On the Solutions of the Equation \( x^3 + Ax = B \) in \( \mathbb{Z}_3^* \) with Coefficients from \( \mathbb{Q}_3 \)

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ABSTRACT

Recall that in [1] it is obtained the criteria solvability of the Equation \( x^3 + ax = b \) in \( \mathbb{Z}_p^* \), \( \mathbb{Z}_p \) and \( \mathbb{Q}_p \) for \( p > 3 \). Since any \( p \)-adic number \( x \) has a unique form \( x = p^k \cdot x^* \), where \( x^* \in \mathbb{Z}_p^* \) and \( k \in \mathbb{Z} \), in [1] it is also shown that from the criteria in \( \mathbb{Z}_p^* \) it follows the criteria in \( \mathbb{Z}_p \) and \( \mathbb{Q}_p \). In this paper we provide the algorithm of finding the solutions of the Equation \( x^3 + ax = b \) in \( \mathbb{Z}_3^* \) with coefficients from \( \mathbb{Q}_3 \).

KEYWORDS

\( p \)-Adic Numbers; Solvability of Equation; Congruence

1. Introduction

In the present time description of different structures in mathematics are studying over field of \( p \)-adic numbers. In particular, \( p \)-adic analysis is one of the intensive developing directions of modern mathematics. Numerous applications of \( p \)-adic numbers have found their own reflection in the theory of \( p \)-adic differential equations, \( p \)-adic theory of probabilities, \( p \)-adic mathematical physics, algebras over \( p \)-adic numbers and others.

The field of \( p \)-adic numbers were introduced by German mathematician K. Hensel at the end of the 19th century [2]. The investigation of \( p \)-adic numbers were motivated primarily by an attempt to bring the ideas and techniques of the power series into number theory. Their canonical representation is similar to expansion of analytical functions in power series, which is analogy between algebraic numbers and algebraic functions. There are several books devoted to study \( p \)-adic numbers and \( p \)-adic analysis [3-6].

Classification of algebras in small dimensions plays important role for the studying of properties of varieties of algebras. It is known that the problem of classification of finite dimensional algebras involves a study on equations for structural constants, \textit{i.e.} to the decision of some systems of the Equations in the corresponding field. Classifications of complex Leibniz algebras have been investigated in [7-10] and many other works. In similar complex case, the problem of classification in \( p \)-adic case is reduced to the solution of the Equations in the field. The classifications of Leibniz algebras over the field of \( p \)-adic numbers have been obtained in [11-13].

In the field of complex numbers the fundamental Abel’s theorem about insolvability in radicals of general Equation of \( n \)-th degree \( (n > 5) \) is well known. In this field square equation is solved by discriminant, for cubic Equation Cardano’s formulas were widely applied. In the field of \( p \)-adic numbers square equation does
not always has a solution. Note that the criteria of solvability of the Equation \( x^2 = a \) is given in [6,14,15] we can find the solvability criteria for the Equation \( x^q = a \), where \( q \) is an arbitrary natural number.

In this paper we consider \( p \)-adic cubic equation \( y^3 + ry^2 + sy + t = 0 \). By replacing \( y = x - \frac{r}{3} \), this equation become the so-called depressed cubic equation

\[
x^3 + ax = b. \quad (1)
\]

The solvability criterion for the cubic equation \( x^3 + ax = b \) over \( 3 \)-adic numbers is different from the case \( p > 3 \). Note that solvability criteria for \( >3 \) is obtained in [1]. The problem of finding a solvability criteria of the cubic equation for the case \( p = 3 \) is complicated. This problem was partially solved in [16], namely, it is obtained solvability criteria of cubic equation with condition \( |a|_3 \neq \frac{1}{3} \).

In this paper we obtain solvability criteria of cubic equation for \( p = 3 \) without any conditions. Moreover, the algorithm of finding the solutions of the equation \( x^3 + ax = b \) in \( \mathbb{Z}_3 \) with coefficients from \( \mathbb{Q}_3 \) is provided.

2. Preliminaries

Let \( \mathbb{Q} \) be a field of rational numbers. Every rational number \( x \neq 0 \) can be represented by the form \( x = p^{\gamma(x)} \frac{n}{m}, \) where \( n, \gamma(x) \in \mathbb{Z}, \ m \) is a positive integer, \( (p,n)=1, (p,m)=1 \) and \( p \) is a fixed prime number. In \( \mathbb{Q} \) a norm has been defined as follows:

\[
|x|_p = \begin{cases} 
p^{-\gamma(x)}, & x \neq 0, \\
0, & x = 0.
\end{cases}
\]

The norm \( |x|_p \) is called a \( p \)-adic norm of \( x \) and it satisfies so called the strong triangle inequality. The completion of \( \mathbb{Q} \) with respect to \( p \)-adic norm defines the \( p \)-adic field which is denoted by \( \mathbb{Q}_p \) ([4,6]). It is well known that any \( p \)-adic number \( x \neq 0 \) can be uniquely represented in the canonical form

\[
x = p^{\gamma(x)} \left( x_0 + x_1 p + x_2 p^2 + \cdots \right),
\]

where \( \gamma = \gamma(x) \in \mathbb{Z} \) and \( x_j \) are integers, \( 0 \leq x_j \leq p-1 \), \( x_0 \neq 0 \), \( (j = 0,1,\cdots) \). \( p \)-Adic number \( x \), for which \( |x|_p \leq 1 \), is called integer \( p \)-adic number, and the set of such numbers is denoted by \( \mathbb{Z}_p \). Integer \( x \in \mathbb{Z}_p \), for which \( |x|_p = 1 \), is called unit of \( \mathbb{Z}_p \), and their set is denoted by \( \mathbb{Z}_p^* \).

For any numbers \( a \) and \( m \) it is known the following result.

**Theorem 2.1** [3]. If \( (a,m)=1 \), then a congruence \( ax \equiv b \pmod{m} \) has one and only one solution.

We also need the following Lemma.

