Higher Genus Characters for Vertex Operator Superalgebras on Sewn Riemann Surfaces

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ABSTRACT

We review our recent results on computation of the higher genus characters for vertex operator superalgebras modules. The vertex operator formal parameters are associated to local parameters on Riemann surfaces formed in one of two schemes of (self- or tori-) sewing of lower genus Riemann surfaces. For the free fermion vertex operator superalgebra we present a closed formula for the genus two continuous orbifold partition functions (in either sewings) in terms of an infinite dimensional determinant with entries arising from the original torus Szegő kernel. This partition function is holomorphic in the sewing parameters on a given suitable domain and possesses natural modular properties. Several higher genus generalizations of classical (including Fay’s and Jacobi triple product) identities show up in a natural way in the vertex operator algebra approach.

Keywords: Vertex Operator Superalgebras; Intertwining Operators; Riemann Surfaces; Szegő Kernel; Modular Forms; Theta-Functions; Frobenius—Fay and Jacobi Product Identities

1. Vertex Operator Superalgebras

In this paper (based on several conference talks of the author) we review our recent results [1-5] on construction and computation of correlation functions of vertex operator superalgebras with a formal parameter associated to local coordinates on a self-sewn Riemann surface of genus $g$ which forms a genus $g+1$ surface. In particular, we review result presented in the papers [1-5] accomplished in collaboration with M. P. Tuite (National University of Ireland, Galway, Ireland).

A Vertex Operator Superalgebra (VOSA) [6-10] is a quadruple $(V,Y,1,\omega)$:

$V = V_{\pi} \oplus V_T = \bigoplus_{n \in \mathbb{Z}} V_n$,

$\dim V_n < \infty$, is a superspace, $Y$ is a linear map

$Y: V \to (\text{End} V)[(z,z^{-1})]$,

so that for any vector (state) $u \in V$ we have

$u(k) 1 = \delta_{k,0} u$, $k \geq -1$,

$Y(u,z) = \sum_{n \in \mathbb{Z}} u(n) z^{n-1}$,

$u(n)V_0 \subset V_{\alpha(p(u))}$, $p(u)$-parity.

The linear operators (modes) $u(n): V \to V$ satisfy creativity

$Y(u,z) 1 = u + O(z)$,

and lower truncation

$u(n)v = 0$,

conditions for $u,v \in V$ and $n \gg 0$.

These axioms imply locality, associativity, commutation and skew-symmetry:

$(z_1 - z_2)^n Y(u,z_1)Y(v,z_2)$

$= (-1)^{p(u,v)} (z_1 - z_2)^n Y(v,z_1)Y(u,z_2)$,

$(z_0 + z_2)^n Y(u,z_0 + z_2)Y(v,z_2)w$

$= (z_0 + z_2)^n Y(Y(u,z_0)v,z_2)w$,

$u(k)Y(v,z) - (-1)^{p(u,v)} Y(v,z)u(k)$

$= \sum_{j \in \mathbb{Z}} \binom{k}{j} Y(u(j)v,z)z^{k-j}$,

$Y(u,z)v = (-1)^{p(u,v)} e^{z(j-1)} Y(v,-z)u$,

for $u,v,w \in V$ and integers $m,n \gg 0$.
\[ p(u,v) = p(u) p(v). \]

The vacuum vector \( 1 \in V_{0,0} \) is such that, \( Y(1,z) = \text{Id}_W \), and \( \omega \in V_{1,0} \) the conformal vector satisfies

\[ Y(\omega,z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \]

where \( L(n) \) form a Virasoro algebra for a central charge \( C \),

\[ L(m)L(n) = (m-n)L(m+n) + \frac{C}{12}(m^3 - m)\delta_{m,-n}, \]

\( L(-1) \) satisfies the translation property

\[ Y(L(-1)u,z) = \partial_z Y(u,z). \]

\( L(0) \) describes a grading with

\[ L(0)u = wt(u)u, \quad \text{and} \quad V_r = \{ u \in V \mid wt(u) = r \}. \]

### 1.1. VOSA Modules

**Definition 1** A \( V \)-module for a VOSA \( V \) is a pair \((W,Y_w)\), \( W \) is a \( \mathbb{C} \)-graded vector space \( W = \bigoplus W_r \), \( \dim W_r < \infty \), \( W_{r+n} = 0 \) for all \( r \) and \( n \ll 0 \).

\[ Y_w : V \rightarrow \text{End}(W) \left[ \left[ z, z^{-1} \right] \right], \]

\[ Y_w(u,z) = \sum_{n \in \mathbb{Z}} u_w(n) z^{-n-1}, \]

for each \( u \in V \), \( u_w : \mathbb{C} \rightarrow W \). \( Y_w(1,z) = \text{Id}_W \), and for the conformal vector

\[ Y_w(\omega,z) = \sum_{n \in \mathbb{Z}} L_w(n) z^{-n-2}, \]

where \( L_w(0)w = rw \), \( w \in W_r \). The module vertex operators satisfy the Jacobi identity:

\[ z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_w(u,z_1) Y_w(v,z_2) = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_w(Y(u,z_0) v, z_2) - \left( -1 \right)^{\rho(w)} z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_w(v,z_2) Y_w(u,z_1), \]

Recall that \( \delta(z) = \sum_{n \in \mathbb{Z}} z^n \). The above axioms imply that \( L_w(n) \) satisfies the Virasoro algebra for the same central charge \( C \) and that the translation property

\[ Y_w(L(-1)u,z) = \partial_z Y_w(u,z). \]

### 1.2. Twisted Modules

We next define the notion of a twisted \( V \)-module \([8,11]\). Let \( g \) be a \( V \)-automorphism \( g \), i.e., a linear map preserving \( 1 \) and \( \omega \) such that

\[ gY(v,z)g^{-1} = Y(gv,z), \]

for all \( v \in V \). We assume that \( V \) can be decomposed into \( g \)-eigenspaces

\[ V = \bigoplus_{\rho \in \mathbb{C}} V^\rho, \]

where \( V^\rho \) denotes the eigenspace of \( g \) with eigenvalue \( e^{2\pi \rho} \).

**Definition 2** A \( g \)-twisted \( V \)-module for a VOSA \( V \) is a pair \((W^g,Y_g)\), \( W^g = \bigoplus W^g_r \), \( \dim W^g_r < \infty \), \( W^g_{r+s} = 0 \), for all \( r \), and \( n \ll 0 \). \( Y_g : V \rightarrow \text{End}(W^g) \left[ \left[ z \right] \right] \), the vector space of \( \text{End}(W^g) \)-valued formal series in \( z \) with arbitrary complex powers of \( z \). For \( \rho \in \mathbb{C} \)

\[ Y_g(v,z) = \sum_{n \in \mathbb{Z}} v_g(n) z^{-n-1}, \]

with \( v_g(\rho + l)w = 0 \), \( w \in W^g \), \( l \in \mathbb{Z} \) sufficiently large.

\[ Y_g(1,z) = \text{Id}_{W^g} \]

\[ Y_g(\omega,z) = \sum_{n \in \mathbb{Z}} L_g(n) z^{-n-2}, \]

where \( L_g(0)w = rw \). The \( g \)-twisted vertex operators satisfy the twisted Jacobi identity:

\[ z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_g(u,z_1) Y_g(v,z_2) = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_g(Y(u,z_0) v, z_2) - \left( -1 \right)^{\rho(w)} z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_g(v,z_2) Y_g(u,z_1), \]

for \( u \in V^\rho \).

### 1.3. Creative Intertwining Operators

We define the notion of creative intertwining operators in \([3]\). Suppose we have a VOA \( V \) with a \( V \)-module \((W,Y_w)\).

**Definition 3** A Creative Intertwining Vertex Operator \( Y \) for a VOA \( V \)-module \((W,Y_w)\) is defined by a linear map

\[ Y(w,z) = \sum_{n \in \mathbb{Z}} w(n) z^{-n-1}, \]

for \( w \in W \) with modes \( w(n) : V \rightarrow W \); satisfies creativity

\[ Y(w,z)1 = w + O(z), \]

for \( w \in W \) and lower truncation

\[ w(n)v = 0, \]

for \( v \in V \), \( w \in W \) and \( n \gg 0 \). The intertwining vertex operators satisfy the Jacobi identity:

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for all $u \in V$ and $w \in W$.

These axioms imply that the intertwining vertex operators satisfy translation, locality, associativity, commutativity and skew-symmetry:

$$
\mathcal{Y}(L_u (-1) w, z) = \partial_v \mathcal{Y}(w, z),
$$

$$
(z_1 - z_2)^n \mathcal{Y}(u, z_1) \mathcal{Y}(w, z_2) = (z_1 - z_2)^n \mathcal{Y}(w, z_1) \mathcal{Y}(u, z_2),
$$

$$
(z_0 + z_2)^n \mathcal{Y}(u, z_0 + z_2) \mathcal{Y}(w, z_2) v = (z_0 + z_2)^n \mathcal{Y}(w, u, z_0) w, z_2) v,
$$

$$
u_w (k) \mathcal{Y}(w, z) = \partial_v \mathcal{Y}(w, z) u(k),
$$

$$
\mathcal{Y}(w, z) v = e^{2\pi i (-1) \rho} \mathcal{Y}(v, -z) w,
$$

for $u, v \in V$, $w \in W$ and integers $m, n \gg 0$.

