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New Fixed Point Theorems of Mixed Monotone Operators

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ABSTRACT

Mixed monotone operator is an important nonlinear operator. It exists extensively in the research of nonlinear differential and integral equations. Generally, the research of mixed monotone operators in partially ordered Banach spaces requires compactness, continuity or concavity-convexity of the operators. In this paper, without any compact and continuous assumption, we obtain some new existence and uniqueness theorems of positive fixed point of e-concave-convex mixed monotone operators in Banach spaces partially ordered by a cone, which extends the existing corresponding results.

KEYWORDS

Mixed Monotone Operators; Cone; Partial Order

1. Introduction

Mixed monotone operators were introduced by Dajun Guo and V. Lakshmikantham in [1] in 1987. Thereafter, many authors have investigated these kinds of operators in Banach spaces and obtained a lot of interesting and important results. They are used extensively in nonlinear differential and integral equations. In this paper, without any compact and continuous assumption, we obtained some new existence and uniqueness theorems of positive fixed point of e-concave-convex mixed monotone operators in Banach spaces partially ordered by a cone.

Let the real Banach space E be partially ordered by a cone P of E, i.e., $x \leq y$ iff $y - x \in P$. $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y, i.e., $u_i, v_i (i = 1, 2) \in P$, $u_1 \leq u_2, v_1 \geq v_2$ implies. $A(u_1, v_1) \leq A(u_2, v_2)$. Element $x \in P$ is called a fixed point of A iff $A(x, x) = x$.

Recall that cone P is said to be solid if the interior $\overset{\circ}{P}$ is nonempty and we denote $x \gg \theta$ if $x \in \overset{\circ}{P}$. P is normal if there exists a positive constant N, such that $\theta \leq x \leq y$ implies $\|x\| \leq N \|y\|$, N is called the normal constant of P.

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$, such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $e > \theta$, we denote by P_e the set $P_e = \{x \in E \mid x \sim e\}$. It is easy to see that $P_e \subset P$ is convex and $\lambda P_e = P_e$ for all $\lambda > 0$. If $P \neq \varnothing$ and $e \in P$, it is clear that $P_e = P$.

All the concepts discussed above can be found in [2-4]. For more facts about mixed monotone operators and other related concepts, the reader could refer to [5-7] and some of the reference therein.

2. Main Results

In this section, we present our main results. To begin with, we give the definition of e-concave-convex operators.

Definition 2.1. We say an operator $A : P_e \times P_e \to P_e$ is an e-concave-convex operators if there exist one positive function $\eta(t, u, v)$ such that

$$A\left(tu,\frac{1}{t}v\right)\geq t\left(1+\eta\left(t,u,v\right)\right)A\left(u,v\right),\forall u,v\in P_e,0<t<1.\tag{2.1}$$

Remark 2.1. (2.1) implies that

$$A\left(su,\frac{1}{s}v\right)\leq s\left(1+\eta\left(\frac{1}{s},su,\frac{1}{s}v\right)\right)^{-1}A\left(u,v\right),\forall u,v\in P_e,s>1.\tag{2.2}$$

Theorem 2.1. Let P be a normal cone of E, and let $A:P_e\times P_e\to P_e$ be a mixed monotone and e-concave-convex operator. In addition, suppose that there exist $u_0,v_0\in P_e,u_0\leq v_0$ (Since $u_0,v_0\in P_e$, we can choose a sufficiently small $\varepsilon\in(0,1)$ such that $u_0\geq\varepsilon v_0$) such that

$$(H_1)u_0\leq A\left(u_0,v_0\right),A\left(v_0,u_0\right)\leq v_0;$$

$$(H_2)\forall t\in(0,1),\overline{\lim_{n\to+\infty}}n\eta\left(t,v_n,u_n\right)\geq\frac{1}{\varepsilon}-1,$$

hold, where $u_n=A\left(u_{n-1},v_{n-1}\right),v_n=A\left(v_{n-1},u_{n-1}\right),n=1,2,\cdots$. Then A has exactly one fixed pint x^* in P_e. Moreover, constructing successively the sequence

$$x_n=A\left(x_{n-1},y_{n-1}\right),y_n=A\left(y_{n-1},x_{n-1}\right),n=1,2,\cdots,$$

for any initial $x_0,y_0\in P_e$, we have

$$\left\|x_n-x^*\right\|\to 0,\left\|y_n-x^*\right\|\to 0,n\to\infty.\tag{2.3}$$

Proof. We divide the proof into 3 steps.

