \rightarrow \infty} [p(\tau_{\bar{\phi}_n} > t) + p(\tau_{\phi_n} > t)] dt$$

then

$$\int_0^\infty p(\tau > t) dt \leq \int_0^\infty \liminf_{n \rightarrow \infty} p(\tau_{\phi_n} > t) dt,$$

where $\widehat{\Phi}_n = (\phi_n, \bar{\phi}_n)$

hence $E(\tau_\Phi) \leq E\left(\liminf_{n \rightarrow \infty} \tau_{\widehat{\Phi}_n}\right)$. By Fatou lemma, we get

$$E(\tau_\Phi) \leq E\left(\liminf_{n \rightarrow \infty} \tau_{\widehat{\Phi}_n}\right) \leq \liminf_{n \rightarrow \infty} E(\tau_{\widehat{\Phi}_n}).$$

Since Φ is sequentially compact (see [2]), then by the same way $\bar{\Phi}$ is sequentially compact. It is known that a lower semi-continuous function over a sequentially compact space attains its minimum.

4. Conclusion

We consider, here, the search for a lost target on one of two disjoint lines, where the target moves randomly according to continuous stochastic process which satisfies some conditions. Theorems conclude that there exists a finite search plan if and only if the expectation value of the initial position of the target is finite. Existence of optimal search plan is proved.

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