1.4. Example: Heisenberg Intertwiners

Consider the Heisenberg vertex operator algebra $M$,

$$
z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) \mathcal{Y}(u, z_1) \mathcal{Y}(v, z_2),
$$

$$
- C(\alpha, \beta) z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) \mathcal{Y}(v, z_1) \mathcal{Y}(u, z_2),
$$

$$
z_0^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) \mathcal{Y}(u, z_0) \mathcal{Y}(v, z_2),
$$

for all $u \otimes e^\alpha, v \otimes e^\beta \in M$.

1.5. Invariant Form for the Extended Heisenberg Algebra

The definitions of invariant forms [8,13] for a VOSA and its $g$-twisted modules were given by Scheithauer [14] and in [2] correspondingly. A bilinear form $\langle \cdot, \cdot \rangle$ on $M$ is said to be invariant if for all $u \otimes e^\alpha, v \otimes e^\beta, w \otimes e^\gamma \in M$ we have

$$
\langle \mathcal{Y}(u \otimes e^\alpha, z) v \otimes e^\beta, w \otimes e^\gamma \rangle = e^{i \alpha \beta} \langle v \otimes e^\beta, \mathcal{Y}'(u \otimes e^\alpha, z) w \otimes e^\gamma \rangle,
$$

$$
\mathcal{Y}'(u \otimes e^\alpha, z) = \mathcal{Y} \left( e^{iz^2 L(0)} \left( - \frac{\lambda}{z} \right)^2 (u \otimes e^\alpha), - \frac{z^2}{z} \right),
$$

where $\mathcal{Y}$ and $\mathcal{Y}'$ are vertex operator $\left( e^z \mathcal{Y}(z, x) \right)$ and $\left( e^z \mathcal{Y}'(z, x) \right)$.

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We are interested in the Möbius map $z \mapsto w = \frac{\rho z}{z}$ associated with the sewing condition so that $\lambda = -\xi \rho^2$, with $\xi \in \{\pm \sqrt{-1}\}$. We prove in [3]

**Theorem 3 (Tuite-Z) The invariant form $\langle , \rangle$ on $\mathcal{M}$ is symmetric, unique and invertible with**

$$
\langle v \otimes e^\alpha, w \otimes e^\beta \rangle = \lambda^{-\alpha} \delta_{\alpha - \beta} \langle v \otimes e^\alpha, w \otimes e^\beta \rangle.
$$

1.6. Rank Two Free Fermionic Vertex Operator Super Algebra

Consider the Vertex Operator Super Algebra (VOSA) generated by

$$
Y(\psi^+ , z) = \sum_{n \in \mathbb{Z}} \psi^+(n) z^{-n-1},
$$

for two vectors $\psi^+$ with modes satisfying anti-commutation relations

$$
\left\{ \psi^+ (m), \psi^- (n) \right\} = \delta_{m-n-1},
$$

$$
\left\{ \psi^+ (m), \psi^+ (n) \right\} = 0.
$$

The VOSA vector space $V = \oplus_{k=0} V_{k/2}$ is a Fock space with basis vectors

$$
\Psi(\kappa, l) = \psi^+ (-k_1) \cdots \psi^+ (-k_i) \psi^- (-l_1) \cdots \psi^- (-l_l) 1,
$$

of weight

$$
wtt[\Psi(\kappa, l)] = \sum_k (k_1 + \frac{1}{2}) + \sum_l (l_1 + \frac{1}{2}),
$$

where $1 \leq k_1 < k_2 < \cdots < k_i$ and $1 \leq l_1 < l_2 < \cdots < l_l$ with $\psi^+ (k_1) 1 = 0$ for all $k \geq 0$.

1.7. Rank Two Fermionic Vertex Operator Super Algebra

The conformal vector is

$$
\omega = \frac{1}{2} \left[ \psi^+ (-2) \psi^- (-1) + \psi^+ (-2) \psi^- (-1) \right] 1,
$$

whose modes generate a Virasoro algebra of central charge 1. $\psi^+$ has $L(0)$-weight $\frac{1}{2}$. The weight 1 subspace of $V$ is $V_1 = \mathbb{C} a$, for normalized Heisenberg bosonic vector $a = \psi^+ (-1) \psi^- (-1) 1$, the conformal vector, and the Virasoro grading operator are

$$
L(0) = a(0)^2 + \sum_{n \geq 0} a(-n)a(n).
$$

2. Sewing of Riemann Surfaces

2.1. Basic Notions

For standard homology basis $a_i$, $b_i$ with $i = 1, \cdots, g$ on a genus $g$ Riemann surface $[15,16]$ consider the normalized differential of the second kind which is a symmetric meromorphic form with $\oint a_i \omega^{(g)} (z) = 0$, has the form

$$
\omega^{(g)} ( z_1 , z_2 ) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} \text{ for local coordinates } z_1 - z_2.
$$

A normalized basis of holomorphic 1-forms $\nu_i$, the period matrix $\Omega^{(g)}_y$, and normalized differential of the third kind are given by

$$
\nu_i^{(g)} (z) = \oint a_i \omega^{(g)} (z), \quad \oint a_i \nu_j^{(g)} = 2\pi i \delta_{ij},
$$

$$
\Omega^{(g)}_{y} = \frac{1}{2\pi i} \oint a_i \nu_j^{(g)}, \quad \omega^{(g)}_{a_i a_j} (z) = \oint a_i \omega^{(g)} (z),
$$

where $\oint a_i \nu_j^{(g)} = 0$, $\omega^{(g)}_{a_i a_j} (z) = (\frac{-1}{2})^{g} dz$ for $z - p_a$, $a = 1, 2$.

2.2. Period Matrix

$\Omega^{(g)}$ is symmetric with positive imaginary part i.e. $\Omega^{(g)} \in \mathbb{H}_g$, the Siegel upper half plane. The canonical intersection form on cycles is preserved under the action of the symplectic group $Sp(2g, \mathbb{Z})$ where

$$
\begin{pmatrix} b \\ a \end{pmatrix} \rightarrow \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{Z}).
$$

This induces the modular action on $\mathbb{H}_g$

$$
\Omega^{(g)} \rightarrow \Omega^{(g)} = (A \Omega^{(g)} + B)(C \Omega^{(g)} + D)^{-1}.
$$

2.3. Sewing Two Tori to Form a Genus Two Riemann Surface

Consider $[1,17]$ two oriented torus $\Sigma^{(i)}_u = \mathbb{C}/\Lambda_u$ with $a = 1, 2$ for $\Lambda_u = 2\pi i (\mathbb{T} \oplus \tau_u \mathbb{Z})$ for $\tau_u \in \mathbb{H}_1$, the complex upper half plane. For $z_u \in \Sigma^{(i)}_u$ the closed disk $|z_u| \leq r_u$ is contained in $\Sigma^{(i)}_u$ provided $r_u < \frac{1}{2} D(\tau_u)$ where

$$
D(\tau_u) = \min_{z_u} |z| = \text{mininal lattice distance}.
$$

Introduce a sewing parameter $\varepsilon \in \mathbb{C}$ and excise the disks $|z_1| \leq |\varepsilon|/r_1$ and $|z_2| \leq |\varepsilon|/r_2$ where

$$
|\varepsilon| \leq r_1 r_2 < \frac{1}{4} D(\tau_1) D(\tau_2).
$$
Identify the annular regions \( |\epsilon|/r_2 \leq |z| \leq r_1 \) and \( |\epsilon|/r_1 \leq |z| \leq r_2 \) via the sewing relation

\[
gives a genus two Riemann surface \( \Sigma^{(2)} \) parameterized by the domain

\[
D^2 = \left\{ (r_1, r_2, \epsilon) \in \mathbb{H} \times \mathbb{H} \times \mathbb{C} \mid |\epsilon| < \frac{1}{4} D(r_1) D(r_2) \right\}.
\]

2.4. Torus Self-Sewing to Form a Genus Two Riemann Surface

In [1] we describe procedures of sewing Riemann surfaces [17]. Consider a self-sewing of the oriented torus

\[
\Sigma^{(1)} = \mathbb{C}/\Lambda, \quad \Lambda = 2\pi i (\mathbb{Z} \oplus \mathbb{Z}), \quad \tau \in \mathbb{H}.
\]

Define the annuli \( A_a \), \( a = 1, 2 \) centered at \( z = 0 \) and \( z = w \) of \( \Sigma^{(1)} \) with local coordinates \( z_1 = z \) and \( z_2 = z - w \) respectively. We use the convention \( \tilde{1} = 2 \), \( \tilde{2} = 1 \). Take the outer radius of \( A_a \) to be

\[
r_a \leq \frac{1}{2} D(q) = \min_{a, b, \lambda, s} |\lambda|.
\]

Introduce a complex parameter \( \rho \), \( |\rho| \leq r_1 r_2 \). Take inner radius to be \( |\rho|/r_2 \), with \( |\rho| \leq r_1 r_2 \). \( r_1, r_2 \) must be sufficiently small to ensure that the disks do not intersect. Excise the disks

\[
\left\{ z \mid |z| \leq |\rho|^{-1} \right\} \subset \Sigma^{(1)},
\]

to form a twice-punctured surface

\[
\Sigma^{(1)} = \Sigma^{(1)} \setminus \bigcup_{a=1,2} \left\{ z \mid |z| \leq |\rho|^{-1} \right\}.
\]

Identify the annular regions \( A_a \subset \Sigma^{(1)} \),

\[
A_a = \left\{ z \mid |\rho|^{-1} \leq |z| \leq r_a \right\}
\]
as a single region \( A = A_1 = A_2 \) via the sewing relation

\[
z_1 z_2 = \rho,
\]
to form a compact genus two Riemann surface

\[
\Sigma^{(2)} = \Sigma^{(1)} \setminus \{ A_1 \cup A_2 \} \cup A,
\]

parameterized by

\[
D^2 = \left\{ (r, w, \rho) \in \mathbb{H} \times \mathbb{H} \times \mathbb{C} \mid |w - \lambda| > 2 |\rho|^{-1} > 0, \lambda \in \Lambda \right\}.
\]