**Lemma 2.1** [14]. The following is true:

\[
\left( \sum_{i=0}^{g} x_i p^i \right)^q = x_0^q + \sum_{i=1}^{g} \left( q x_0^{q-1} x_i + N_k \left( x_0, x_1, \cdots, x_{k-1} \right) \right) p^i,
\]

where \( x_0 \neq 0, \ 0 \leq x_j \leq p-1, \ N_i = 0 \) and for \( k \geq 2 \)

\[
N_k = N_k \left( x_0, \cdots, x_{k-1} \right) = \sum_{m_0, m_1, \cdots, m_k-1, \sum_{j=0}^{k-1} m_j = 0, \ m_j \leq \sum_{j=0}^{k-1} m_{j-1}} \frac{q!}{m_0! \cdots m_k!} x_0^{m_0} x_1^{m_1} \cdots x_{k-1}^{m_{k-1}}.
\]

From Lemma 2.1 by \( q = 3 \) we have

\[
\left( \sum_{i=0}^{g} x_i p^i \right)^3 = x_0^3 + \sum_{i=1}^{g} \left( 3 x_0^2 x_i + N_k \left( x_0, x_1, \cdots, x_{k-1} \right) \right) p^i.
\]

For \( j \leq k \) we put
Also the following identity is true:
\[
\left( \sum_{i=0}^{k} a_i p^i \right) \left( \sum_{j=0}^{\infty} x_j p^j \right) = \sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} a_i x_{k-i} \right) p^k.
\] (2)

3. The Main Result

In this paper we study the cubic Equation (1) over the field 3-adic numbers, i.e.,
\[ a, b, x \in \mathbb{Q}_3. \]

Put
\[ x = 3^{\gamma(x)} (x_0 + x_1 3 + \cdots), a = 3^{\gamma(a)} (a_0 + a_1 3 + \cdots), b = 3^{\gamma(b)} (b_0 + b_1 3 + \cdots), \]
where \( x_j, a_j, b_j \in \{0, 1, 2\}, x_0, a_0, b_0 \neq 0, (j = 0, 1, \ldots) \).

Since any 3-adic number \( x \) has a unique form \( x = 3^k x^\ast \), where \( x^\ast \in \mathbb{Z}_3^\ast \) and \( k \in \mathbb{Z} \), we will be limited to search a decision from \( \mathbb{Z}_3^\ast \), i.e. \( \gamma(x) = 0 \).

Putting the canonical form of \( a, b \) and \( x \) in (1), we get
\[
\left( \sum_{x=0}^{k} x_k 3^k \right)^3 + 3^{\gamma(a)} \left( \sum_{x=0}^{k} x_k 3^k \right) \left( \sum_{x=0}^{k} x_k 3^k \right) = 3^{\gamma(b)} \left( \sum_{x=0}^{k} b_k 3^k \right).
\] (3)

Proposition 3.1 If one of the following conditions:

1) \( \gamma(a) = 0 \) and \( \gamma(b) < 0 \);  2) \( \gamma(a) > 0 \) and \( \gamma(b) > 0 \);
3) \( \gamma(a) > 0 \) and \( \gamma(b) < 0 \);  4) \( \gamma(a) < 0 \) and \( \gamma(b) = 0 \);
5) \( \gamma(a) < 0 \) and \( \gamma(b) > 0 \),
is fulfilled, then the Equation (1) has not a solution in \( \mathbb{Z}_3^\ast \).

Proof. 1) Let \( \gamma(a) = 0 \) and \( \gamma(b) < 0 \). Multiplying Equation (3) by \( 3^{-\gamma(b)} \), we get the following congruence \( b_0 \equiv 0 \) (mod 3), which is not correct. Consequently, Equation (1) has no solution in \( \mathbb{Z}_3^\ast \).
2) Let \( \gamma(a) > 0 \) and \( \gamma(b) > 0 \). Then from (3) it follows a congruence \( x_0 \equiv 0 \) (mod 3), which has no a nonzero solution. Therefore, in \( \mathbb{Z}_3^\ast \) Equation (1) does not have a solution.

In other cases, we analogously get the congruences
\[ b_0 \equiv 0 \) (mod 3) \] or \[ a_0 x_0 \equiv 0 \) (mod 3),
which are not hold. Therefore, in \( \mathbb{Z}_3^\ast \) there is no solution.

From the Proposition 3.1 we have that the cubic equation may have a solution if one of the following four cases

1) \( \gamma(a) = 0 \), \( \gamma(b) = 0 \);  2) \( \gamma(a) = 0 \), \( \gamma(b) > 0 \);
3) \( \gamma(a) < 0 \), \( \gamma(b) < 0 \);  4) \( \gamma(a) > 0 \), \( \gamma(b) = 0 \)
is hold.

In the following theorem we present an algorithm of finding of the Equation \( x^3 + ax = b \) for the first case.

Theorem 3.1 Let \( \gamma(a) = \gamma(b) = 0 \) and \( a_0 = 1 \). Then \( x \) to be a solution of the Equation (1) in \( \mathbb{Z}_3^\ast \) if and
only if the congruences
\begin{align*}
&x_0^3 + a_0 x_0 = b_0 \pmod{3}, \\
x_i a_0 + x_0 a_i + N_i(x_0) + M_i(x_0) = b_i \pmod{3}, \\
x_k a_0 + \cdots + x_0 a_k + x^2 x_{k-1} + N_{k-1}(x_0, x_1, \ldots, x_{k-1}) + M_{k-1}(x_0, x_1, \ldots, x_{k-1}) = b_{k-1} \pmod{3}, k \geq 2
\end{align*}
are fulfilled, where integers \( M_i(x_0, \ldots, x_{k-1}) \) are defined consequently from the following correlations
\begin{align*}
x_k a_0 + x_k a_i + x_i a_k + x_0 a_{k-1} + x^2 x_{k-2} + N_{k-2}(x_0, x_1, \ldots, x_{k-2}) = b_{k-2} - M_{k-2}(x_0, x_1, \ldots, x_{k-2}) + M_{k-1}(x_0, x_1, \ldots, x_{k-1}) \cdot 3, k \geq 3.
\end{align*}

