An Improvement of a Known Unique Common Fixed Point Result for Four Mappings on 2-Metric Spaces*

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ABSTRACT

In this paper, we introduce a new class $\Gamma$, which is weaker than a known class $\Psi$, of real continuous functions defined on $[0, +\infty)$, and use another method to prove the known unique common fixed point theorem for four mappings with $\gamma$ -contractive condition instead of $\psi$ -contractive condition on 2-metric spaces.

Keywords: 2-Metric Space; Class $\Gamma$; Class $\Psi$; Common Fixed Point

1. Introduction

The second author has obtained an unique common fixed point theorem for four mappings with $\psi$ -contractive condition [1,2] on 2-metric spaces in [1], where $\psi$ is a continuous and non-decreasing real function on $[0, +\infty)$ satisfying that $\psi(t) < t$ for all $t > 0$. The result generalizes and improves many corresponding results.

Here, we introduce a new class $\Gamma$ of real functions defined on $[0, +\infty)$, and reprove the well known unique common fixed point theorem for four mappings with $\psi$ -contractive condition replaced by $\gamma$ -contractive condition on 2-metric spaces. The method used in this paper is very different from that in [1].

At first, we give well known definitions and results.

Definition 1.1. ([3,4]) A 2-metric space $(X, d)$ consists of a nonempty set $X$ and a function
$$d : X \times X \times X \rightarrow [0, +\infty)$$
such that
1) for distant elements $x, y \in X$, there exists an $u \in X$ such that $d(x, y, u) \neq 0$;
2) $d(x, y, z) = 0$ if and only if at least two elements in $\{x, y, z\}$ are equal;
3) $d(x, y, z) = d(u, v, w)$, where $\{u, v, w\}$ is any permutation of $\{x, y, z\}$;
4) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$ for all $x, y, z, u \in X$.

Definition 1.2. ([3,4]) A sequence $\{x_n\}_{n \in \mathbb{N}}$ in 2-metric space $(X, d)$ is said to be cauchy sequence, if for each $\varepsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $x \in X$ and $n, m > N$.

Definition 1.3. ([5,6]) A sequence $\{x_n\}_{n \in \mathbb{N}}$ is said to be convergent to $x \in X$, if for each $a \in X$,
$$\lim_{n \rightarrow \infty} d(x_n, x, a) = 0.$$ And write $x_n \rightarrow x$ and call $x$ the limit of $\{x_n\}_{n \in \mathbb{N}}$.

Definition 1.4. ([5,6]) A 2-metric space $(X, d)$ is said to be complete, if every cauchy sequence in $X$ is convergent.

Definition 1.5. ([7,8]) Let $f$ and $g$ be two self-mappings on a set $X$. If $w = fx = gx$ for some $x \in X$, then $x$ is called a coincidence point of $f$ and $g$, and $w$ is called a point of coincidence of $f$ and $g$.

Definition 1.6. ([9]) Two mappings $f, g : X \rightarrow X$ are said to be weakly compatible if, for every $x \in X$, holds $fgx =gfx$ whenever $fx = gx$.

The following three lemmas are known results.

Lemma 1.7. ([3-6]) Let $(X, d)$ be a 2-metric space and $\{x_n\}_{n \in \mathbb{N}}$ a sequence. If there exists $h \in [0,1)$ such that
$$d(x_{n+2}, x_{n+1}, a) \leq h d(x_{n+1}, x_n, a)$$
for all $a \in X$ and $n \in \mathbb{N}$, then $d(x_n, x_m) = 0$ for all $n, m, l \in \mathbb{N}$, and $\{x_n\}_{n \in \mathbb{N}}$ is a cauchy sequence.

Lemma 1.8. ([3-6]) If $(X, d)$ is a 2-metric space and
sequence \( \{x_n\}_{n \in \mathbb{N}} \to x \in X \), then
\[
\lim_{n \to +\infty} d(x_n, b, c) = d(x, b, c)
\]
for each \( b, c \in X \).