3. Elliptic Functions

3.1. Weierstrass \( \wp \)-function periodic in \( z \) with periods \( 2\pi i \) and \( 2\pi \tau \) is

\[
\wp(z, \tau) = \frac{1}{z^2} + \sum_{m \neq 0, \omega \neq 0} \left[ \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right]
\]

\[
= \frac{1}{z^2} + \sum_{n \geq 1} \left( n - 1 \right) E_n(z^2) \zeta_n^2,
\]

for \( (z, \tau) \in \mathbb{H} \times \mathbb{H} \), \( \omega_n = 2\pi i (m\tau + n) \). We define for

\[
k \geq 1,
\]

\[
P_k(z, \tau) = \left( \frac{1}{k-1} \right) \partial^{k-1} P_i(z, \tau)
\]

\[
= \frac{1}{z^2} + (-1)^k \sum_{n \geq 1} \left( n - 1 \right) E_n(z^2) \zeta_n^{2k}.
\]

Then

\[
P_k(z, \tau) = \wp(z, \tau) + E_2(z, \tau).
\]

\( P_k \) has periodicities

\[
P_k(z + 2\pi i, \tau) = P_k(z, \tau),
\]

\[
P_k(z + 2\pi \tau, \tau) = P_k(z, \tau) - \delta_{k1}.
\]

3.2. Eisenstein Series

The Eisenstein series \( E_n(\tau) \) is equal to 0 for \( n \) odd, and for \( n \)

\[
E_n(\tau) = \frac{B_n(0)}{n!} + \frac{2}{(n-1)!} \sum_{q=1}^{\infty} \frac{r^{n-1} q^q}{1 - q^q},
\]

where \( B_n(0) \) is the \( n \)th Bernoulli number. If \( n \geq 4 \) then

\( E_n(\tau) \) is a holomorphic modular form of weight \( n \) on \( \text{SL}(2, \mathbb{Z}) \)

\[
E_n(\gamma \cdot \tau) = (c\tau + d)^n E_n(\tau),
\]

for all \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \), where \( \gamma \cdot \tau = \frac{a\tau + b}{c\tau + d} \).
$E_z(\tau)$ is a quasimodular form

$$E_z(\gamma \cdot \tau) = (c\tau + d)^2 E_z(\tau) - \frac{c(c\tau + d)}{2\pi i},$$

having the exceptional transformation law.

### 3.3. The Theta Function

We recall the definition of the theta function with real characteristics [18]

$$\varphi^{(\beta)}\left[\alpha, \beta\right](z\Omega(s)) =$$

$$\sum_{m \in \mathbb{Z}^g} \exp\left(i\pi (m + \alpha) \cdot \Omega(s) \cdot (m + \alpha) + (m + \alpha) \cdot (z + 2\pi i \beta)\right),$$

for

$$\alpha = (\alpha_j), \beta = (\beta_j) \in \mathbb{R}^g, z = (z_j) \in \mathbb{C}^g,$$

$$\theta_j = -e^{-2\pi i \beta_j}, \phi_j = -e^{2\pi i \alpha_j}, j = 1, \ldots, g,$$

$$\varphi^{(\beta)}\left[\alpha, \beta\right](z + 2\pi i \Omega(s) \cdot r + s) \Omega(s) \right) =$$

$$= e^{2\pi i \alpha \cdot r} e^{2\pi i \alpha \cdot \beta} \varphi^{(\beta)}\left[\alpha, \beta\right](z\Omega(s)),

= e^{2\pi i \alpha \cdot r + s} \varphi^{(\beta)}\left[\alpha + s, \beta + s\right](z\Omega(s))$$

for $r, s \in \mathbb{Z}^g$.

### 3.4. Twisted Elliptic Functions

Let $(\theta, \phi) \in U(1) \times U(1)$ denote a pair of modulus one complex parameters with $\phi = \exp(2\pi i \lambda)$ for $0 \leq \lambda < 1$. For $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ we define “twisted” Weierstrass functions for $k \geq 1$ [19,20]

$$P_k\left[\theta, \phi\right](z, \tau) = \frac{(-1)^k}{(k-1)! \sum_{q \in \mathbb{Z}^g}} \sum_{r \in \mathbb{Z}^g} \sum_{\gamma \in \mathbb{Z}^g} q^{n-1} q^r q^{n-1},$$

for $q = q_{2\pi i \tau}$ where $\sum$ means we omit $n = 0$ if

$$(\theta, \phi) = (1, 1), P_k\left[\theta, \phi\right](z, \tau)$$

converges absolutely and uniformly on compact subsets of the domain $|q| < q_\lambda < 1$ [20].

**Lemma 1 (Mason-Tuite-Z)** For $(\theta, \phi) \neq (1, 1)$,

$$P_k\left[\theta, \phi\right](z, \tau)$$

is periodic in $z$ with periods $2\pi i \tau$ and $2\pi i$ with multipliers $\theta$ and $\phi$ respectively.

### 3.5. Modular Properties of Twisted Weierstrass Functions

Define the standard left action of the modular group for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = SL(2, \mathbb{Z})$$

on $(z, \tau) \in \mathbb{C} \times \mathbb{H}$ with

$$\gamma \cdot (z, \tau) = (\gamma \cdot z, \gamma \cdot \tau) = \begin{pmatrix} z & a \tau + b \\ c \tau + d \end{pmatrix}.$$

We also define a left action of $\Gamma$ on $(\theta, \phi)$

$$\gamma \left[\theta, \phi\right] = \left[\theta^\phi, \phi^\theta\right].$$

Then we obtain:

**Theorem 4 (Mason-Tuite-Z)** For $(\theta, \phi) \neq (1, 1)$ we have

$$P_k\left[\gamma \left[\theta, \phi\right]\right](z, \tau) = (c\tau + d)^k P_k\left[\theta, \phi\right](z, \tau).$$

### 3.6. Twisted Eisenstein Series

We introduce twisted Eisenstein series for $n \geq 1$,

$$E_n\left[\theta, \phi\right](\tau) = \frac{B_n(\lambda)}{n!} + \frac{1}{(n-1)! \sum_{r \in \mathbb{Z}^g}} \sum_{\gamma \in \mathbb{Z}^g} (r + 1) \delta_n \theta^r q^{r, \lambda}$$

$$+ \delta_n \sum_{r \in \mathbb{Z}^g} (r - \lambda) n! \theta q^{r, \lambda},$$

where $\sum$ means we omit $r = 0$ if $(\theta, \phi) = (1, 1)$ and

where $B_n(\lambda)$ is the Bernoulli polynomial defined by

$$\frac{q^t}{q - 1} = 1 + \sum_{n = 1}^\infty B_n(\lambda) z^{n-1}.$$

In particular

$$B_n(\lambda) = \lambda - \frac{1}{2}.$$

Note that

$$E_n\left[1\right](\tau) = E_n(\tau),$$

the standard Eisenstein series for even $n \geq 2$, whereas

$$E_n\left[1\right](\tau) = -B_n(0) \delta_n z^{n-1} = \frac{1}{2} \delta_n z^{n-1}$$

for $n$ odd.

**Theorem 5 (Mason-Tuite-Z)** We have

$$P_k\left[\theta, \phi\right](z, \tau) = \frac{1}{z^k} + \sum_{n = 1}^{n-1} \sum_{a \in \mathbb{Z}^g, c \in \mathbb{Z}^g} E_n\left[\theta, \phi\right](\tau) z^{n-1}.$$
Theorem 6 (Mason-Tuite-Z) For \((\theta, \phi) \neq (1,1)\),
\[ E_k(\theta, \phi) \] is a modular form of weight \( k \) where
\[ E_k(\gamma, \phi)(\tau) = (c\tau + d)^k E_k(\phi)(\tau). \]

3.7. Twisted Elliptic Functions

In particular,
\[ P_i(\theta, \phi)(z, \tau) = \frac{1}{z} - \sum_{k \geq 1} \left[ \sum_{n \geq 1} \frac{\theta^n \phi^n}{(m^2 + n)^k} \right] \]
\[ P_i(\theta, \phi)(z - z', \tau) = \frac{1}{z - z'} + \sum_{k, j \geq 1} C(k, l) \theta(k, l) \zeta^k \zeta'^l, \]
where
\[ C(k, l, \tau) = (-1)^k \left( \frac{k + l - 2}{k - 1} \right) E_{k+l-1}(\theta)(\tau), \]
and
\[ D(k, l, \tau, z) = (-1)^{k+l} \left( \frac{k + l - 2}{k - 1} \right) P_{k+l-1}(\theta)(\tau, z). \]

4. The Prime Form

There exists a (nonsingular and odd) character \([\gamma, \delta]\) such that [18,21,22]
\[ \delta \left[ \begin{array}{c} \gamma \\ \delta \end{array} \right] (0|\Omega^{(\gamma)}) = 0, \]
\[ \partial_{\gamma} \delta \left[ \begin{array}{c} \gamma \\ \delta \end{array} \right] (0|\Omega^{(\gamma)}) \neq 0. \]

Let
\[ \zeta(z) = \sum_{\gamma} \delta \left[ \begin{array}{c} \gamma \\ \delta \end{array} \right] (0|\Omega^{(\gamma)}) \psi^{(\gamma)}(z), \]
be a holomorphic 1-form, and let \(\zeta(z)^{1/2}\) denote the form of weight \(1/2\) on the double cover \(\Sigma^{(\gamma)}\) of \(\Sigma^{(\gamma)}\).

