Doubly and Triply Periodic Waves Solutions for the KdV Equation*

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ABSTRACT
Based on the arbitrary constant solution, a series of explicit doubly periodic solutions and triply periodic solutions for the Korteweg-de Vries (KdV) equation are first constructed with the aid of the Darboux transformation method.

Keywords: KdV Equation; Doubly Periodic Solution; Triply Periodic Solution; Darboux Transformation

1. Introduction
The famous KdV equation
\[ u_t + 6uu_x + u_{xxx} = 0 \]  (1)
is a shallow water wave equation early derived by Korteweg de and Vries, its first application was discovered in the study of collision-free hydro-magnetic waves in 1960. Subsequently, it has arisen in a number of physical contexts, such as stratified internal waves, ion-acoustic waves, plasma physics, lattice dynamics and so on. Following the further studies of these physical problems, its exact solutions have attracted much attention and have been extensively studied [1-7]. However, in contrast to solitary wave solutions, the analytic periodic solutions represent only a small subclass of its known solutions, and multi-periodic solutions are scarce. It is always useful to seek more and various multi-periodic solutions for recovering interactions among some simple periodic waves in a nonlinear medium.

We know that the Darboux transformation method is the main method to construct exact multi-soliton solutions, and this method is scarcely used for solving multi-periodic solutions [8-10]. In the paper, not only explicit doubly periodic solutions are available, but also a group of explicit triply periodic solutions is obtained by means of the Darboux transformation method.

2. Doubly Periodic Solutions
According to [11], the linear system
\[
\begin{align*}
\Phi &= \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix} \Phi, \\
\Phi &= \begin{pmatrix} u_x - (4\lambda + 2u) \\ A - u_x \end{pmatrix} \Phi,
\end{align*}
\]  (2)
is the Lax pair for Equation (1), with the Darboux matrix
\[
D(x,t,\lambda) = \begin{pmatrix} -\sigma_i & 1 \\ \lambda - \lambda_i + \sigma_i^2 & -\sigma_i \end{pmatrix},
\]  (3)
where \( A = -(4\lambda + 2u)(\lambda - u) + u_{xx} \), \( \lambda, \lambda_i (i = 0,1,2) \) are the spectral parameters. The monograph [11] further points out, if \( u_i \) is a known solution to Equation (1), then
\[ u_{i+1} = 2\lambda_i - u_i - 2\sigma_i^2 \]  (4)
becomes new solution generated from \( u_i \), with
\[ \sigma_i = \frac{a_i^{(1)}(x,t,\lambda_i)\mu_i + a_i^{(2)}(x,t,\lambda_i)\gamma_i}{a_i^{(1)}(x,t,\lambda_i)\mu_i + a_i^{(2)}(x,t,\lambda_i)\gamma_i}, \]  (5)
where, \( \mu_i \) and \( \gamma_i \) are arbitrary constants, but \( \mu_i^2 + \gamma_i^2 \neq 0 \), and \( \Phi_i(x,t,\lambda) = \left( a_i^{(1)}(x,t,\lambda), a_i^{(2)}(x,t,\lambda) \right) \) is the fundamental solution matrix to the lax pair on \( 2u_i \).

Only solving the fundamental solution matrix of the lax pair corresponding to constant solution \( u_0 \), it is possible to construct multi-periodic solutions to the KdV Equation (1). Substituting \( u_0 \) into the system (2) yields

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If setting $\xi = x - (4\lambda + 2u_0)t$, then we can assert that both the system (6) and the following linear system

$$
\Phi_\xi = \begin{pmatrix} 0 & 1 \\ \lambda - u_0 & 0 \end{pmatrix} \Phi,
$$

$$
\Phi_{\xi} = -\left(4\lambda + 2u_0\right) \begin{pmatrix} 0 & 1 \\ \lambda - u_0 & 0 \end{pmatrix} \Phi.
$$

(6)

have exactly the same solutions. Under the condition for $u_0 > \lambda$, by the eigenvalue method, we obtain the complex-valued fundamental solution matrix to the above system

$$
\begin{pmatrix} e^{ia\xi} & e^{-ia\xi} \\ iae^{ia\xi} & -iae^{-ia\xi} \end{pmatrix},
$$

(7)

where $a = a(\lambda) = \sqrt{u_0 - \lambda}$. Because the real and imaginary parts of a complex-valued solution are also solutions, we thus take

$$
\Phi_\theta (x, t, \lambda) = \begin{pmatrix} \cos \theta & \sin \theta \\ -a \sin \theta & a \cos \theta \end{pmatrix}
$$

(8)

as the fundamental solution matrix to the system (6), where $\theta = \theta(\lambda) = a\xi$.