Step 1. We prove A has a fixed point in P_e.

Construct successively the sequences

$$u_n=A\left(u_{n-1},v_{n-1}\right),v_n=A\left(v_{n-1},u_{n-1}\right),n=1,2,\cdots.$$

It follows from (H_1) and the mixed monotonicity of A that

$$u_0\leq u_1\leq\cdots\leq u_n\leq\cdots v_n\leq\cdots\leq v_1\leq v_0.\tag{2.4}$$

Let

$$r_n=\sup\left\{r>0\big|u_n\geq rv_n\right\},n=0,1,2,\cdots.$$

Thus we have $u_n\geq r_nv_n,n=0,1,2,\cdots$, and then

$$u_{n+1}\geq u_n\geq r_nv_n\geq r_nv_{n+1},n=1,2,\cdots.$$

Therefore, $r_{n+1}\geq r_n$, i.e., $\{r_n\}$ is increasing with $\{r_n\}\subset(0,1]$.

Suppose $r_n\to r^*$ as $n\to\infty$. Then $r^*=1$. Indeed, suppose to the contrary that $r_n\leq r^*<1$. By (H_1), (2.1) and the mixed monotonicity of A, we have

$$u_{n+1}=A\left(u_n,v_n\right)\geq A\left(r_nv_n,\frac{1}{r_n}u_n\right)=A\left(\frac{r_n}{r^*}r^*v_n,\frac{r^*}{r_n}\frac{1}{r^*}u_n\right)\geq\frac{r_n}{r^*}A\left(r^*v_n,\frac{1}{r^*}u_n\right)$$

$$\geq\frac{r_n}{r^*}r^*\left(1+\eta\left(r^*,v_n,u_n\right)\right)A\left(v_n,u_n\right)=r_n\left(1+\eta\left(r^*,v_n,u_n\right)\right)v_{n+1}$$

Therefore

$$r_{n+1}\geq r_n\left(1+\eta\left(r^*,v_n,u_n\right)\right),$$

$$r_{n+1}\geq r_0\left(1+\eta\left(r^*,v_0,u_0\right)\right)\left(1+\eta\left(r^*,v_1,u_1\right)\right)\cdots\left(1+\eta\left(r^*,v_n,u_n\right)\right),n=0,1,\cdots\tag{2.5}$$

By (2.1) and the mixed monotonicity of A, we know that $0<\eta\left(t,u,v\right)\leq\frac{1-t}{t},\forall u,v\in P_e$, and thus $\overline{\lim_{n\to+\infty}}\eta\left(r^*,v_n,u_n\right)$ exists. We distinguish two cases.

Case 1: $\overline{\lim\limits_{n\to+\infty}}\eta\left(r^*,v_n,u_n\right)=\eta_0\neq 0$. In this case we know that $0<\eta_0\leq\dfrac{1-r^*}{r^*}$. So, we have

$r^*=\overline{\lim\limits_{n\to+\infty}}r_n\geq r^*\left(1+\eta_0\right)$, which is a contradiction.

Case 2: $\overline{\lim\limits_{n\to+\infty}}\eta\left(r^*,v_n,u_n\right)=0$. In this case, it is easy to see that $\underline{\lim\limits_{n\to+\infty}}\eta\left(r^*,v_n,u_n\right)=0$. For convenience's sake,

let $\eta_n=\eta\left(r^*,v_n,u_n\right)$. Since $\underline{\lim\limits_{n\to+\infty}}\eta_n=0$, there is a nonincreasing subsequence $\left\{\eta_{nk}\right\}$ of $\left\{\eta_n\right\}$ such that

$\lim\limits_{n\to+\infty}\eta_{nk}=0$. Without loss of generality, we may still use $\left\{\eta_n\right\}$ to stand for $\left\{\eta_{nk}\right\}$. From (2.5), we obtain

$$r_{n+1}\geq r_0\left(1+\eta_n\right)^n\geq r_0\left(1+n\eta_n\right),n=0,1,\cdots.$$

Hence

$$1>r^*\geq r_0\left(1+\overline{\lim\limits_{n\to+\infty}}n\eta_n\right)\geq 1,$$

which is also a contradiction. Thus $r^*=1$.

For any natural number p we have

$$\theta\leq u_{n+p}-u_n\leq v_n-u_n\leq\left(1-r_n\right)v_0,$$
$$\theta\leq v_{n+p}-v_n\leq v_n-u_n\leq\left(1-r_n\right)v_0.$$

Since P is normal, we have $\left\|u_{n+p}-u_n\right\|\leq N\left(1-r_n\right)\left\|v_0\right\|$, $\left\|v_{n+p}-v_n\right\|\leq N\left(1-r_n\right)\left\|v_0\right\|$. Here N is the normality constant.