We define the prime form
\[ E^{(\gamma)}(z, z') = \frac{\delta \left[ \begin{array}{c} \gamma \\ \delta \end{array} \right] (0|\Omega^{(\gamma)})}{\zeta(z)^{1/2} \zeta(z')^{1/2}} \]
\[ - (z - z') \zeta(z) \zeta(z')^{-1/2} \]
for \(z \neq z'.\)

The prime form is anti-symmetric,
\[ E^{(\gamma)}(z, z') = -E^{(\gamma)}(z', z), \]
and a holomorphic differential of weight
\[ \left( \frac{-1}{2}, -1 \right) \]
on \(\Sigma \times \Sigma\),
and has multipliers \(1\) and \(e^{-i\pi l y} \sum_{j=1}^{J_0} \delta_{j}(\gamma)\) along the \(a_i\) and \(b_j\) cycles in \(z\) [21]. The normalized differentials of the second and third kind can be expressed in terms of the prime form [18]
\[ \omega^{(\gamma)}(z, z') = \partial_{\gamma} \partial_{\gamma} \log E^{(\gamma)}(z, z') dz dz', \]
\[ \omega^{(\gamma)}_{p,q}(z) = \partial_{\gamma} \log \frac{E^{(\gamma)}(z, p)}{E^{(\gamma)}(z, q)} dz. \]

Conversely, we can also express the prime form in terms of \(\omega^{(\gamma)}_{p,q}\) by [22]
\[ E^{(\gamma)}(z, z') = \lim_{p \to \infty} \left[ \sqrt{(z - p)(q - z')} \exp \left( -\frac{1}{2} \sum_{j=1}^{J_0} \delta_{j}(\gamma) \right) \right] \frac{1}{2} \frac{1}{2}. \]

Torus Prime Form

The prime form on torus [18]
\[ E^{(\gamma)}(z, z') = K^{(\gamma)}(z - z', \tau) dz \zeta(z)^{1/2} \zeta(z')^{1/2}, \]
\[ K^{(\gamma)}(z, \tau) = \partial_{\gamma} \partial_{\gamma}(0, \tau), \]
for \(z \in \mathbb{C}\) and \(\tau \in \mathbb{H}_1\) and where
\[ \partial_{\gamma}(z, \tau) = \partial \left[ \begin{array}{c} 1 \\ 2 \end{array} \right](z, \tau). \]

We have
\[ K^{(\gamma)}(z, \tau) = \exp(-P_0(z, \tau)), \]
\[ P_0(z, \tau) = -\log(z) + \sum_{k \geq 1} K_{\gamma}(z, \tau)^k, \]
\[ P_i(z,\tau) = -\frac{d}{dz} P_0(z,\tau) = \frac{1}{z} - \sum_{k=2} E_k(\tau) z^{k-1}. \]

\[ K^{(1)}(z,\tau) \text{ has periodicities} \]
\[ K^{(1)}(z + 2\pi i,\tau) = -K^{(1)}(z,\tau), \]
\[ K^{(1)}(z + 2\pi r,\tau) = -q z^{-1/2} K^{(1)}(z,\tau). \]

5. The Szegö Kernel
The Szegö Kernel \[ [18,21,22] \] is defined by
\[ S^{(g)}[\theta](z,z') \Omega^{(g)} = \frac{g}{\alpha(0)} \left( \int_{\Sigma^g} E^{(g)}(z,z') \right) \frac{1}{z - z'} \quad \text{for } z = z', \]
where \( g[\alpha(0) \neq 0] \) \( \theta_j = -e^{2\pi i \beta_j}, \phi_j = -e^{2\pi i \alpha_j}, \)
\( j = 1, \ldots, g, \) where \( E^{(g)}(z_1, z_2) \) is the genus \( g \) prime form. The Szegö kernel has multipliers along the \( a_j \) and \( b_j \) cycles in \( z \) given by \( \phi \) and \( \theta \) respectively
\[ S^{(g)}[\theta]\phi](z,z') = -S^{(g)}[\theta^{-1}](z,z') \]
where \( \theta^{-1} = (\theta_1^{-1}) \) and \( \phi^{-1} = (\phi_1^{-1}) \).

Finally, we describe the modular invariance of the Szegö kernel under the symplectic group \( Sp(2g,\mathbb{Z}) \) where we find [21]
\[ S^{(g)}[\tilde{\theta}][\tilde{\phi}](z,z') \tilde{\Omega}^{(g)} = S^{(g)}[\theta][\phi](z,z') \Omega^{(g)}, \]
with \( \tilde{\theta}_j = -e^{2\pi i \beta_j}, \tilde{\phi}_j = -e^{2\pi i \alpha_j}, \)
\[ \left( -\tilde{\beta} \right) \left( A \ B \right) \left( -\tilde{\beta} \right) \left( C \ D \right) \left( \alpha \right) + \frac{1}{2} \left( \diag \left( AB^T \right) \right) \]
\[ \tilde{\Omega}^{(g)} = \left( A \Omega^{(g)} + B \right) \left( C \Omega^{(g)} + D \right)^{-1}, \]
where \( \diag(M) \) denotes the diagonal elements of a matrix \( M \).

5.1. Modular Properties of the Szegö Kernel
Finally, we describe the modular invariance of the Szegö kernel under the symplectic group \( Sp(2g,\mathbb{Z}) \) where we find [21]
\[ S^{(g)}[\theta](z',z) \tilde{\Omega}^{(g)} = S^{(g)}[\tilde{\theta}](z',z) \tilde{\Omega}^{(g)}, \]
where \( \tilde{\theta} = -e^{2\pi i \beta_j}, \tilde{\phi} = -e^{2\pi i \alpha_j} \)
\[ \left( -\tilde{\beta} \right) \left( A \ B \right) \left( -\tilde{\beta} \right) \left( C \ D \right) \left( \alpha \right) + \frac{1}{2} \left( \diag \left( AB^T \right) \right) \]
\[ \tilde{\Omega}^{(g)} = \left( A \Omega^{(g)} + B \right) \left( C \Omega^{(g)} + D \right)^{-1}, \]
\[ \text{for Weierstrass function} \]
\[ \phi(z,\tau) = \frac{1}{2} \sum_{k=4} (k-1) E_k(\tau) z^{k-2}, \]
\[ \phi(z,\tau) = \frac{1}{2} \sum_{k=4} (k-1) E_k(\tau) z^{k-2}, \]
and Eisenstein series for $k \geq 2$

$$E_k(\tau) = \frac{1}{(2\pi i)^k} \sum_{n=1}^{\infty} \frac{1}{n^k} \sum_{m=-n}^{n} \frac{1}{(m\tau + n)^k}.$$  

$E_k$ vanishes for odd $k$ and is a weight $k$ modular form for $k \geq 4$. $E_2$ is a quasi-modular form. Expanding

$$P_2(z_1 - z_2, \tau) = \frac{1}{(z_1 - z_2)^2} + \sum_{k,l=1}^{\infty} C(k,l) z_1^{k-1} z_2^{l-1},$$

$$C(k,l) = C(k,l,\tau) = (-1)^{k+1} \frac{(k+l-1)!}{(k-1)!(l-1)!} E_{k+l}(\tau),$$

we compute $\omega^{(2)}(z_1, z_2)$ in the sewing scheme in terms of the following genus one data, $a = 1, 2$

$$A_a(k,l,\tau, \epsilon) = \frac{\epsilon^{(k+l)/2}}{\sqrt{kl}} C(k,l,\tau_a).$$

6.2. A Determinant and the Period Matrix

Consider the infinite matrix $I - A_1 A_2$ where $I$ is the infinite identity matrix and define $\det(I - A_1 A_2)$ by

$$\log \det(I - A_1 A_2) = \Tr \log(I - A_1 A_2)$$

$$= -\sum_{n \geq 1} \frac{1}{n} \Tr \left( (A_1 A_2)^n \right),$$

as a formal power series in $\epsilon$ [23].

**Theorem 7 (Mason-Tuite)**

(a) The infinite matrix

$$(I - A_1 A_2)^{-1} = \sum_{n \geq 0} (A_1 A_2)^n,$$

is convergent for $(\tau_1, \tau_2, \epsilon) \in D^\times$.

(b) $\det(I - A_1 A_2)$ is non-vanishing and holomorphic on $D^\times$.

Furthermore we may obtain an explicit formula for the genus two period matrix $\Omega = \Omega^{(2)}$ on $\Sigma^{(2)}$ [23].

**Theorem 8 (Mason-Tuite)**

$\Omega = \Omega(\tau_1, \tau_2, \epsilon) = $ holomorphic on $D^\times$ and is given by

$$2\pi i \Omega_{11} = 2\pi i \tau_1 + \epsilon \left( A_2 (I - A_1 A_2)^{-1} \right)(1,1),$$

$$2\pi i \Omega_{22} = 2\pi i \tau_2 + \epsilon \left( A_1 (I - A_1 A_2)^{-1} \right)(1,1),$$

$$2\pi i \Omega_{12} = -\epsilon (I - A_1 A_2)^{-1}(1,1).$$

Here $(1,1)$ refers to the $(1,1)$-entry of a matrix.