Proof. Let \( x_0 + x_1 \cdot 3 + x_2 \cdot 3^2 + \cdots, 0 \leq x_j \leq 2, x_0 \neq 0, (j = 0, 1, \ldots) \) is a solution of Equation (1), then Equality (3) becomes
\begin{align*}
x_0^3 + \sum_{k=1}^{\infty} \left( 3x_0^2 x_k + N_k(x_0, x_1, \ldots, x_{k-1}) \right) 3^k + a_0 x_0 + \sum_{k=1}^{\infty} \left( \sum_{i=0}^{k} x_i a_{k-i} \right) 3^k = b_0 + \sum_{k=1}^{\infty} b_k 3^k.
\end{align*}
So we have
\begin{align*}
x_0^3 + a_0 x_0 + \left( x_i a_0 + x_0 a_i + N_i(x_0) \right) 3 + \sum_{k=2}^{\infty} \left( \sum_{i=0}^{k} x_i a_{k-i} \right) 3^k = b_0 + \sum_{k=1}^{\infty} b_k 3^k,
\end{align*}
from which it follows the necessity in fulfilling the congruences of the theorem.

Now let \( x \) is satisfied the congruences of the theorem. Since \( (a_0, 3) = 1 \), then by Theorem 2.1 it implies that these congruences have the solutions \( x_k \).

Then
\begin{align*}
x_0^3 + \sum_{k=1}^{\infty} \left( 3x_0^2 x_k + N_k(x_0, x_1, \ldots, x_{k-1}) \right) 3^k + a_0 x_0 + \sum_{k=1}^{\infty} \left( \sum_{i=0}^{k} x_i a_{k-i} \right) 3^k
&= x_0^3 + a_0 x_0 + \left( N_i + x_i a_i + a_0 x_0 \right) 3 + \sum_{k=2}^{\infty} \left( x_i^2 x_{k-1} + N_{k-1}(x_0, x_1, \ldots, x_{k-1}) + x_0 a_k + x_k a_{k-1} + \cdots + x_{k-2} a_i + x_i a_0 \right) 3^k
&= b_0 + M_i(x_0) \cdot 3 + \left( b_0 - M_i(x_0) + M_2(x_0, x_1) \cdot 3 \right) 3 + \sum_{k=2}^{\infty} \left( b_k - M_k(x_0, x_1, \ldots, x_{k-1}) + M_{k-1}(x_0, x_1, \ldots, x_{k-1}) \cdot 3 \right) 3^k
&= b_0 + \sum_{k=1}^{\infty} b_k 3^k.
\end{align*}

Therefore, we show that \( x = \sum_{k=0}^{\infty} x_k 3^k \) is a solution of the Equation (1). \( \blacksquare \)

Let us examine a case \( \gamma(a) = 0, \gamma(b) > 0 \) and get necessary and sufficient conditions for a solution of Equation (1).

**Theorem 3.2** Let \( \gamma(a) = 0, \gamma(b) = m > 0 \) and \( a_0 = 2 \). Then \( x \) to be a solution of Equation (1) in \( \mathbb{Z}_3^* \) if and only if the congruences
\[ x_0^3 + a_0 x_0 = 0 \pmod{3}, \]
\[ x_1 a_0 + x_0 a_1 + N_1 (x_0) + M_1 (x_0) = 0 \pmod{3}, \]
\[ x_1 a_0 + \ldots + x_{m-1} a_{m-1} + N_1 (x_0, \ldots, x_{m-1}) + M_1 (x_0, \ldots, x_{m-1}) = 0 \pmod{3}, \]
\[ x_0 x_1 a_0 + \ldots + x_{m-1} a_{m-1} + N_1 (x_0, \ldots, x_{m-1}) + M_1 (x_0, \ldots, x_{m-1}) = b_{k-m} \pmod{3}, \]

are fulfilled, where integers \( M_1 (x_0, x_1, \ldots, x_{m-1}) \) are defined consequently from the following correlations

\[ x_0 + 2 x_0 = M_1 (x_0) \cdot 3, \]
\[ x_1 a_0 + x_0 a_1 + N_1 (x_0) = -M_1 (x_0) + 3 M_2 (x_0, x_1), \]
\[ x_1 a_0 + x_0 a_1 + \ldots + x_{n-1} a_{n-1} + N_1 (x_0, \ldots, x_{n-1}) + M_1 (x_0, \ldots, x_{n-1}) = -M_1 (x_0, x_1, \ldots, x_{n-1}) + 3 M_{k+1} (x_0, x_1, \ldots, x_{n-1}), 2 \leq k \leq m-1, \]
\[ x_0 x_1 a_0 + x_0 a_1 + \ldots + x_{n-1} a_{n-1} + N_1 (x_0, \ldots, x_{n-1}) + M_1 (x_0, \ldots, x_{n-1}) = b_{k-m} - M_1 (x_0, x_1, \ldots, x_{n-1}) + 3 M_{k+1} (x_0, x_1, \ldots, x_{n-1}), k \geq m. \]

**Proof.** Let \( x \) is a solution of the Equation (1), then Equality (3) becomes

\[
\begin{aligned}
x_0^3 + 2 \sum_{k=1}^{3^k} (3 x_0^2 x_k + N_1 (x_0, x_1, \ldots, x_{k-1})) 3^k + a_0 x_0 + \sum_{k=0}^{3^k} \sum_{i=0}^{3^k} x_i a_{k-i} 3^i &= 3^{n+1} \left( b_0 + \sum_{k=1}^{3^k} b_k 3^k \right).
\end{aligned}
\]

Therefore, we have

\[
\begin{aligned}
x_0^3 + a_0 x_0 + \left( x_1 a_0 + x_0 a_1 + N_1 (x_0) \right) 3^k + \sum_{k=2}^{3^k} \sum_{i=0}^{3^k} x_i a_{k-i} 3^i &= 3^{n+1} \left( b_0 + \sum_{k=1}^{3^k} b_k 3^k \right),
\end{aligned}
\]

from which it follows the necessity in fulfilling the congruences of the theorem.