**Lemma 1.9.** (7,8) Let \( f, g : X \to X \) be weakly compatible. If \( f \) and \( g \) have a unique point of coincidence \( w = f^k = g^k \), then \( w \) is the unique common fixed point of \( f \) and \( g \).

**2. Main Results**

Denote by \( \Gamma \) the set of functions \( \gamma : [0, +\infty) \to [0, +\infty) \) satisfying the following:

(i) \( \gamma \) is continuous; (ii) \( \gamma(t) < t \) for all \( t > 0 \).

Denote by \( \Psi^{[l]} \) the set of functions
\[
\psi : [0, +\infty) \to [0, +\infty),
\]
satisfying the following:

(i) \( \psi \) is continuous and non-decreasing; (ii) \( \psi(t) < t \) for all \( t > 0 \).

Obviously, \( \psi \) is stronger than \( \gamma \).

\[
d(Sx, Ty, a) \leq q\gamma\left\{ \max\left\{ d(Jx, Ly, a), d(Jx, Sx, a), d(Ly, Ty, a), \frac{d(Jx, Ty, a)}{2}, \frac{d(Ly, Sx, a)}{2} \right\} \right\},
\]
where \( 0 < q < 1 \) and \( \gamma \in \Gamma \). If one of
\[S(X), T(X), I(X), J(X)\]
and \( J(X) \) is complete, then \( T \) and \( I \), \( S \) and \( J \) have an unique point of coincidence in \( X \). Further, \( \{I, T\} \) and \( \{S, J\} \) are weakly compatible respectively, then \( S, T, I, J \) have an unique common fixed point in
\[
d(y_{2n}, y_{2n+1}, a) = d(Sx_{2n}, Tx_{2n+1}, a)
\]
\[
\leq q\gamma\left\{ \max\left\{ d(Jx_{2n}, Tx_{2n+1}, a), d(Jx_{2n}, Sx_{2n}, a), d(Ly_{2n+1}, Tx_{2n+1}, a), \frac{d(Jx_{2n}, Tx_{2n+1}, a)}{2}, \frac{d(Ly_{2n+1}, Sx_{2n}, a)}{2} \right\} \right\}.
\]

If
\[
\max\left\{ d(y_{2n-1}, y_{2n}, a), d(y_{2n}, y_{2n+1}, a), \frac{d(y_{2n-1}, y_{2n+1}, a)}{2} \right\} = 0
\]
for some \( a \in X \), then \( d(y_{2n}, y_{2n+1}, a) = 0 \), hence we have that
\[
d(y_{2n}, y_{2n+1}, a) = 0
\]
\[
d(y_{2n}, y_{2n+1}, a) \leq qd(y_{2n-1}, y_{2n}, a).
\]
Hence we can assume now that
\[
\max\left\{ d(y_{2n-1}, y_{2n}, a), d(y_{2n}, y_{2n+1}, a), \frac{d(y_{2n-1}, y_{2n+1}, a)}{2} \right\} > 0
\]

**Example 2.1.** Define \( \gamma(x) : [0, +\infty) \to [0, +\infty) \) as follow:
\[
\gamma(x) = \begin{cases} 
\frac{1}{2}x, & \text{for } 0 \leq x \leq 1 \\
\frac{1}{2}x + 1, & \text{for } 1 < x \leq \frac{4}{3} \\
\frac{1}{3}, & \text{for } x > \frac{4}{3}
\end{cases}
\]

Obviously, \( \gamma \in \Gamma \), but since \( \gamma(1) = \frac{1}{2} > \frac{1}{3} = \gamma(2) \), so \( \gamma \notin \mathcal{P} \).

The following is the main conclusion in this paper.