6.3. Genus Two Szegö Kernel on $\Sigma^{(2)}$ in the $\epsilon$-Formalism

We may compute $S^{(2)}[\theta_1 \phi_1](z, z')$ for $\theta = (\theta_1, \theta_2)$ in the sewing scheme in terms of the genus one data

$$F_2(k,l) = F_2 \left[ \delta_a \left( k, l, \tau_a, \epsilon \right) \right] = \epsilon^{\frac{1}{2}(k+l-1)} C \left[ \delta_a \left( k, l, \tau_a \right) \right].$$

$S^{(2)}$ is described in terms of the infinite matrix $I - Q$ for

$$Q = \begin{bmatrix} 0 & \xi F \left[ \theta_1 \phi_1 \right] \\ -\xi^2 F_2 \left[ \theta_2 \phi_2 \right] & 0 \end{bmatrix}, \quad \xi = -1.$$  

**Theorem 9 (Tuite-Z)**

(a) The infinite matrix $(I - Q)^{-1} = \sum_{n \geq 0} Q^n$ is convergent for $(\tau_1, \tau_2, \epsilon) \in D^\times$.

(b) $\det(I - Q)$ is non-vanishing and holomorphic on $D^\times$.

6.4. Genus Two Szegö Kernel in the $\rho$-Formalism

It is convenient to define $\kappa \in \left[ -\frac{1}{2}, \frac{1}{2} \right]$ by $\phi_\kappa = -e^{2\pi i \kappa}$.

Then we prove [1] the following

**Theorem 10 (Tuite-Z)**

$S^{(2)}_\kappa(x, y) = S^{(2)}_\kappa(x, y) + O(\rho)$, for $x, y \in \Sigma^{(2)}$ where $S^{(2)}_\kappa(x, y)$ is defined for $\kappa \neq -\frac{1}{2}$, by

$$S^{(2)}_\kappa \left[ \frac{\alpha_1}{\beta_1} \right](x, y) = \left( \frac{\delta_1(x-w, \tau) \delta_1(y, \tau)}{\delta_1(x, \tau) \delta_1(y-w, \tau)} \right)^\kappa \cdot \frac{\alpha_1}{\beta_1} \left( \frac{1}{(x-y+\kappa w, \tau)} \right)^\frac{1}{2} \frac{1}{(x-y-\kappa w, \tau)}^{\frac{1}{2}}$$

with similar expression for $S^{(2)}_\kappa(x, y)$ for $\kappa = -\frac{1}{2}$.

Let $k_a = k + (-1)^a \kappa$, for $a = 1, 2$ and integer $k \geq 1$.

We introduce the moments for $S^{(2)}_\kappa(x, y)$:
with associated infinite matrix \( G = (G_{ab}(k,l)) \). We define also half-order differentials

\[
\begin{align*}
    h_a(k,x) &= h_a \left[ \phi^{(1)}(k;\kappa,k,l) \right], \\
    \bar{h}_a(k,y) &= \bar{h}_a \left[ \phi^{(1)}(k;\kappa,k,l) \right],
\end{align*}
\]

and let \( h(x) = (h_a(k,x)) \) and \( \bar{h}(y) = (\bar{h}_a(k,y)) \), denote the infinite row vectors indexed by \( a, k \). From the sewing relation \( z_iz_i = \rho \) we have

\[
\frac{1}{\pi} \frac{dz_i}{z_i} = (\xi \rho^2)^{\frac{1}{2}} \frac{dz_i}{x_i},
\]

for \( \xi \in \{ \pm 1 \} \), depending on the branch of the double cover of \( \Sigma^{(1)} \) chosen. It is convenient to define

\[
T = \xi GD^\rho,
\]

with an infinite diagonal matrix

\[
D^\rho(k,l) = \left[ \begin{array}{cc} \theta^{-1} & 0 \\ 0 & -\theta \end{array} \right] \delta(k,l).
\]

Defining \( \det(I-T) \) by the formal power series in \( \rho \)

\[
\log \det(I-T) = \Tr \log(I-T) = -\sum_{n \geq 1} \frac{1}{n} \Tr(T^n),
\]

we prove in [1].

**Theorem 11 (Tuite-Z)**

a) \( (I-T)^{-1} = \sum_{n \geq 0} T^n \) is convergent for \( |\rho| < r I \),

b) \( \det(I-T) \) is non-vanishing and holomorphic in \( \rho \) on \( D^\rho \).

**Theorem 12 (Tuite-Z)**

\( S^{(2)}(x,y) \) is given by

\[
S^{(2)}(x,y) = S_{(1)}^{(1)}(x,y) + \xi h(x) D^\rho (I-T)^{-1} \bar{h}^T(y).
\]

### 7. Genus One Partition and n-Point Functions

#### 7.1. The Torus Partition Function for a Heisenberg VOA

For a VOA \( V = \bigoplus_{n \geq 0} V_n \) of central charge \( c \) define the genus one partition (trace or characteristic) function by

\[
Z_V^{(1)}(q) = \Tr q^{L(0) - \frac{c}{24}} = \sum_{n \geq 0} \dim V_n q^{n - \frac{c}{24}},
\]

for the Heisenberg VOA \( M \) commutation relations with modes

\[
\left[ a(m), a(n) \right] = m \delta_{m-n},
\]

\[
Z_M^{(1)}(q) = \frac{1}{\eta(q)} \text{ for } \eta(q) = q^{\frac{1}{24}} \prod_{\alpha \geq 1} (1 - q^\alpha).
\]

### 7.2. Genus One Twisted Graded Dimension

We define the genus one partition function for the Vosa by the supertrace

\[
Z_V^{(1)}(\tau) = \STr q^{L(0) - \frac{1}{24}} = \Tr q^{L(0) - \frac{1}{24}}
\]

\[
= q^{-\frac{1}{24}} \prod_{\alpha \geq 1} \left( 1 - q^{\frac{1}{24}} \right)^2,
\]

where \( \sigma u = e^{2\pi i \tau} u \).

More generally, we can construct a \( \sigma g \)-twisted module \( M_{\sigma g} \) for any automorphism \( g = e^{2\pi i \theta} \) generated by the Heisenberg state \( a \in V' \). We introduce the second automorphism \( h = e^{2\pi i \alpha} \) and define the orbifold \( \sigma g \)-twisted trace by

\[
Z_V^{(1)}(\tau) = \STr_{M_{\sigma g}} h q^{L(0) - \frac{1}{24}}(q),
\]

to find for \( \theta = e^{-2\pi \alpha} \),

\[
Z_V^{(1)}(\tau) = q^{(\rho(I-T)^{-1} \frac{1}{24})} \prod_{\alpha \geq 1} \left( 1 - \theta^{-1} q^{\rho(I-T)^{-1}} \right) \left( 1 - \theta q^{\rho(I-T)^{-1}} \right).
\]
7.3. Genus One Fermionic One-Point Functions

Each orbifold 1-point function can found from a generalized Zhu reduction formulas as a determinant.

**Theorem 13 (Mason-Tuite-Z)**

For a Fock vector
\[
\Psi_{\lambda, \nu}^{\gamma}, \Phi^{\gamma}, \nu \in \mathcal{V}
\]

\[
Z_{\nu}^{\gamma} \left( h \right) \left( \Psi_{\lambda, \nu}^{\gamma}, \Phi^{\gamma}, \nu \right) = \det \left( C^{\lambda} \Phi^{\gamma} \right) Z_{\nu}^{\gamma} \left( h \right) \left( \nu \right),
\]

where for \( i, j = 1, 2, \ldots, n \)

\[
C^{\lambda} \left( i, j \right) = C^{\lambda} \left( k_{i}, l_{j}, \tau \right).
\]

7.4. Genus One \( n \)-Point Functions for VOA

In general, we can define the genus one orbifold \( n \)-point function for \( \nu_{1}, \ldots, \nu_{n} \in \mathcal{V} \) by

\[
\left[ a(\alpha), Y(\nu, h, z) \right] = \sum_{\nu = 0}^{m} \left( M_{j} \right) Y(\alpha(j), h, z) z^{-\nu},
\]

expansions for \( P_{\nu} \)-functions:

\[
P_{\nu} \left( \theta \right) = \frac{1}{z_{1} - z_{2} + \sum_{k, \in \mathbb{Z}} C^{\theta} \left( k, l \right) \frac{1}{z_{1}^{k} z_{2}^{l}}},
\]

\[
P_{\nu} \left( \theta \right) \left( z, \tau \right) = \frac{(-1)^{\nu} \delta_{\nu, 1} \delta_{1, \mathbb{Z}}}{(k-1)^{\nu} z_{1}^{k-1} z_{2}^{l}},
\]

**Theorem 14 (Mason-Tuite-Z)**

For any \( \nu_{1}, \ldots, \nu_{n} \in \mathcal{V} \) we have

\[
Z^{(0)} \left( \nu, \nu_{1}, \ldots, \nu_{n}; \tau \right) = \sum_{r=0}^{n} \left( \nu^{(0)} \right) \left( \nu_{1}, \ldots, \nu_{n}; \tau \right) P_{\nu} \left( \theta \right) \left( \nu, \nu_{1}, \ldots, \nu_{n}; \tau \right)
\]

\[
+ \delta_{\nu, 1} \delta_{1, \mathbb{Z}} \mathcal{S} \mathcal{T} \mathcal{R} \left( o(\nu) Y_{M} \left( q_{1}^{\nu}, \nu_{1}, q_{1} \right) \ldots Y_{M} \left( q_{n}^{\nu}, \nu_{n}, q_{n} \right) q_{1}^{\nu} q_{2}^{\nu} \right),
\]

where \( p_{a_{1}, \ldots, a_{n}} \) is given by

\[
p \left( A, B_{\nu}, \ldots, B_{\nu} \right) = \begin{cases} 1, & \text{for } r = 1 \\ (-1)^{p} \delta_{p_{a_{1}}, \ldots, p_{a_{n}}}, & \text{for } r > 1. \end{cases}
\]