Now let \( x \) is satisfied the congruences of the theorem. Since \((a_0, 3) = 1\), then by Theorem 2.1 there are solutions \( x_1 \) of the congruences.

Putting element \( x \) to Equality (3), we have

\[
\begin{aligned}
x_0^3 + a_0 x_0 + \left( x_1 a_0 + x_0 a_1 + N_1 (x_0) \right) 3^k + \sum_{k=2}^{3^k} \sum_{i=0}^{3^k} x_i a_{k-i} 3^i &= 3^{n+1} \left( b_0 + \sum_{k=1}^{3^k} b_k 3^k \right).
\end{aligned}
\]

Therefore, we show that \( x \) is a solution of Equation (1). □

The following theorem gives necessary and sufficient conditions for a solution of Equation (1) for the case \( \gamma (a) < 0 \) and \( \gamma (b) < 0 \).

**Theorem 3.3** Let

\[ \gamma (a) = \gamma (b) = -m < 0, (m > 0). \]

Then \( x \) to be a solution of Equation (1) in \( \mathbb{Z}_3^* \) if and only if the next congruences...
are fulfilled, where integers \( M_k(x_0, x_1, \ldots, x_{k-1}) \) are defined consequently from the equalities
\[
a_0, x_0 = b_0 \pmod{3},
\]
\[
x_ka_0 + x_{k-1}a_1 + \cdots + x_0a_k + M_k(x_0, x_1, \ldots, x_{k-1}) = b_k \pmod{3}, 1 \leq k \leq m-1,
\]
\[
x_ka_0 + x_{k-1}a_1 + \cdots + x_0a_k + x_0^2 + M_k(x_0, x_1, \ldots, x_{k-1}) = b_m \pmod{3},
\]
\[
x_ka_0 + x_{k-1}a_1 + \cdots + x_0a_k + x_0^2 + M_{m+k}(x_0, x_1, \ldots, x_{m+k}) = b_m \pmod{3},
\]
\[
x_ka_0 + x_{k-1}a_1 + \cdots + x_0a_k + x_0^2 + x_0^2 + N_k(x_0, x_1, \ldots, x_{k-1}) + M_k(x_0, x_1, \ldots, x_{k-1}) = b_k \pmod{3}, k \geq m + 2
\]
are fulfilled, where integers \( M_k(x_0, x_1, \ldots, x_{k-1}) \) are defined consequently from the equalities
\[
a_0, x_0 = b_0 + M_k(x_0) \cdot 3,
\]
\[
x_ka_0 + x_{k-1}a_1 + \cdots + x_0a_k = b_k - M_k(x_0, x_1, \ldots, x_{k-1}) + 3M_{k+1}(x_0, x_1, x_2), 1 \leq k \leq m-1,
\]
\[
x_ka_0 + x_{k-1}a_1 + \cdots + x_0a_k + x_0^3 + M_{k+1}(x_0, x_1, x_2) = b_m - M_{m+k}(x_0, x_1, x_2) + 3M_{m+k+1}(x_0, x_1, x_2),
\]
\[
x_ka_0 + x_{k-1}a_1 + \cdots + x_0a_k + x_0^3 + x_0^2 + x_0^2 + M_{m+k}(x_0, x_1, x_2) = b_m - M_{m+k+1}(x_0, x_1, x_2) + 3M_{m+k+2}(x_0, x_1, x_2),
\]
\[
x_ka_0 + x_{k-1}a_1 + \cdots + x_0a_k + x_0^3 + x_0^2 + x_0^2 + N_{m+k}(x_0, x_1, x_2) + M_{m+k}(x_0, x_1, x_2) = b_m - M_{m+k+1}(x_0, x_1, x_2) + 3M_{m+k+2}(x_0, x_1, x_2), k \geq m + 2.
\]

**Proof.** The proof of the Theorem can be obtained by similar way to the proofs of Theorems 3.1 and 3.2. Examining various cases of \( \gamma(a) \) and \( \gamma(b) \) we need to study only the case \( \gamma(a) > 0 \) and \( \gamma(b) = 0 \). Because of appearance of uncertainty of a solution, we divide this case to \( \gamma(a) > 1 \) and \( \gamma(a) = 1 \).

**Theorem 3.4** Let \( \gamma(a) = 2, \gamma(b) = 0 \) and \( (b_0, b_1) = (1, 0) \) or \( (2, 2) \). Then \( x \) to be a solution of Equation (1) in \( \mathbb{Z}_3 \) if and only if he next congruences
\[
x_0^3 = b_0 \pmod{3},
\]
\[
x_0^5 = b_0 + b_1 \cdot 3 \pmod{9},
\]
\[
x_0^3 + x_0a_0 + M_1(x_0) = b_2 \pmod{3},
\]
\[
x_0^3 + x_0a_0 + M_1(x_0) = b_2 \pmod{3},
\]
\[
x_0^3 + P^k(x_0, x_1) + x_0a_0 + x_0a_1 + x_0^3 + M_2(x_0, x_1) = b_3 \pmod{3},
\]
\[
x_0^3 + P^k(x_0, x_1) + x_0a_0 + x_0a_1 + x_0^3 + M_2(x_0, x_1) = b_3 \pmod{3},
\]
\[
x_0^3 + x_0^3 + P^{k+1}(x_0, x_1, x_2) + 2x_0x_1x_2 + x_0^3a_0 + x_0^3a_1 + \cdots + x_0^3a_{k-2} + M_{k+1}(x_0, x_1, x_2) = b_3 \pmod{3}, k \geq 4.
\]