**Theorem 2.2.** Let \((X, d)\) be a 2-metric space, \(S, T, I, J : X \to X\) four mappings satisfying that
\[S(X) \subset I(X) \text{ and } T(X) \subset J(X).\]
Suppose that for each \( x, y, a \in X \),
\[
d(x, y, a) \leq q\gamma\left\{ \max\left\{ d(Jx, Ly, a), d(Jx, Sx, a), d(Ly, Ty, a), \frac{d(Jx, Ty, a)}{2}, \frac{d(Ly, Sx, a)}{2} \right\} \right\},
\]
where \( 0 < q < 1 \) and \( \gamma \in \Gamma \). If one of
\[S(X), T(X), I(X), J(X)\]
and \( J(X) \) is complete, then \( T \) and \( I \), \( S \) and \( J \) have an unique point of coincidence in \( X \). Further, \( \{I, T\} \) and \( \{S, J\} \) are weakly compatible respectively, then \( S, T, I, J \) have an unique common fixed point in
\[
d(y_{2n}, y_{2n+1}, a) = d(Sx_{2n}, Tx_{2n+1}, a)
\]
\[
\leq q\gamma\left\{ \max\left\{ d(Jx_{2n}, Tx_{2n+1}, a), d(Jx_{2n}, Sx_{2n}, a), d(Ix_{2n+1}, Tx_{2n+1}, a), \frac{d(Jx_{2n}, Tx_{2n+1}, a)}{2}, \frac{d(Ix_{2n+1}, Sx_{2n}, a)}{2} \right\} \right\}.
\]
If
\[
\max\left\{ d(y_{2n-1}, y_{2n}, a), d(y_{2n}, y_{2n+1}, a), \frac{d(y_{2n-1}, y_{2n+1}, a)}{2} \right\} = 0
\]
for some \( a \in X \), then \( d(y_{2n}, y_{2n+1}, a) = 0 \), hence we have that
\[
d(y_{2n}, y_{2n+1}, a) = 0
\]
\[
d(y_{2n}, y_{2n+1}, a) \leq qd(y_{2n-1}, y_{2n}, a).
\]
Hence we can assume now that
\[
\max\left\{ d(y_{2n-1}, y_{2n}, a), d(y_{2n}, y_{2n+1}, a), \frac{d(y_{2n-1}, y_{2n+1}, a)}{2} \right\} > 0
\]
for all \( a \in X \).

If
\[
\max \left\{ d(y_{2n-1}, y_{2n}, a), d(y_{2n}, y_{2n+1}, a), \frac{d(y_{2n-1}, y_{2n+1}, a)}{2} \right\} = d(y_{2n}, y_{2n+1}, a)
\]
for some \( a \in X \), then (2) becomes that
\[
d(y_{2n}, y_{2n+1}, a) \leq q \left( d(y_{2n}, y_{2n+1}, a) \right) < qd(y_{2n}, y_{2n+1}, a),
\]
which is a contradiction since \( q < 1 \). Hence we have that
\[
\max \left\{ d(y_{2n-1}, y_{2n}, a), d(y_{2n}, y_{2n+1}, a), \frac{d(y_{2n-1}, y_{2n+1}, a)}{2} \right\} = \max \left\{ d(y_{2n-1}, y_{2n}, a), \frac{d(y_{2n-1}, y_{2n+1}, a)}{2} \right\}
\]
for all \( a \in X \).

If \( d(y_{2n-1}, y_{2n}, a) \geq \frac{d(y_{2n-1}, y_{2n+1}, a)}{2} \) for some \( a \in X \), then from (2),
\[
d(y_{2n}, y_{2n+1}, a) \leq q \left( d(y_{2n-1}, y_{2n}, a) \right) < qd(y_{2n-1}, y_{2n}, a).
\]

(3)

If \( d(y_{2n-1}, y_{2n}, a) \leq \frac{d(y_{2n-1}, y_{2n+1}, a)}{2} \) for some \( a \in X \), then from (2),
\[
d(y_{2n}, y_{2n+1}, a) \leq q \left( d(y_{2n-1}, y_{2n}, a) \right) < \frac{qd(y_{2n-1}, y_{2n+1}, a)}{2}
\]
\[
\leq \frac{q \left[ d(y_{2n-1}, y_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n}, a) + d(y_{2n}, y_{2n+1}, a) \right]}{2},
\]
(4)