7.6. General Genus One Fermionic \( n \)-Point Functions

The generating two-point function (for \( (\theta, \phi) \neq (1,1) \)) is given by

\[
Z^{(0)} \left( \nu, \nu_{1}, \ldots, \nu_{n}; \tau \right) = P_{\nu} \left( \theta \right) \left( \nu, \nu_{1}, \ldots, \nu_{n}; \tau \right) Z^{(0)} \left( \nu, \nu_{1}, \ldots, \nu_{n}; \tau \right)
\]

**Theorem 15 (Mason-Tuite-Z)**

\[
Z^{(0)} \left( \nu, \nu_{1}, \ldots, \nu_{n}; \tau \right) = \det Z_{\nu}^{(0)} \left( h \right) \left( \nu \right)
\]

**Theorem 16 (Mason-Tuite-Z)**

For \( \nu_{1}, \ldots, \nu_{n} \) Fock vectors

\[
\Psi^{(\alpha)} = \Psi^{(\alpha)} \left[ -k^{(\alpha)}; -l^{(\alpha)} \right]
\]

\[
\Psi^{(\alpha)}_{\nu} = \Psi^{(\alpha)} \left[ -k^{(\alpha)}; -l^{(\alpha)} \right]_{\nu}
\]
for \( k^{(a)} = k^{(a)}_1, \ldots, k^{(a)}_n \) and \( l^{(a)} = l^{(a)}_1, \ldots, l^{(a)}_n \) with
\[ a = 1, \ldots, n. \]
Then for \( (\theta, \phi) \neq (1,1) \) the corresponding \( n \)-point functions are non-vanishing provided
\[ \sum_{a=1}^{n} (s_a - t_a) = 0, \]
and
\[ Z_f^{(i)} \left[ f \left[ \begin{array}{c} \Psi^{(a)}_1, z_1, \ldots, \Psi^{(a)}_n, z_n \end{array} \right], r \right] \]
\[ = \varepsilon \det M \cdot Z_f^{(i)} \left[ f \left[ \begin{array}{c} g \left[ \begin{array}{c} \Psi^{(a)}_1, z_1, \ldots, \Psi^{(a)}_n, z_n \end{array} \right], r \right] \right] , \]
where \( \varepsilon \) is certain parity factor. Here \( M \) is the block matrix
\[ M = \left[ \begin{array}{cccc} C^{(11)} & D^{(12)} & \cdots & D^{(1n)} \\ D^{(21)} & C^{(22)} & \cdots & D^{(2n)} \\ \vdots & \vdots & \ddots & \vdots \\ D^{(n1)} & \cdots & \cdots & C^{(nn)} \end{array} \right] , \]
with
\[ Z^{(i)} \left[ f \left[ \begin{array}{c} g \left[ \begin{array}{c} u, z_1, \ldots, z_n \end{array} \right] \right] \right] \]
\[ = \text{Str}_{\mathcal{E}_N} \left( f_i \Psi \left( q^L_{u}v, q^L_{u} \right) Y \left( q^L_{u}v, q^L_{u} \right) \right) \]
\[ = \text{Str}_{\mathcal{E}_N} \left( f_i \Psi \left( q^L_{u}v, q^L_{u} \right) Y \left( q^L_{u}v, q^L_{u} \right) \right) \]
\[ = \text{Str}_{\mathcal{E}_N} \left( f_i \Psi \left( q^L_{u}v, q^L_{u} \right) Y \left( q^L_{u}v, q^L_{u} \right) \right) \]
for \( s_a, t_a \geq 1 \) with \( 1 \leq a \leq n \) and
\[ D^{(ab)}(i,j) = D \left[ \begin{array}{c} \theta \phi \left( k^{(a)}_j, l^{(b)}_j, z_{ab} \right) \right] \left( 1 \leq i \leq s_a, 1 \leq j \leq t_b \right) , \]
for \( s_a, t_a \geq 1 \) with \( 1 \leq a, b \leq n \) and \( a \neq b \). \( \varepsilon \) is the sign of the permutation associated with the reordering of \( \Psi^+ \) to the alternating ordering.

Furthermore, the \( n \)-point function is an analytic function in \( z_a \) and converges absolutely and uniformly on compact subsets of the domain \( |q| < q_{ab} < 1 \).

### 7.7. Torus Intertwined \( n \)-Point Functions


Let \( g_1, f_j, i = 1,2 \) be VOSA \( V \) automorphisms commuting with \( \sigma v = (-1)^{p(v)} v \). For \( u \in \mathcal{E}_{g_2} \) and the states \( v_1, \ldots, v_n \in V \) we define the intertwined \( n \)-point function [4] on the torus by
\[ \text{Str}_{\mathcal{E}} \left( f_i \Psi \left( q^L_{u}v, q^L_{u} \right) Y \left( q^L_{u}v, q^L_{u} \right) \right) \]
\[ = \text{Str}_{\mathcal{E}} \left( f_i \Psi \left( q^L_{u}v, q^L_{u} \right) Y \left( q^L_{u}v, q^L_{u} \right) \right) \]
\[ = \text{Str}_{\mathcal{E}} \left( f_i \Psi \left( q^L_{u}v, q^L_{u} \right) Y \left( q^L_{u}v, q^L_{u} \right) \right) \]
\[ = \text{Str}_{\mathcal{E}} \left( f_i \Psi \left( q^L_{u}v, q^L_{u} \right) Y \left( q^L_{u}v, q^L_{u} \right) \right) \]
\[ = \text{Str}_{\mathcal{E}} \left( f_i \Psi \left( q^L_{u}v, q^L_{u} \right) Y \left( q^L_{u}v, q^L_{u} \right) \right) \]

trace over a \( V \)-module \( N \) is defined by
\[ \text{Str}_{\mathcal{E}} \left( f_i \Psi \left( q^L_{u}v, q^L_{u} \right) Y \left( q^L_{u}v, q^L_{u} \right) \right) \]
\[ = \text{Str}_{\mathcal{E}} \left( f_i \Psi \left( q^L_{u}v, q^L_{u} \right) Y \left( q^L_{u}v, q^L_{u} \right) \right) \]
\[ = \text{Str}_{\mathcal{E}} \left( f_i \Psi \left( q^L_{u}v, q^L_{u} \right) Y \left( q^L_{u}v, q^L_{u} \right) \right) \]

For an element \( u \in \mathcal{E}_{g_2} \) of a VOSA \( g \)-twisted \( V \)-module we introduce also the differential form
\[ \mathcal{F}^{(i)} \left[ f \right] \left[ \begin{array}{c} u, z_1, \ldots, z_n \end{array} \right] \]
\[ = Z^{(i)} \left[ f \left[ \begin{array}{c} u, z_1, \ldots, z_n \end{array} \right] \right] \]
\[ = Z^{(i)} \left[ f \left[ \begin{array}{c} u, z_1, \ldots, z_n \end{array} \right] \right] \]
\[ = Z^{(i)} \left[ f \left[ \begin{array}{c} u, z_1, \ldots, z_n \end{array} \right] \right] \]
\[ = Z^{(i)} \left[ f \left[ \begin{array}{c} u, z_1, \ldots, z_n \end{array} \right] \right] \]

associated to the torus intertwined \( n \)-point function.

### 7.8. Torus Intertwined Two-Point Function

The rank two free fermion VOSA intertwined torus \( n \)-point function is parameterized by \( \theta_1 = e^{-2\pi i/\alpha_1} \), \( \phi_1 = e^{2\pi i/\alpha_1} \), and \( \phi_2 = e^{2\pi i/\alpha_1} \), [2, 4] where
\[ \sigma f_i = e^{2\pi i/\alpha_1} f_i, \quad \sigma g_1 = e^{-2\pi i/\alpha_1} g_1, \quad \sigma g_2 = e^{2\pi i/\alpha_1} g_2, \]
for real valued \( \alpha_1, \beta_1, \kappa, (\theta_1, \phi_1) \neq (1,1) \).

For \( u = 1 \otimes e^x \equiv e^x \in \mathcal{E}_{g_2} \) and \( v_i = 1, i = 1, \ldots, n \) we obtain [4] the basic intertwined two-point function on the torus
\[ C^{(a)}(i,j) = C \left[ \begin{array}{c} \theta \phi \left( k^{(a)}_i, l^{(a)}_j, x \right) \right] \cdot \left( 1 \leq i \leq s_a, 1 \leq j \leq t_a \right), \]
for \( s_a, t_a \geq 1 \) with \( 1 \leq a \leq n \) and
\[ D^{(ab)}(i,j) = D \left[ \begin{array}{c} \theta \phi \left( k^{(a)}_j, l^{(b)}_j, x \right) \right] \left( 1 \leq i \leq s_a, 1 \leq j \leq t_b \right), \]
for \( s_a, t_a \geq 1 \) with \( 1 \leq a, b \leq n \) and \( a \neq b \). 