**Proof.** Analogously to the proof of Theorem 3.1. Let \( \gamma(a) = 3, \gamma(b) = 0 \) and \( (b_0, b_1) = (1, 0) \) or \( (2, 2) \). Then \( x \) to be a solution of the Equation (1) in \( \mathbb{Z}_3 \) if and only if he next congruences
\[
x_0^3 = b_0 \pmod{3},
\]
\[
x_0^3 = b_0 + b_1 \cdot 3 \pmod{9},
\]
\[
x_0^3 + M_1(x_0) = b_2 \pmod{3},
\]
\[
x_0^3 + P^k(x_0, x_1) + x_0a_0 + x_0a_1 + x_0^3 + M_2(x_0, x_1) = b_3 \pmod{3},
\]
\[
x_0^3 + P^{k+1}(x_0, x_1, x_2) + 2x_0x_1x_2 + x_0^3a_0 + x_0^3a_1 + \cdots + x_0^3a_{k-2} + M_{k+1}(x_0, x_1, x_2) = b_3 \pmod{3}, k \geq 4.
\]
are fulfilled, where integers \( M_k(x_0, x_1, \ldots, x_{k-1}) \) are defined from the equalities
\[
x^0 = b_0 + b_1 \cdot 3 + M_1(x_0) \cdot 9, \\
x^0_1 = b_2 - M_1(x_0) + 3M_2(x_0, x_1), \\
x^0_2 = b_3 + P^1_2(x_0, x_1) + 3M_4(x_0, x_1, x_2), \\
x^0_3 = \ldots \\
x^0_{k-1} + P^k_{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2x_0x_{k-1}a_0 + x_{k-1}a_1 + \ldots + x_{k-1}a_{k-3} = b_k - M_{k-1}(x_0, x_1, \ldots, x_{k-2}) + 3M_k(x_0, x_1, \ldots, x_{k-1}), k \geq 4.
\]

**Proof.** Analogously to the proof of Theorem 3.1.■

Similarly to Theorem 3.4, it is proved the following

**Theorem 3.6** Let \( \gamma(a) = m \geq 4, \gamma(b) = 0 \) and \((b_0, b_1) = (1, 0) \) or \((2, 2)\). Then \( x \) to be a solution of the Equation (1) in \( \mathbb{Z}_3^* \) if and only if the next congruences
\[
x^0_1 = b_0 \pmod{3}, \\
x^0_1 = b_0 + b_1 \cdot 3 \pmod{9}, \\
x^0_2 = b_1 + M_1(x_0) = b_2 \pmod{3}, \\
x^0_2 = b_3 + P^1_2(x_0, x_1) + 3M_4(x_0, x_1) = b_3 \pmod{3}, \\
x^0_2 = b_4 + P^2_2(x_0, x_1, x_2) + 2x_0x_1 = b_5 \pmod{3}, \\
x^0_2 = b_5 \pmod{3}, k \geq m + 1
\]

are fulfilled, where integers \( M_k(x_0, x_1, \ldots, x_{k-1}) \) are defined from the equalities
\[
x^0 = b_0 + b_1 \cdot 3 + M_1(x_0) \cdot 9, \\
x^0_1 = b_2 - M_1(x_0) + 3M_2(x_0, x_1), \\
x^0_2 = b_3 + P^1_2(x_0, x_1) + 3M_4(x_0, x_1, x_2), \\
x^0_3 = \ldots \\
x^0_{k-1} + P^k_{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2x_0x_{k-1}a_0 + x_{k-1}a_1 + \ldots + x_{k-1}a_{k-3} = b_k - M_{k-1}(x_0, x_1, \ldots, x_{k-2}) + 3M_k(x_0, x_1, \ldots, x_{k-1}), k \geq m + 1.
\]

Now we consider Equality (3) with \( \gamma(a) = 1, \gamma(b) = 0 \). Put
\[
A_0 = x^0_0 + a_0, A_k = \frac{A_{k-1}}{A_k} + a_k + R_k, \text{ where } R_k = \sum_{j=0}^{k} x_j x_{k-j}, k \geq 1,
\]

\[
N_j = \begin{cases} 
\frac{N_{j-1}}{3} + x^3_j, & j = 3s - 1, \\
\frac{N_{j-1}}{3} + x^3_j, & j = 3s, \\
\frac{N_{j-1}}{3} - x^3_{j-1}, & j = 3s + 1
\end{cases}
\]

\[
S_j = \begin{cases} 
\frac{P^{j-1}}{3}, & j = 3s - 1, \\
\frac{P^{j-1}}{3} + x^3_j, & j = 3s, \\
\frac{P^{j-1}}{3} - x^3_{j-1}, & j = 3s + 1
\end{cases}
\]

**Theorem 3.7** Let \( \gamma(a) = 1, \gamma(b) = 0 \) and \( x \in \mathbb{Z}_3^* \) to be so that \( A_0 = x^0_0 + a_0 \neq 0 \pmod{3} \). Then \( x \) to be a solution of Equation (1) in \( \mathbb{Z}_3^* \) if and only if the congruences
\[ x_0^3 = b_1 \pmod{3}, \]
\[ x_0^2 + M_1(x_0) \equiv b_1 \pmod{3}, \]
\[ (x_0^2 + a_0)x_{k-1} + x_{k-2}a_1 + \cdots + x_0a_{k-1} + N_k(x_0, \ldots, x_{k-2}) + M_k(x_0, \ldots, x_{k-2}) \equiv b_k \pmod{3}, k \geq 2 \]

are faithfully, where \( M_1(x_0) = \frac{x_0^3 - b_0}{3} \) and integers \( M_k(x_0, \ldots, x_{k-2}), (k \geq 2) \) are defined from the equalities

\[ x_0 a_0 + M_1(x_0) = b_1 + M_2(x_0) \cdot 3, \]
\[ (x_0^2 + a_0)x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-2}) + x_{k-2}a_1 + \cdots + x_0a_{k-1} + M_k(x_0, \ldots, x_{k-2}) \equiv b_k + M_{k+1}(x_0, \ldots, x_{k-1}) \cdot 3, k \geq 2. \]

**Proof.** Let the congruences \( x_0^3 = b_0 \pmod{3}, x_0 a_0 + \frac{x_0^2 - b_0}{3} = b_1 \pmod{3} \) has a solution \( x_0 \). Then denote by \( M_2(x_0) \) the number satisfying the equality \( 3M_2(x_0) = x_0 a_0 + M_1(x_0) - b_1 \).