So $\left\{u_n\right\}$ and $\left\{v_n\right\}$ are Cauchy sequences. Because E is complete, there exist u^*,v^* such that $u_n\to u^*,v_n\to v^*\left(n\to+\infty\right)$, By (2.4) we know that $u_n\leq u^*\leq v^*\leq v_n$ and

$$\theta\leq v^*-u^*\leq\left(1-r_n\right)v_0.$$

Further

$$\left\|v^*-u^*\right\|\leq N\left(1-r_n\right)\left\|v_0\right\|,$$

And thus $v^*=u^*$. Let $x^*:=v^*=u^*$, we obtain

$$u_{n+1}=A\left(u_n,v_n\right)\leq A\left(x^*,x^*\right)\leq A\left(v_n,u_n\right)=v_{n+1}.$$

Let $n\to+\infty$, we get $x^*=A\left(x^*,x^*\right)$. That is, x^* is a fixed point of A in P_e.

Step 2. We prove that x^* is the unique fixed point of A in P_e.

In fact, suppose $\overline{x}$ is another fixed point of A in P_e. Since $\overline{x},x^*\in P_e$, there exist positive numbers α,β such that $x^*\geq\alpha\overline{x},\overline{x}\geq\beta x^*$. Let

$$\gamma_0=\sup\left\{\gamma>0\,\middle|\,\gamma\overline{x}\leq x^*\leq\frac{1}{\gamma}\overline{x}\right\}. \tag{2.6}$$

Evidently, $\gamma_0\in\left(0,1\right]$. We now prove $\gamma_0=1$. If otherwise, $0<\gamma_0<1$. From (2.1), we obtain

$$x^*=A\left(x^*,x^*\right)\geq A\left(\gamma_0\overline{x},\frac{1}{\gamma_0}\overline{x}\right)\geq\gamma_0\left(1+\eta\left(\gamma_0,\overline{x},\overline{x}\right)\right)A\left(\overline{x},\overline{x}\right)=\gamma_0\left(1+\eta\left(\gamma_0,\overline{x},\overline{x}\right)\right)\overline{x};$$

$$\overline{x}=A\left(\overline{x},\overline{x}\right)\geq A\left(\gamma_0 x^*,\frac{1}{\gamma_0}x^*\right)\geq\gamma_0\left(1+\eta\left(\gamma_0,x^*,x^*\right)\right)A\left(x^*,x^*\right)=\gamma_0\left(1+\eta\left(\gamma_0,x^*,x^*\right)\right)x^*.$$

Which contradicts the definition of γ_0. Hence $\gamma_0=1$, thus $x^*=\overline{x}$. Therefore A has a unique fixed point x^* in P_e.

Step 3. We prove (2.3).

For any $x_0,y_0\in P_e$, we can choose a small number $t_0\in\left(0,1\right)$ such that

$$t_0 x^*\leq x_0\leq\frac{1}{t_0}x^*,t_0 x^*\leq y_0\leq\frac{1}{t_0}x^*. \tag{2.7}$$

Let

$$a_0 = t_0 x^*, b_0 = \frac{1}{t_0} x^*, a_n = A(a_{n-1}, b_{n-1}), b_n = A(b_{n-1}, a_{n-1}). \tag{2.8}$$

Using (2.7), (2.8) and the mixed monotonicity of A, we have

$$a_n \le x^* \le b_n, a_n \le x_n \le b_n, a_n \le y_n \le b_n, n = 1, 2, \cdots$$
$$a_0 \le a_1 \le \cdots \le a_n \le \cdots \le b_n \le \cdots \le b_1 \le b_0. \tag{2.9}$$

Let

$$\lambda_n = \sup\{t > 0 | tx^* \le a_n, tb_n \le x^*\}, n = 1, 2, \cdots. \tag{2.10}$$

From (2.8) and (2.9), we obtain

$$\lambda_n x^* \le a_n, \lambda_n b_n \le x^*, n = 1, 2, \cdots,$$
$$0 \le t_0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_n \le \cdots \le 1. \tag{2.11}$$