The rank two free fermion VOSA intertwined torus \( n \)-point function is generated by \( \Psi^+ \) with
\[ \left[ \Psi^+ (m), \Psi^+ (n) \right] = \delta_{m,-n-1}, \left[ \Psi^+ (m), \Psi^+ (n) \right] = 0, \]
\[ \left[ \Psi^- (m), \Psi^- (n) \right] = 0. \]
We then consider the differential form

\[ Z^{(1)} \left[ \frac{f_1}{g_1} \right] (e^x, z_1^2; e^{-x}, z_1^2; \tau) = \text{STr}_{\text{reg}} \left( f_1 \mathcal{Y} \left( q_{z_1}, e^x, q_{z_1} \right) \mathcal{Y}^* \left( q_{z_1}, e^{-x}, q_{z_1} \right) q^{(0)}(\tau) \right). \]

We then consider the differential form

\[ G^{(1)}_n \left[ \frac{f_1}{g_1} \right] (x_1, y_1, \ldots, x_n, y_n) \equiv \mathcal{X}^{(1)} \left[ \frac{f_1}{g_1} \right] (e^x, w; \psi^+; x_1; \psi^-, \ldots; x_n; \psi^-, y_1, \ldots, 0; \tau), \]

associated to the torus intertwined \(2n\)-point function

\[ Z^{(1)} \left[ \frac{f_1}{g_1} \right] (e^x, w; \psi^+; x_1; \psi^-, \ldots; x_n; \psi^-, y_n; e^{-x}, 0; \tau), \]

with alternatively inserted \(n\) states \(\psi^+\) and \(n\) states \(\psi^-\) distributed on the resulting genus two Riemann surface \(\Sigma(2)\) at points \(x_i, y_i \in \Sigma(2), i = 1, \ldots, n\). We then prove in [4].

Theorem 17 (Tuite-Z) For the rank two free fermion vertex operator superalgebra \(V\) and for \((\theta, \phi) \neq (1, 1)\) the generating form is given by

\[ Z^{(2)}(\tau, \tau_2, \epsilon) = \frac{1}{\eta(\tau) \eta(\tau_2)} \left( \text{det}(I - A_1 A_2) \right)^{1/2}; \]

b) \(Z^{(2)}(\tau, \tau_2, \epsilon)\) is holomorphic on the domain \(D^+\);

c) \(Z^{(2)}(\tau, \tau_2, \epsilon)^2\) is automorphic of weight \(-1\);

d) \(Z^{(2)}(\tau, \tau_2, \epsilon)\) has an infinite product formula.

8. Genus Two Partition and \(n\)-Point Functions

8.1. Genus Two Partition Function in \(\epsilon\)-Formalism

We define the genus two partition function in the earlier sewing scheme in terms of data coming from the two tori, namely the set of 1-point functions \(Z^{(1)}_v(u, \tau)\) for all \(u \in V\). We assume that \(V\) has a nondegenerate invariant bilinear form—the Li-Zamolodchikov metric. Define

\[ Z^{(2)}_v(\tau, \tau_2, \epsilon) = \sum_{n \geq 0} \sum_{w \varepsilon [1]} Z^{(1)}(u, \tau) Z^{(1)}(\overline{u}, \tau_2). \]

The inner sum is taken over any basis and \(\overline{u}\) is dual to \(u\) wrt to the Li-Zamolodchikov metric.

8.2. Genus Two Partition Function for the Heisenberg VOA

We can compute \(Z^{(2)}_v\) using a combinatorial-graphical technique based on the explicit Fock basis and recalling the infinite matrices \(A_1, A_2\).

Theorem 18 (Mason-Tuite) a) The genus two partition function for the rank one Heisenberg VOA is

\[ Z^{(2)}_w(\tau, \tau_2, \epsilon) = \frac{1}{\eta(\tau) \eta(\tau_2)} \left( \text{det}(I - A_1 A_2) \right)^{1/2}; \]

b) \(Z^{(2)}_w(\tau, \tau_2, \epsilon)\) is holomorphic on the domain \(D^+\);

c) \(Z^{(2)}_w(\tau, \tau_2, \epsilon)^2\) is automorphic of weight \(-1\);

d) \(Z^{(2)}_w(\tau, \tau_2, \epsilon)\) has an infinite product formula.

8.3. Genus Two Fermionic Partition Function

Following the definition for the bosonic VOA we define for \(h, g\)

\[ Z^{(2)} \left[ \frac{h}{g} \right] (q_1, q_2, \epsilon) \]

\[ = \sum_{n \geq 0} \sum_{w \varepsilon [1]} Z^{(1)} \left[ \frac{h}{g} \right] (u, q_1) Z^{(1)} \left[ \frac{h}{g} \right] (\overline{u}, q_2). \]

The inner sum is taken over any \(V_{[n]}\) basis and \(\overline{u}\) is dual to \(u\) with respect to the Li-Zamolodchikov square
9. Generalizations of Classical Identities

9.1. Bosonization

The genus one orbifold partition function can be alternatively computed by decomposing the VOSA into Heisenberg modules $M \otimes \mathbb{C}[\alpha]$ indexed by $\alpha(0)$ integer eigenvalues $m$, i.e., a $\mathbb{Z}$ lattice [26]. Let $\alpha_1, \ldots, \alpha_n$ be lattice elements of the rank one even lattice, $\alpha_1 + \ldots + \alpha_n = 0$, and $\varepsilon(\alpha, \alpha')$-cocycle. Then

**Theorem 20 (Tu}
Then the genus one twisted partition function is given by

\[ Z\left[ \frac{h}{g} \right](\tau) = \sum_{m \in \mathbb{Z}} (-1)^m e^{2\pi i m^2} \text{Tr}_{M \times M} \left\{ \frac{\mathcal{Z}(0)}{Z^{(1)}} \right\} \theta \left[ \begin{array}{c} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{array} \right](\tau). \]

Comparing to the fermionic product formula we obtain the classical Jacobi triple product formula:

\[ \prod_{n=0}^{\infty} (1-q^n)(1+zq^{n+1})(1+z^{-1}q^n) = \sum_{m \in \mathbb{Z}} q^{m(m-1)/2}. \]

9.2. Genus Two Jacobi Triple Product Formula

The genus two partition function can similarly be computed in the bosonic formalism to obtain a genus two Riemann theta function [19]

\[ Z^{(2)}\left[ \frac{h}{g} \right](q_1, q_2, \epsilon) = \Theta^{(2)}\left[ \begin{array}{c} a \\ b \end{array} \right] \left( \Omega^{(2)} \right)(q_1, q_2), \]

for an appropriate character valued genus two Riemann theta function

\[ \Theta^{(2)}\left[ \begin{array}{c} a \\ b \end{array} \right] \left( \Omega^{(2)} \right) = \sum_{m \in \mathbb{Z}^2} e^{a(m+1/2)(m+1/2)-b(m+1/2)}. \]

Comparing with the fermionic result we thus find that

\[ G^{(1)}_{2n, \beta}(f; z_1, \ldots, z_n, z'_1, \ldots, z'_n, \tau) \]

\[ = \frac{e^{2\pi i (\alpha+1/2)\beta + 1/2}}{\eta(\tau)} \left[ \begin{array}{c} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{array} \right] \prod_{j \neq j'} K^{(1)}(z_j - z_{j'}, \tau) K^{(1)}(z'_j - z'_{j'}, \tau). \]

Comparing this to fermionic expressions for \((\theta, \phi) \neq (1,1)\), we obtain the classical Frobenius elliptic function version of generalized Fay’s trisecant identity [21]:

**Corollary 1 (Mason-Tuite-Z)** For \((\theta, \phi) \neq (1,1)\), we have
9.5. Generalized Fay’s Trisecant Identity

We may generalize these identities using \([26]\). Consider the general lattice \(n\)-point function. We have \([19]\). For integers \(m_i, n_j \geq 0\) satisfying \(\sum_{i=1}^{k} m_i = \sum_{j=1}^{l} n_j\), we have

\[
Z^{(1)}_z \left( f; \begin{bmatrix} 1 \otimes e^{m_1}, z_1, \ldots, 1 \otimes e^{m_k}, z_k \end{bmatrix}, \begin{bmatrix} 1 \otimes e^{-n_1}, z'_1, \ldots, 1 \otimes e^{-n_l}, z'_l \end{bmatrix}; \tau \right)
= \frac{e^{2\pi i (\alpha \tau + \beta)}}{\eta(\tau)}^g \left[ \begin{bmatrix} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{bmatrix} \left( \sum_{i=1}^{k} m_i z_i - \sum_{j=1}^{l} n_j z'_j, \tau \right) \prod_{1 \leq i < j \leq k} K^{(1)}(z_i - z_j, \tau) K^{(1)}(z'_i - z'_j, \tau) \right].
\]

Comparing this to the expression for \(n\)-point functions we obtain a new elliptic generalization of Fay’s trisecant identity:

\[
de(\ell) \left[ \begin{bmatrix} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{bmatrix} (0, \tau) \right] \prod_{1 \leq i < j \leq k} K^{(1)}(z_i - z_j, \tau) K^{(1)}(z'_i - z'_j, \tau).
\]

Corollary 2 (Mason-Tuite-Z) For \((\theta, \phi) \neq (1,1)\), we have

\[
de(\ell) \left[ \begin{bmatrix} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{bmatrix} (0, \tau) \right] \prod_{1 \leq i < j \leq k} K^{(1)}(z_i - z_j, \tau) K^{(1)}(z'_i - z'_j, \tau).
\]

10. Genus Two Intertwined Partition and \(n\)-Point Functions

In \([4]\) we then prove:

**Theorem 22 (Tuite-Z)** Let \(V_{\sigma_{g_i}}, i = 1, 2\) be \(\sigma_{g_i}\)-twisted \(V\)-modules for the rank two free fermion vertex operator superalgebra \(V\). Let \((\theta, \phi) \neq (1,1)\). Then the partition function on a genus two Riemann surface obtained in the \(\rho\)-self-sewing formalism of the torus is a non-vanishing holomorphic function on \(D^\rho\) given by

\[
Z^{(1)} \left[ f_i \right] (w, \rho) = Z^{(0)} \left[ f_i \right] (e^w, e^s, 0; \tau) \det(1 - \tau),
\]

for \(1 \leq a \leq r\) and \(1 \leq b \leq s\), and \(D\)-functions are given by the expansion

\[
P_i = \sum_{k,l \in Z} \left[ \begin{bmatrix} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{bmatrix} \left( k, l, z \right) z_i^{-1} z_j^{-1} \right].
\]
where $Z^{[0]}_{g_1}(f_1,g_1)\left(e^x,w;e^{-x},0;\tau\right)$ is the intertwined $V$-module $V_{g_1}$ torus basic two-point function, and $b(\kappa)$ is some function.