Using Theorem 2.1, we have existence of solutions \( x_k \) of the congruences

\[ (x_0^2 + a_0)x_{k-1} + x_{k-2}a_1 + \cdots + x_0a_{k-1} + N_k(x_0, \ldots, x_{k-2}) + M_k(x_0, \ldots, x_{k-2}) \equiv b_k \pmod{3}, k \geq 2. \]

The next chain of equalities

\[ x_0^2 + a_0 x_0 \cdot 3 + \sum_{k=2}^{\infty} \left( (x_0^2 + a_0)x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-2}) + x_{k-2}a_1 + \cdots + x_0a_{k-1} \right) 3^k \]

\[ = b_0 + M_1(x_0) \cdot 3 + (b_1 - M_1(x_0)) + M_2(x_0) \cdot 3 \cdot 3 + \sum_{k=2}^{\infty} (b_k - M_k(x_0, x_1, \ldots, x_{k-2}) + M_{k+1}(x_0, x_1, \ldots, x_{k-1}) \cdot 3) \cdot 3^k \]

\[ = b_0 + \sum_{k=1}^{\infty} b_k 3^k, \]

shows that \( x \) is a solution of Equation (1). \( \blacksquare \)

From the proof of Theorem 3.7, it is easy to see that if \( A_0 = x_0^3 + a_0 = 0 \pmod{3} \), then we have the following congruences and appropriate equalities

a) \( x_0^3 \equiv b_0 \pmod{3} \), i.e. \( x_0^3 = b_0 + M_1(x_0) \cdot 3; \)

b) \( x_0a_0 + M_1(x_0) \equiv b_1 \pmod{3} \), then \( x_0a_0 + M_1(x_0) = b_1 + M_2(x_0) \cdot 3; \)

c) \( x_0a_1 + M_2(x_0) \equiv b_2 \pmod{3} \), then

\[ x_0a_1 + M_2(x_0) = b_2 + M_3(x_0) \cdot 3; \quad (4) \]

d) \( \frac{A_0}{3} x_1 + x_0 a_1 + x_0a_2 + x_0x_1^2 + x_1^3 + M_3(x_0) \equiv b_3 \pmod{3} \), it follows that

\[ \frac{A_0}{3} x_1 + x_0 a_1 + x_0a_2 + x_0x_1^2 + x_1^3 + M_3(x_0) = b_3 + M_4(x_0, x_1) \cdot 3. \]

Since \( A_1 = \frac{A_0}{3} + a_1 + 2x_0x_1 \), then the congruence d) can be written in the form

\[ (A_1 - x_0x_1) x_1 + x_0a_2 + x_1^3 + M_3(x_0) \equiv b_3 \pmod{3}, \]

and so we have

\[ (A_1 - x_0x_1) x_1 + x_0a_2 + x_1^3 + M_3(x_0) = b_3 + M_4(x_0, x_1) \cdot 3. \]
If for any natural number $k$ we have $A_k \equiv 0 \pmod{3}$, then we could establish the criteria of solvability for Equation (1). However, if there exists $k$, such that $A_k \not\equiv 0 \pmod{3}$, then the criteria of solvability can be found, and therefore, we need the following

**Lemma 3.1** Let $\gamma(a) = 1$, $\gamma(b) = 0$ and $x \in \mathbb{Z}_3^*$ to be so that $A_{i-j} \equiv 0 \pmod{3}, 1 \leq j \leq k$, $A_i \not\equiv 0 \pmod{3}$ for some fixed $k$. If $x$ be a solution of Equation (1), then it is true the following system of the congruences

$$x_0^i \equiv b_0 \pmod{3},$$
$$x_0a_0 + M_i (x_0) \equiv b_i \pmod{3},$$
$$x_{j-1}a_j + x_{j-2}a_{j+1} + \cdots + x_0a_{j-i} + S^j_{i-1} + M_i \left(x_0, x_1, \cdots, x_{j-1}\right) \equiv b_{j-1} \pmod{3},$$

$$\left(A_j - x_0x_j\right)x_j + x_{j-1}a_{j+1} + \cdots + x_0a_{j-i} + S^{j-1}_{i-1} + b_{j-1} \left(x_0, x_1, \cdots, x_{j-1}\right) \equiv b_{j-1} \pmod{3},$$

$$A_i x_{k-i} + x_{k-i-2}a_{k} + \cdots + x_0a_{k+1} + S^{k-i}_{k-i} + M_{i} \left(x_0, x_1, \cdots, x_{k-i}\right) \equiv b_{k-i} \pmod{3},$$

where $1 \leq j \leq k$ and integers $M_i \left(x_0, \cdots, x_{k-i}\right)$ are defined from the equalities

$$3 \cdot M_i (x_0) = x_0^i - b_0,$$
$$3 \cdot M_2 (x_0) = x_0a_0 + M_i (x_0) - b_i,$$
$$3 \cdot M_{2j+1} (x_0, \cdots, x_{j-1}) = x_{j-1}a_j + \cdots + x_0a_{j-i} + S^j_{i-1} + M_i \left(x_0, \cdots, x_{j-1}\right) - b_{j-1},$$
$$3 \cdot M_{2j+2} (x_0, x_1, \cdots, x_{j-i}) = A_i x_{k-i} + x_{k-i-2}a_{k} + \cdots + x_0a_{k+1} + S^{k-i}_{k-i} + b_{k-i} \left(x_0, x_1, \cdots, x_{k-i}\right) - b_{k-i},$$