For simplicity, we setting $a_i = a(\lambda), b_i = \theta(\lambda), \Gamma_{ij} = a_j^2 - a_i^2 = \lambda_j - \lambda_1, (i, j = 0, 1, 2)$.

From (5), we have

$$
\sigma_0 = a_0 \frac{-\mu_0 \sin \theta_0 + \gamma_0 \cos \theta_0}{\mu_0 \cos \theta_0 + \gamma_0 \sin \theta_0},
$$

in the above formula, choosing $\mu_0 = 1, \gamma_0 = 0$ and $\mu_0 = 0, \gamma_0 = 1$, respectively, we get

$$
\sigma_{0t} = -a_0 \tan \theta_0
$$

(9)

and

$$
\sigma_{0c} = a_0 \cot \theta_0,
$$

(10)

respectively, with (4), the periodic wave solutions

$$
u_{t,1} = u_0 - 2a_0^2 \sec^2 \theta_0
$$

and

$$
v_{t,2} = u_0 - 2a_0^2 \csc^2 \theta_0
$$

are obtained.

Now we construct the doubly periodic solutions generated from $u_t$, thanks to (4), we see that

$$
u_2 = 2\lambda_1 - \left(2\lambda - u_0 - 2\sigma_0^2\right) - 2\sigma_2^2,
$$

(11)

we first give $\sigma_1$, then substitute $\sigma_0$ and $\sigma_1$ into (11). Again according to [11], we can obtain the fundamental solution matrix to the lax pair associated with the known periodic wave solution $u_t$ in the following manner

$$
\Phi_t (x, t, \lambda) = \begin{pmatrix} -\sigma_0 \\ \lambda - \lambda_1 + \sigma_1 \end{pmatrix},
$$

$$
\Phi_\theta (x, t, \lambda) = \begin{pmatrix} -\sigma_0 \cos \theta - a \sin \theta \\ -\sigma_0 \sin \theta + a \cos \theta \end{pmatrix}
$$

(12)

where $P = \left(\lambda - \lambda_1 + \sigma_1^2\right) \cos \theta + \sigma_0 a \sin \theta$.

$$
Q = \left(\lambda - \lambda_1 + \sigma_0^2\right) \sin \theta - \sigma_0 a \cos \theta.
$$

After combining (5) and (12), choosing $\mu_1 = 1, \gamma_1 = 0$, we get

$$
\sigma_{1c} = -\frac{(\lambda - \lambda_1 + \sigma_0^2) + \sigma_0 a \tan \theta_1}{\sigma_0 + a \tan \theta_1},
$$

(13)

Substituting (9) and (13) into (11), we have new doubly periodic solution

$$
u_{2,1} = u_0 + \frac{2\Gamma_{01} \left(a_0^2 \sec^2 \theta_0 - a_1^2 \csc^2 \theta_0\right)}{(a_0 \tan \theta_0 - a_1 \tan \theta_1)^2},
$$

(14)

Again substituting (10) and (13) into (11), we obtain another new doubly periodic solution

$$
u_{2,2} = u_0 + \frac{2\Gamma_{01} \left(a_0^2 \csc^2 \theta_0 - a_1^2 \sec^2 \theta_0\right)}{(a_0 \cot \theta_0 + a_1 \tan \theta_1)^2}.
$$

(15)

Similarly, choosing $\mu_1 = 0, \gamma_1 = 1$, we have

$$
\sigma_{1c} = -\frac{(\lambda - \lambda_0 + \sigma_1^2) - \sigma_0 a \cot \theta_1}{\sigma_0 a \cot \theta_1},
$$

(16)

which implies the doubly periodic solutions

$$
u_{2,3} = u_0 + \frac{2\Gamma_{01} \left(a_0^2 \sec^2 \theta_0 - a_1^2 \csc^2 \theta_0\right)}{(a_0 \tan \theta_0 + a_1 \cot \theta_1)^2},
$$

(17)

and

$$
u_{2,4} = u_0 + \frac{2\Gamma_{01} \left(a_0^2 \csc^2 \theta_0 - a_1^2 \sec^2 \theta_0\right)}{(a_0 \cot \theta_0 - a_1 \cot \theta_1)^2}.
$$

(18)

Specially, although $u_{2,3}$ is a doubly periodic solution, its structure is very similar to a given two-soliton solution in [1].