If \( d(y_{2n-1}, y_{2n}, y_{2n+1}) > 0 \), then
\[
d(y_{2n-1}, y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1}, y_{2n-1})
\]
\[
\leq q \psi \left( \max \left\{ d(Jx_{2n}, Jx_{2n+1}, y_{2n-1}), d(Jx_{2n}, Sx_{2n}, y_{2n-1}), d(Jx_{2n+1}, Tx_{2n+1}, y_{2n-1}), \frac{d(Jx_{2n}, Tx_{2n+1}, y_{2n-1})}{2} \right\} \right)
\]
\[
= q \psi \left( \left. \frac{d(Jx_{2n}, Sx_{2n}, y_{2n-1})}{2} \right) \right)< qd(y_{2n-1}, y_{2n}, y_{2n+1}),
\]
which is a contradiction since \( 0 < q < 1 \). hence
\[
d(y_{2n-1}, y_{2n}, y_{2n+1}) = 0.
\]
So (4) becomes that
\[
d(y_{2n}, y_{2n+1}, a) < q \left[ d(y_{2n-1}, y_{2n}, a) + d(y_{2n}, y_{2n+1}, a) \right].
\]
(5)

Hence we obtain that
\[
\frac{d(y_{2n}, y_{2n+1}, a)}{2} < q \left( d(y_{2n-1}, y_{2n}, a) + d(y_{2n}, y_{2n+1}, a) \right).
\]
\[
d(y_{2n}, y_{2n+1}, a) \leq \frac{q}{2-q} d(y_{2n-1}, y_{2n}, a),
\]
(6)

By (3) and (6), we obtain that
\[
d(y_{2n}, y_{2n+1}, a) \leq \max \left\{ q, \frac{q}{2-q} \right\} d(y_{2n-1}, y_{2n}, a)
\]
\[
= qd(y_{2n-1}, y_{2n}, a), \forall a \in X.
\]
(7)
Similarly, we can obtain that for each \( n = 0,1,\ldots \),
\[
d(y_{2n+1}, y_{2n+2}, a) \leq qd(y_{2n}, y_{2n+1}, a), \forall a \in X. \tag{8}
\]
Combining (7) and (8), we have that
\[
d(y_{n+1}, y_{n+2}, a) \leq qd(y_{n}, y_{n+1}, a), \forall a \in X. \tag{9}
\]
Hence \( \{y_n\} \) is Cauchy sequence by Lemma 1.7.

Suppose that \( I(X) \) is complete, then there exists \( u \in I(X) \) and \( v \in X \) such that
\[
y_{2n} = Sx_{2n} = Jx_{2n+1} \rightarrow u = Iv.
\]

Let \( n \rightarrow \infty \), then by Lemma 1.8, the above becomes
\[
d(u, Tv, a) \leq qd(u, Tv, a).
\]

If \( d(u, Tv, a) > 0 \) for some \( a \in X \), then we obtain that
\[
d(u, Tv, a) < qd(u, Tv, a),
\]
which is a contradiction since \( 0 < q < 1 \). Hence \( d(u, Tv, a) = 0 \) for all \( a \in X \), so \( Tv = Iv \), i.e., \( u \) is a point of coincidence of \( T \) and \( I \), and \( v \) is a coincidence point of \( T \) and \( I \).

On the other hand, since \( u = Tv \in T(X) \subset J(X) \), there exists \( w \in X \) such that \( u = Jw \). By (1), for any \( a \in X \),
\[
d(Sw, u, a) \leq d(Sw, y_{2n+1}, a) + d(y_{2n+1}, u, a) + d(y_{2n+1}, Sw) = d(Sw, Tx_{2n+1}, a) + d(y_{2n+1}, u, a) + d(y_{2n+1}, Sw)
\]
which is a contradiction since \( 0 < q < 1 \), so \( d(Sw, u, a) = 0 \) for all \( a \in X \). Hence \( Sw = u = Jw \), i.e., \( u \) is a point of coincidence of \( S \) and \( J \), and \( w \) is a coincidence point of \( S \) and \( J \).