In what follows, we will prove that $\lim\limits_{n\to\infty} \lambda_n = 1$. If not, that is, $\lim\limits_{n\to\infty} \lambda_n = \lambda^* < 1$, then from (2.8) and (2.11), we have

$$a_{n+1} = A(a_n, b_n) \ge A\left(\lambda_n x^*, \frac{1}{\lambda_n} x^*\right) = A\left(\frac{\lambda_n}{\lambda^*} \lambda^* x^*, \frac{\lambda^*}{\lambda_n} \frac{1}{\lambda^*} x^*\right)$$
$$\ge \frac{\lambda_n}{\lambda^*} A\left(\lambda^* x^*, \frac{1}{\lambda^*} x^*\right) \ge \lambda_n \left(1 + \eta(\lambda^*, x^*, x^*)\right) x^*; \tag{2.12}$$

$$b_{n+1} = A(b_n, a_n) \le A\left(\frac{1}{\lambda_n} x^*, \lambda_n x^*\right) = A\left(\frac{\lambda^*}{\lambda_n} \frac{1}{\lambda^*} x^*, \frac{\lambda_n}{\lambda^*} \lambda^* x^*\right)$$
$$\le \frac{\lambda^*}{\lambda_n} A\left(\frac{1}{\lambda^*} x^*, \lambda^* x^*\right) \le \frac{1}{\lambda_n}\left(1 + \eta\left(\lambda^*, \frac{1}{\lambda^*} x^*, \lambda^* x^*\right)\right)^{-1} x^*. \tag{2.13}$$

Let $d = \min\left\{\eta(\lambda^*, x^*, x^*), \eta\left(\lambda^*, \frac{1}{\lambda^*} x^*, \lambda^* x^*\right)\right\}$. It follows from (2.11), (2.12) and (2.13) that

$\lambda_{n+1} \ge \lambda_n (1+d) \ge \cdots \ge \lambda_1 (1+d)^n$, which is a contradiction. That is $\lim\limits_{n\to\infty} \lambda_n = 1$.

From (2.9) and (2.11), we have

$$\lambda_n x^* \le x_n \le \frac{1}{\lambda_n} x^*, \lambda_n x^* \le y_n \le \frac{1}{\lambda_n} x^*, n = 1, 2, \cdots.$$

Thus,

$$(\lambda_n - 1) x^* \le x_n - x^* \le \left(\frac{1}{\lambda_n} - 1\right) x^*, (\lambda_n - 1) x^* \le y_n - x^* \le \left(\frac{1}{\lambda_n} - 1\right) x^*, n = 1, 2, \cdots.$$

By using the normality of P, we know that (2.2) holds.

3. Concerned Remarks and Corollaries

Using Theorem 2.1, we have the following corollaries.

Corollary 3.1. Let P be a normal cone of E, and let $A: P_e \times P_e \to P_e$ be a mixed monotone and e-concave-convex operator. In addition, suppose that there exist $u_0, v_0 \in P_e, u_0 \le v_0$ (Since $u_0, v_0 \in P_e$, we can choose a sufficiently small $\varepsilon \in (0,1)$, such that $u_0 \ge \varepsilon v_0$), such that

$$(H_1) u_0 \le A(u_0, v_0), A(v_0, u_0) \le v_0;$$
$$(H_2') \forall t \in (0,1), \varlimsup_{n\to+\infty} n\eta(t, v_n, u_n) = +\infty,$$

hold, where $u_n = A(u_{n-1}, v_{n-1}), v_n = A(v_{n-1}, u_{n-1}), n = 1, 2, \cdots$. Then A has exactly one fixed pint x^* in P_e. Moreover, constructing successively the sequence

$$x_n = A(x_{n-1}, y_{n-1}), y_n = A(y_{n-1}, x_{n-1}), n = 1, 2, \cdots,$$

for any initial $x_0, y_0 \in P_e$, we have

$$\left\| x_n - x^* \right\| \to 0, \left\| y_n - x^* \right\| \to 0, n \to \infty.$$

Corollary 3.2. Let P be a normal cone of E, and let $A : P_e \times P_e \to P_e$ be a mixed monotone and e-concave-convex operator. In addition, suppose that

(H_3) there exist $u_0, v_0 \in P_e$, such that $u_0 \leq A(u_0, v_0), A(v_0, u_0) \leq v_0$;
(H_4) for all $t \in (0,1)$, there exist $x_t, y_t \in [u_0, v_0]$, such that $\eta(t, x, y) \geq \eta(t, x_t, y_t), \quad \forall x, y \in P_e$ hold.
Then A has exactly one fixed pint x^* in P_e. Moreover, constructing successively the sequence

$$x_n = A(x_{n-1}, y_{n-1}), y_n = A(y_{n-1}, x_{n-1}), n = 1, 2, \cdots,$$

for any initial $x_0, y_0 \in P_e$, we have

$$\left\| x_n - x^* \right\| \to 0, \left\| y_n - x^* \right\| \to 0, n \to \infty.$$