**Proof.** We will prove Theorem by induction. Let $k = 1$, *i.e.*

$$A_0 = x_0^2 + a_0 \equiv 0 \pmod{3}, A_1 = \frac{A_0}{3} + a_1 + 2x_0x_1 \not\equiv 0 \pmod{3},$$

then the system of the congruences (4) are true. Note that $S^1_0 = 0$, $S^1_1 = x_0^2$. From (3) it is easy to get

$$A_0 \left(x_{i-2} + x_{i-3}a_2 + x_{i-3}a_1 + \cdots + x_0a_{i-1} + S_{i-2} + 2x_0x_{i-2} + M_i \left(x_0, \cdots, x_{i-3}\right)\right) \equiv b_i \pmod{3}, i \geq 4.$$

Therefore,

$$A_i x_{i-2} + x_{i-3}a_2 + \cdots + x_0a_{i-1} + S_{i-2} + M_i \left(x_0, \cdots, x_{i-3}\right) \equiv b_i \pmod{3}, i \geq 4,$$

where

$$3 \cdot M_{i+1} (x_0, \cdots, x_{i-2}) = A_i x_{i-2} + x_{i-3}a_2 + \cdots + x_0a_{i-1} + S_{i-2} + M_i \left(x_0, \cdots, x_{i-3}\right) - b_i, i \geq 4.$$
3 \cdot M_5 (x_0, x_1) = x_1 a_2 + x_0 a_3 + S_4^2 (x_0, x_1) - b_4;

f) \frac{A_3}{3} \sum x_2 + x_2 \alpha_2 + x_1 a_3 + x_0 a_4 + S_4^1 + M_4 (x_0, x_1) \equiv b_5 \pmod{3}, \text{ it follows}

3 \cdot M_6 (x_0, x_1, x_2) = \frac{A_3}{3} x_2 + x_2 a_2 + x_1 a_3 + x_0 a_4 + S_4^1 + M_4 (x_0, x_1) - b_5;

h) \frac{A_3}{3} \sum x_{i-2} + x_{i-2} a_2 + x_{i-3} a_3 + \cdots + x_0 a_i + S_{i-1}^{x_1} + M_{i+1} (x_0, x_1, \ldots, x_{i-3}) \equiv b_{i+1} \pmod{3},

where \( i \geq 5 \) and \( M_{i+2} (x_0, x_1, \ldots, x_{i-2}) \) are defined by equalities

\[ 3 \cdot M_{i+2} (x_0, x_1, \ldots, x_{i-2}) = \frac{A_3}{3} \sum x_{i-2} + x_{i-2} a_2 + x_{i-3} a_3 + \cdots + x_0 a_i + S_{i-1}^{x_1} + M_{i+1} (x_0, x_1, \ldots, x_{i-3}) - b_{i+1}. \]

Since \( S_2^2 = 0, S_2^3 = 0, S_3^4 = x_0^2 + x_1^2, S_4^{x_1} = S_2^{x_1} + x_2^2 x_{i-2} + 2 x_0 x_2 x_{i-2}, \) we denote by \( i = t - 4 \) and have
e) \( x_1 a_2 + x_1 a_3 + M_4 (x_0, x_1) \equiv b_4 \pmod{3},
f) \left( A_4 - x_0 x_1 \right) x_2 + x_1 a_3 + x_0 a_4 + M_4 (x_0, x_1) \equiv b_5 \pmod{3},
h) A_2^i x_{i+2} + x_1 a_2 + x_0 a_3 + \cdots + x_0 a_i + S_{i-1}^{x_1} + M_{i+3} (x_0, x_1, \ldots, x_{i-3}) \equiv b_{i+3} \pmod{3}, \text{ where}

\[ 3 \cdot M_5 (x_0, x_1) = x_1 a_2 + x_0 a_3 + M_4 (x_0, x_1) - b_5, \]

\[ 3 \cdot M_6 (x_0, x_1, x_2) = \left( A_4 - x_0 x_2 \right) x_2 + x_1 a_3 + x_0 a_4 + M_5 (x_0, x_1) - b_5, \]

\[ 3 \cdot M_{i+3} (x_0, x_1, \ldots, x_{i-2}) = A_2^i x_{i+2} + x_1 a_2 + \cdots + x_0 a_i + S_{i-1}^{x_1} + M_{i+3} (x_0, x_1, \ldots, x_{i-3}) - b_{i+3}. \]

So we showed that the statement of Lemma is true for \( k = 2 \).

Let the system of congruences (5) and (6) is true for \( k \). Since \( A_t = 0 \pmod{3} \), then from the congruences

\[ A_4 x_{k+1} + x_{k+1} a_1 + x_{k+1} a_2 + \cdots + x_0 a_{2k+1} + S_{2k+1}^{x_1} + M_{2k+1} (x_0, x_1, \ldots, x_{k+1}) \equiv b_{2k+1} \pmod{3}, i \geq 1 \]

we derive

\[ x_1 a_{k+1} + x_{k+1} a_{k+2} + \cdots + x_0 a_{2k+1} + S_{2k+1}^{x_1} + M_{2k+1} (x_0, x_1, \ldots, x_k) \equiv b_{2k+2} \pmod{3}, \]

\[ \frac{A_3}{3} x_{k+1} + x_1 a_{k+2} + x_{k+2} a_{k+3} + \cdots + x_0 a_{2k+2} + S_{2k+2}^{x_1} + M_{2k+2} (x_0, x_1, \ldots, x_k) \equiv b_{2k+3} \pmod{3}, \]

\[ \frac{A_3}{3} x_{k+1} + x_{k+1} a_{k+2} + x_{k+1} a_{k+2} + \cdots + x_0 a_{2k+2} + S_{2k+2}^{x_1} + M_{2k+2} (x_0, x_1, \ldots, x_k) \equiv b_{2k+3} \pmod{3}, i \geq 1. \]

It is easy to check that

\[ S_{2k+2}^{x_1} = S_{2k+1}^{x_1} + R_{2k+1} x_{k+1} - x_0 x_{k+1}^2, S_{2k+1}^{x_1} = S_{2k+1}^{x_1} + R_{2k+1} x_{k+1}, i \geq 1. \]