If \( z = Sx = Jx \) is another point of coincidence of \( S \) and \( J \), then there exists \( a \in X \) such that \( d(z, u, a) > 0 \), and we have that
\[
d(z, u, a) = d(Sx, Tv, a)
\]
which is a contradiction since \( 0 < q < 1 \), so \( d(Sw, u, a) = 0 \) for all \( a \in X \). Hence \( Sw = u = Jw \), i.e., \( u \) is a point of coincidence of \( S \) and \( J \), and \( w \) is a coincidence point of \( S \) and \( J \).
which is a contradiction. So \( d(z,u,a) = 0 \) for all \( a \in X \), hence \( z = u \), i.e., \( u \) is the unique point of coincidence of \( S \) and \( J \). Similarly, we can prove that \( u \) is also the unique point of coincidence of \( T \) and \( I \).

By Lemma 1.9, \( u \) is the unique common fixed point \( \{S, J\} \) and \( \{T, I\} \) respectively, hence \( u \) is the unique common fixed point of \( S, T, I, J \).

If \( J(X) \) or \( T(X) \) is complete, then we can also use similar method to prove the same conclusion. We omit the part.

The following particular form of Theorem 2.2 for \( \Psi \)-condition is the main result in [1]. The detailed proof can be found in [1].

**Theorem 2.3.** Let \((X,d)\) be a 2-metric space, \( S, T, I, J : X \to X \) four mappings satisfying that \( S(X) \subset I(X) \) and \( T(X) \subset J(X) \). Suppose that for each \( x, y \in X \),

\[
 d(Sx,Ty,a) \leq q\gamma \left\{ \max \left\{ d(Jx,Iy,a), d(Sx,a), d(Iy,Ty,a), \frac{d(Jx,Iy,a)}{2}, \frac{d(Iy,Sx,a)}{2} \right\} \right\}, \forall a \in X, \tag{10}
\]

where \( 0 < q < 1 \) and \( \gamma \in \Gamma \). If one of \( S(X), T(X), I(X) \) and \( J(X) \) is complete, then \( T \) and \( I \), \( S \) and \( J \) have an unique point of coincidence in \( X \).

Further, \( \{I, T\} \) and \( \{S, J\} \) are weakly compatible respectively, then \( S, T, I, J \) have an unique common fixed point in \( X \).

Using Theorem 2.2, we can give many different type fixed point or common fixed point theorems. But we give only the next two contractive or quasi-contractive versions of Theorem 2.2 for two mappings.

**Theorem 2.4.** Let \((X,d)\) be a 2-metric space, \( S, T : X \to X \) two mappings satisfying that for each \( x, y, a \in X \),

\[
 d(Sx,Ty,a) \leq q\gamma \left\{ \max \left\{ d(x,y,a), d(x,a), d(y,Ty,a), \frac{d(x,y,a)}{2}, \frac{d(y,Sx,a)}{2} \right\} \right\}, \forall a \in X, \tag{11}
\]

where \( 0 < q < 1 \) and \( \gamma \in \Gamma \). If one of \( S(X) \) and \( T(X) \) is complete, then \( S \) and \( T \) have an unique common fixed point in \( X \).

**Theorem 2.5.** Let \((X,d)\) be a complete 2-metric space, \( I, J : X \to X \) two surjective mappings. If for each \( x, y, a \in X \),

\[
 d(x,y,a) \leq q\gamma \left\{ \max \left\{ d(Jx,Iy,a), d(Jx,a), d(Iy,y,a), \frac{d(Jx,Iy,a)}{2}, \frac{d(Iy,x,a)}{2} \right\} \right\}, \forall a \in X, \tag{12}
\]

where \( 0 < q < 1 \) and \( \gamma \in \Gamma \). Then \( I \) and \( J \) have an unique common fixed point in \( X \).

**REFERENCES**