3. Triply Periodic Solutions

As shown in [11], the fundamental solution matrix to the lax pair associated with the doubly periodic wave solution $u_2$ can be given by

$$
\Phi_t (x, t, \lambda) = \begin{pmatrix} -\sigma_1 \\ \lambda - \lambda_0 + \sigma_1^2 \end{pmatrix},
$$

(19)

substituting (12) into (19), in exactly the same manner as in Section 2, we get
\[ \sigma_{2i} = \frac{(\lambda_i - \lambda_2)(\sigma_0 + a_2 \tan \theta_i)}{\lambda_2 - \lambda_0 + (\sigma_0 + \sigma_1)(\sigma_0 + a_2 \tan \theta_2)} - \sigma_i \]

and

\[ \sigma_{2c} = \frac{(\lambda_i - \lambda_2)(\sigma_0 - a_2 \cot \theta_i)}{\lambda_2 - \lambda_0 + (\sigma_0 + \sigma_1)(\sigma_0 - a_2 \cot \theta_2)} - \sigma_i. \]

Owing to (4) and (11), we have

\[ u_3 = 2(\lambda_0 - u_0 - \sigma_0^2) + 2(\lambda_2 - \lambda_4 + \sigma_1^2 - \sigma_2^2) + u_0. \quad (20) \]

Here, we set \( F_i = a_i \tan \theta_i \), \( G_i = a_i \cot \theta_i \), \( i = 0, 1, 2 \).

Substituting \( \sigma_0, \sigma_1 \) and \( \sigma_{2i} \) into (20), we obtain triply periodic solution

\[ u_{3i} = u_0 + 2a_i^2 \sec^2 \theta_0 + \frac{2\Gamma_{12} \Gamma_0 \left(a_i^2 \sec^2 \theta_0 - a_i^2 \sec^2 \theta_1 \right) (F_0 - F_2)}{\left[ \Gamma_0 (F_1 - F_0) + \Gamma_{10} (F_0 - F_2) \right]^2} + \frac{2\Gamma_{12} \Gamma_0 \left(a_i^2 \sec^2 \theta_3 - a_i^2 \sec^2 \theta_0 \right) (F_1 - F_0)}{\left[ \Gamma_0 (F_1 - F_0) + \Gamma_{10} (F_0 - F_2) \right]^2}. \]

Similarly, we have

\[ u_{3c} = u_0 + 2a_i^2 \sec^2 \theta_0 + \frac{2\Gamma_{12} \Gamma_0 \left(a_i^2 \sec^2 \theta_0 - a_i^2 \sec^2 \theta_1 \right) (G_0 + F_2)}{\left[ \Gamma_0 (G_1 + G_0) - \Gamma_{10} (G_0 + F_2) \right]^2} + \frac{2\Gamma_{12} \Gamma_0 \left(a_i^2 \sec^2 \theta_3 - a_i^2 \sec^2 \theta_0 \right) (G_1 + F_0)}{\left[ \Gamma_0 (G_1 + G_0) - \Gamma_{10} (G_0 + F_2) \right]^2}. \]

\[ u_{3s} = u_0 + 2a_i^2 \sec^2 \theta_0 + \frac{2\Gamma_{12} \Gamma_0 \left(a_i^2 \csc^2 \theta_0 - a_i^2 \csc^2 \theta_1 \right) (F_0 - F_2)}{\left[ \Gamma_0 (G_1 + G_0) + \Gamma_{10} (G_0 + F_2) \right]^2} + \frac{2\Gamma_{12} \Gamma_0 \left(a_i^2 \csc^2 \theta_3 - a_i^2 \csc^2 \theta_0 \right) (G_1 + F_0)}{\left[ \Gamma_0 (G_1 + G_0) + \Gamma_{10} (G_0 + F_2) \right]^2}. \]

\[ u_{3t} = u_0 + 2a_i^2 \csc^2 \theta_0 + \frac{2\Gamma_{12} \Gamma_0 \left(a_i^2 \csc^2 \theta_0 - a_i^2 \csc^2 \theta_1 \right) (G_0 + F_2)}{\left[ \Gamma_0 (G_1 + G_0) + \Gamma_{10} (G_0 + F_2) \right]^2} + \frac{2\Gamma_{12} \Gamma_0 \left(a_i^2 \csc^2 \theta_3 - a_i^2 \csc^2 \theta_0 \right) (G_1 + F_0)}{\left[ \Gamma_0 (G_1 + G_0) + \Gamma_{10} (G_0 + F_2) \right]^2}. \]

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