By these correlations we deduce

\[ \frac{A_3}{3} x_{k+1} + x_{k+1} a_{k+2} + x_{k+1} a_{k+2} + \cdots + x_0 a_{2k+2} + S_{2k+2}^{x_1} + M_{2k+2} (x_0, x_1, \ldots, x_k) \]

\[ = \left( A_{k+1} - x_0 x_{k+1} \right) x_{k+1} + x_1 a_{k+2} + \cdots + x_0 a_{2k+2} + S_{2k+2}^{x_1} + M_{2k+2} (x_0, x_1, \ldots, x_k). \]

For \( i \geq 1 \) we get

\[ \frac{A_3}{3} x_{k+1} + x_{k+1} a_{k+2} + \cdots + x_0 a_{2k+2} + S_{2k+2}^{x_1} + M_{2k+2} (x_0, x_1, \ldots, x_k). \]

Consequently, we have
\[ x_t a_{k+1} + x_{k+1} a_{k+2} + \cdots + x_0 a_{2 k+1} + S_{2 k+1}^d + M_{2 k+1} (x_0, \ldots, x_k) \equiv b_{2 k+1} \pmod{3}, \]

\[ (A_{k+1} - x_0 a_{k+1}) x_{k+1} + x_{k+1} a_{k+2} + \cdots + x_0 a_{2 k+1} + S_{2 k+1}^d + M_{2 k+1} (x_0, \ldots, x_k) \equiv b_{2 k+1} \pmod{3}, \]

\[ A_{k+1} x_{k+1} + x_{k+1} a_{k+2} + \cdots + x_0 a_{2 k+1} + S_{2 k+1}^d + M_{2 k+1} (x_0, x_k, \ldots, x_k) \equiv b_{2 k+1} \pmod{3}, \]

So we established that the system of congruences (5)-(6) is true for \( k + 1. \)

Using the Lemma 3.1 we obtain the following Theorems.

**Theorem 3.8** Let \( \gamma(a) = 1, \ \gamma(b) = 0 \) and \( x \in \mathbb{Z}_3^* \) to be such that \( A_{k+1} = 0 \pmod{3}, 1 \leq j \leq k, \ A_k \neq 0 \pmod{3} \) for some fixed \( k \geq 1. \) Then \( x \) is a solution of the Equation (1) in \( \mathbb{Z}_3^* \) if and only if the system of the congruences

\[ x_j a_j + x_{j+1} a_{j+1} + \cdots + x_0 a_{2 j+1} + S_2^j + M_2 (x_0, x_1, \ldots, x_{j+1}) \equiv b_2, \]

\[ (A_j - x_0) x_j + x_{j+1} a_{j+1} + x_{j+2} a_{j+2} + \cdots + x_0 a_{2 j+1} + S_2^j + M_2 (x_0, x_1, \ldots, x_{j+1}) \equiv b_{j+1} \pmod{3}, \]

has a solution, where \( 1 \leq j \leq k \) and integers \( M_k (x_0, x_1, \ldots, x_{k+1}) \) are defined from the equalities

\[ 3 \cdot M_k (x_0) = x_0^3 - b_0, \]

\[ 3 \cdot M_2 (x_0) = x_0 a_0 + M_1 (x_0) + b_1, \]

\[ 3 \cdot M_2 (x_{j+1}) (x_0, x_1, \ldots, x_{j+1}) = x_0 a_0 + x_{j+1} a_{j+1} + \cdots + x_0 a_{2 j+1} + S_2^j + M_2 (x_0, x_1, \ldots, x_{j+1}) - b_2 j, \]

\[ 3 \cdot M_2 (x_{j+2}) (x_0, x_1, \ldots, x_{j+2}) = (A_j - x_0) x_j + x_{j+1} a_{j+1} + x_{j+2} a_{j+2} + \cdots + x_0 a_{2 j+1} + S_2^j + M_2 (x_0, x_1, \ldots, x_{j+1}) - b_{j+1}. \]

**Theorem 3.9** Let \( \gamma(a) = 1, \ \gamma(b) = 0 \) and \( x \in \mathbb{Z}_3^* \) to be such that \( A_k = 0 \pmod{3} \) for all \( k \in \mathbb{N}. \) Then \( x \) is a solution of the Equation (1) in \( \mathbb{Z}_3^* \) if and only if the system of the congruences

\[ x_0 a_0 + M_1 (x_0) = b_1 \pmod{3}, \]

\[ x_j a_j + x_{j+1} a_{j+1} + \cdots + x_0 a_{2 j+1} + S_2^j + M_2 (x_0, x_1, \ldots, x_{j+1}) \equiv b_2 j, \]

\[ (A_j - x_0) x_j + x_{j+1} a_{j+1} + x_{j+2} a_{j+2} + \cdots + x_0 a_{2 j+1} + S_2^j + M_2 (x_0, x_1, \ldots, x_{j+1}) = b_{j+1} \pmod{3}, \]

has a solution, where \( j \geq 1 \) and integers \( M_k (x_0, x_1, \ldots, x_{k+1}) \) are defined from the equalities

\[ 3 \cdot M_k (x_0) = x_0^3 - b_0, \]

\[ 3 \cdot M_2 (x_0) = x_0 a_0 + M_1 (x_0) - b_1, \]

\[ 3 \cdot M_2 (x_{j+1}) (x_0, x_1, \ldots, x_{j+1}) = x_0 a_0 + x_{j+1} a_{j+1} + \cdots + x_0 a_{2 j+1} + S_2^j + M_2 (x_0, x_1, \ldots, x_{j+1}) - b_2 j, \]

\[ 3 \cdot M_2 (x_{j+2}) (x_0, x_1, \ldots, x_{j+2}) = (A_j - x_0) x_j + x_{j+1} a_{j+1} + x_{j+2} a_{j+2} + \cdots + x_0 a_{2 j+1} + S_2^j + M_2 (x_0, x_1, \ldots, x_{j+1}) - b_{j+1}. \]

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