Corollary 3.3. Let P be a normal cone of E, and let $A : P_e \times P_e \to P_e$ be a mixed monotone and e-concave-convex operator. In addition, suppose that $\forall t \in (0,1)$, $\eta(t, x, y)$ are monotone on x and y. Then a necessary and sufficient condition for A to have exactly one fixed pint x^* in P_e is that (H_3) holds. Moreover, constructing successively the sequence

$$x_n = A(x_{n-1}, y_{n-1}), y_n = A(y_{n-1}, x_{n-1}), n = 1, 2, \cdots,$$

for any initial $x_0, y_0 \in P_e$, we have

$$\left\| x_n - x^* \right\| \to 0, \left\| y_n - x^* \right\| \to 0, n \to \infty.$$

Proof. Corollary 3.1 ensures the sufficiency of Corollary 2.3, so we have only to prove the necessity of Corollary 2.3.

Suppose that x^* is the unique fixed point of A in P_e. For any $t_0 \in (0,1)$, let $u_0 = t_0 x^*$, $v_0 = \dfrac{1}{t_0} x^*$. It follows from (1.1), (1.2) and the mixed monotonicity of A that

$$A(u_0, v_0) = A\left(t_0 x^*, \frac{1}{t_0} x^*\right) \geq t_0 \left(1 + \eta\left(t_0, x^*, x^*\right)\right) A\left(x^*, x^*\right) \geq t_0 x^* = u_0;$$

$$A(v_0, u_0) = A\left(\frac{1}{t_0} x^*, t_0 x^*\right) \leq \frac{1}{t_0} \left(1 + \eta\left(t_0, \frac{1}{t_0} x^*, t_0 x^*\right)\right)^{-1} A\left(x^*, x^*\right) \leq \frac{1}{t_0} x^* = v_0.$$

Therefore, (H_3) holds.

Corollary 3.4. Let P be a normal cone of E, and let $A : P_e \times P_e \to P_e$ be a mixed monotone operator. In addition, suppose that $\forall t \in (0,1)$, there exists $\eta(t) > 0$ such that

$$A\left(tx, \frac{1}{t} y\right) \geq t(1 + \eta(t)) A(x, y), x, y \in P_e, \tag{3.1}$$

holds. Then a necessary and sufficient condition for A to have exactly one fixed pint x^* in P_e is that (H_3) holds. Moreover, constructing successively the sequence

$$x_n = A(x_{n-1}, y_{n-1}), y_n = A(y_{n-1}, x_{n-1}), n = 1, 2, \cdots,$$

for any initial $x_0, y_0 \in P_e$, we have

$$\left\| x_n - x^* \right\| \to 0, \left\| y_n - x^* \right\| \to 0, n \to \infty.$$

Corollary 3.5. Let P be a normal cone of E, and let $A : P_e \times P_e \to P_e$ be a mixed monotone operator with

property (2.1). In addition, suppose that $\lim\limits_{t \to 0^+} \eta(t) = +\infty$ holds. Then A has exactly one fixed pint x^* in P_e.

Proof. In fact, by corollary 2.4, we have only to prove that (H_3) holds.

For any $x_0 \in P_e$, it follows from $\lim\limits_{t \to 0^+} \eta(t) = +\infty$ that there exists $t_0 \in (0,1)$, such that

$\dfrac{1}{1+\eta(t_0)} x_0 \le A(x_0, x_0) \le \left(1 + \eta(t_0)\right) x_0$. Let $u_0 = t_0 x_0, v_0 = \dfrac{1}{t_0} x_0$. Following from (3.1) and the mixed monoto-

nicity of A, we get

$$A(u_0, v_0) = A\left(t_0 x_0, \frac{1}{t_0} x_0\right) \ge t_0 \left(1 + \eta(t_0)\right) A(x_0, x_0) \ge t_0 x_0 = u_0;$$

$$A(v_0, u_0) = A\left(\frac{1}{t_0} x_0, t_0 x_0\right) \le \frac{1}{t_0} \left(1 + \eta(t_0)\right)^{-1} A(x_0, x_0) \le \frac{1}{t_0} x_0 = v_0.$$

The Corollary 3.5 is thus proved.

Acknowledgements

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