We may similarly compute the genus two partition function in the $\rho$-formalism for the original rank one fermion VOSA $V\left(H,\mathbb{Z}+\frac{1}{2}\right)$ in which case we can only construct a $\sigma$-twisted module. Then we have [4] the following:

**Corollary 3 (Tuite-Z)** Let $V$ be the rank one free fermion vertex operator superalgebra and $f_i, g_i \in \{\sigma, 1\}$, $a=1,2$, be automorphisms. Then the partition function for $V$-module $V_{g_i}$ on a genus two Riemann surface obtained from $\rho$-formalism of a self-sewn torus $\Sigma^{(1)}$ is given by

$$Z^{[0]}_{g_1}(f_1,g_1)\left(e^x,w;e^{-x},0;\tau\right)\det(I-T)^{1/2},$$

where $Z^{[0]}_{g_1}(f_1,g_1)\left(e^x,w;e^{-x},0;\tau\right)$ is the rank one fermion intertwined partition function on the original torus.

### 10.1. Genus Two Generating Form

In [4] we define matrices

$$S^{(2)} =\left(S^{(2)}(x_i,y_j)\right), \quad S^{(1)} =\left(S^{(1)}(x_i,y_j)\right),$$

$$H^+ =\left(\left(h(x_i)(k,a)\right)\right), \quad H^- =\left(\left(h(y_i)(l,b)\right)\right).$$

$S^{(2)}$ and $S^{(1)}$ are finite matrices indexed by $x_i$, $y_j$ for $i,j=1,\ldots,n$; $H^+$ is semi-infinite with $n$ rows indexed by $x_i$ and columns indexed by $k \geq 1$ and $a=1,2$ and $H^-$ is semi-infinite with rows indexed by $l \geq 1$ and $b=1,2$ and with $n$ columns indexed by $y_j$. We then prove

**Lemma 2 (Tuite-Z)**

$$\det S^{(1)}_{x_k} \xi H^+ D^2 =\det \left(S^{(2)}(I-T)\right),$$

with $T$, $D^2$. Introduce the differential form

$$\mathcal{G}_n^{[2]}(f)\left(x_i,y_j,\ldots,x_n,y_n\right)$$

$$=\mathcal{F}^{[2]}(g)\left(\psi^+,\psi^-;\ldots,\psi^+;\tau,w,\rho\right),$$

associated to the rank two free fermion VOSA genus two $2n$-point function

$$Z^{[2]}_{\rho}(f)\left(\psi^+,\psi^-;\psi^-,\psi^+;\ldots,\psi^+;\tau,w,\rho\right),$$

with alternatively inserted $n$ states $\psi^+$ and $n$ states $\psi^-$. The states are distributed on the genus two Riemann surface $\Sigma^{(2)}$ at points $x_i,y_j \in \Sigma^{(1)}$, $i=1,\ldots,n$. Then we have

**Theorem 23 (Tuite-Z)** All $n$-point functions for rank two free fermion VOSA twisted modules $V_{g_1}$ on self-sewn torus are generated by the differential form

$$\mathcal{G}_n^{[2]}(f)\left(x_i,y_j,\ldots,x_n,y_n\right)^{\tau,w,\rho} \det S^{(2)},$$

where the elements of the matrix

$$S^{(2)} =\left[S^{(2)}(x_i,y_j\tau,w)\right], \quad i,j=1,\ldots,n$$

and $Z^{[2]}_{\rho}(f)\left(\tau,w,\rho\right)$ is the genus two partition function.

### 10.2. Modular Invariance Properties of Intertwined Functions

Following the ordinary case [20,27,40] we would like to describe modular properties of genus two “intertwined” partition and $n$-point generating functions. As in [27], consider $\hat{H} \subset Sp(4,\mathbb{Z})$ with elements

$$\mu(a,b,c) = \begin{pmatrix}1 & 0 & 0 & b \\ a & 1 & b & c \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  

$\hat{H}$ is generated by $A = \mu(1,0,0)$, $B = \mu(0,1,0)$ and $C = \mu(0,0,1)$ with relations


We also define $\Gamma_1 \subset Sp(4,\mathbb{Z})$ where $\Gamma_1 \cong SL(2,\mathbb{Z})$ with elements

$$\gamma_i = \begin{pmatrix}a_i & 0 & b_i & 0 \\ 0 & 1 & 0 & 0 \\ c_i & 0 & d_i & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad a_i d_i - b_i c_i = 1.$$  

Together these groups generate

$$L = \hat{H} \times \Gamma_1 \subset Sp(4,\mathbb{Z}).$$  

From [27] we find that $L$ acts on the domain $\mathcal{D}^\rho$ of

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as follows:
\[
\mu(a,b,c) \cdot (\tau, w, \rho) = (\tau, w + 2\pi ia + 2\pi ib, \rho),
\]
\[
\gamma_1 \cdot (\tau, w, \rho) = \left( a \tau + b, \frac{w}{c \tau + d}, \frac{\rho}{c \tau + d} \right).
\]

We then define \([4]\) a group action of \(\gamma_1 \in SL(2, \mathbb{Z})\) on the torus intertwined two-point function
\[
Z^{(1)} \left[ \frac{f_1}{g_1} \right] (u, w; v, 0; \tau) = Z^{(1)} \left[ \frac{f_1}{g_1} \right] (u, \gamma_1 \cdot w; v, 0; \gamma_1 \cdot \tau),
\]
with the standard action \(\gamma_1 \cdot \tau\) and \(\gamma_1 \cdot w\), and
\[
\gamma_1 \cdot \left[ \frac{f}{g} \right] = \left[ \frac{f^n g^h}{g^n h^f} \right],
\]
and the torus multiplier \(\epsilon^{(1)}_{\gamma_1} \left[ \frac{f}{g} \right] \in U(1)\), [1,19]. Then we have \([4]\)

**Theorem 24 (Tuite-Z)** The torus intertwined two-point function for the rank two free fermion VOSA is a

\[
A \left[ \begin{array}{cc}
  f_1 \\
  f_2 \\
  g_1 \\
  g_2
\end{array} \right] = \left[ \begin{array}{cc}
  f_1 g_2^* \\
  f_2 g_1^* \\
  g_1 \\
  g_2
\end{array} \right], \quad B \left[ \begin{array}{cc}
  f_1 \\
  f_2 \\
  g_1 \\
  g_2
\end{array} \right] = \left[ \begin{array}{cc}
  f_1 g_2 \\
  f_2 g_1 \\
  g_1 \\
  g_2
\end{array} \right], \quad C \left[ \begin{array}{cc}
  f_1 \\
  f_2 \\
  g_1 \\
  g_2
\end{array} \right] = \left[ \begin{array}{cc}
  f_1 g_2 \\
  f_2 g_1 \\
  g_1 \\
  g_2
\end{array} \right].
\]

In a similar way we may introduce the action of \(\gamma \in L\) on the genus two partition function \([4]\)
\[
Z^{(2)} \left[ \frac{f}{g} \right] \gamma(\tau, w, \rho)
\]
\[
= Z^{(2)} \left[ \frac{f}{g} \right] \gamma(\tau, w, \rho),
\]
\[
\gamma_1 \cdot \left[ \frac{f}{g} \right] = \left[ \frac{f_1 g_2^*}{g_1} \right].
\]

We may now describe the modular invariance of the genus two partition function for the rank two free fermion VOSA under the action of \(L\). Define a genus two multiplier \(\epsilon^{(2)}_{\gamma} \left[ \frac{f}{g} \right] \in U(1)\) for \(\gamma \in L\) in terms of the genus one multiplier as follows
\[
\epsilon^{(2)}_{\gamma} \left[ \frac{f}{g} \right] = \epsilon^{(1)}_{\gamma_1} \left[ \frac{f}{g} \right],
\]
for the generator \(\gamma_1 \in \Gamma_1\). We then find \([4]\).

**Theorem 25 (Tuite-Z)** The genus two partition function for the rank two VOSA is modular invariant with respect to \(L\) with the multiplier system, i.e.,
\[
Z^{(2)} \left[ \frac{f}{g} \right] (\tau, w, \rho) = \epsilon^{(2)}_{\gamma} \left[ \frac{f}{g} \right] Z^{(2)} \left[ \frac{f}{g} \right] (\tau, w, \rho).
\]

Finally, we can also obtain modular invariance for the generating form
\[
G^{(2)}_{n} \left[ \frac{f}{g} \right] (x_1, y_1, \ldots, x_n, y_n),
\]
for all genus two \(n\)-point functions \([4]\).

**Theorem 26 (Tuite-Z)** \(G^{(2)}_{n} \left[ \frac{f}{g} \right] (x_1, y_1, \ldots, x_n, y_n),\)
is modular invariant with respect to \(L\) with a multiplier.

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