Some Properties of a Kind of Singular Integral Operator with Weierstrass Function Kernel

Lixia Cao
Department of Information and Computing Sciences, Mathematics College, Northeast Petroleum University, Daqing, China
Email: caolixia98237@163.com

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ABSTRACT

We considered a kind of singular integral operator with Weierstrass function kernel on a simple closed smooth curve in a fundamental period parallelogram. Using the method of complex functions, we established the Bertrand Poincaré formula for changing order of the corresponding integration, and some important properties for this kind of singular integral operator.

Keywords: Weierstrass Function Kernel; Singular Integral Operator; Bertrand Poincaré Formula; Properties

1. Introduction

The properties of singular integral operator with Cauchy or Hilbert kernel on simple closed smooth curve or open arc have been elaborately discussed in [1-3]. Based on these, for the boundary curve is a closed curve or an open arc, the authors discussed the singular integral operators and corresponding equation with Cauchy kernel or Hilbert kernel in [1-3]. In recent years, many authors discussed the numerical solution of a class of systems of Cauchy singular integral equations with constant coefficients, Numerical methods for nonlinear singular Volterra integral equations in [4-6].

In this paper, we consider a kind of singular integral operator with Weierstrass function kernel on a simple closed smooth curve in a fundamental period parallelogram. Our goal is to develop the Bertrand Poincaré formula for changing order of the corresponding integration, and some important properties of the above singular integral operator.

2. Preliminaries

Definition 1 Suppose that \( \omega_1, \omega_2 \) are complex constants with \( \text{Im}(\omega_1/\omega_2) \neq 0 \), and \( P \) denotes the fundamental period parallelogram with vertices \( \pm\omega_1 \pm\omega_2 \). Then the function

\[ \zeta(z) = 1/z + \sum_{m,n} \left[ 1/(\Omega_{mn} + z/\Omega^2_{mn}) \right] \]

is called the Weierstrass \( \zeta \)-function, where

\[ \Omega_{mn} = 2m\omega_1 + 2n\omega_2 \]

\[ \sum' \] denotes the sum of all \( m,n = 0, \pm 1, \pm 2, \cdots \), except for \( m = n = 0 \).

Definition 2 Suppose that \( L_0 \) is a smooth closed curve in the counterclockwise direction, lying entirely in the fundamental period parallelogram \( P \), with \( z_0 (\neq 0) \) and the origin lying in the domain \( S_0 \) enclosed by \( L_0 \). The following operator

\[ K\varphi = a(t_0)\varphi(t_0) + \frac{1}{i\pi} \int_{\gamma_0} \varphi(t)K(t_0,t)\left[ \zeta(t-t_0) + \zeta(t_0-z_0) \right] dt, \quad t_0 \in L_0 \]

is called the singular integral operator with \( \zeta \)-function kernel on \( L_0 \), where \( \varphi(t) \in H(L_0) \) is the unknown function, and

\[ K(t_0,t) \in H(L_0 \times L_0), \quad a(t) \in H(L_0) \]

are the given functions.

Letting \( b(t) = K(t,t) \), then (1) becomes

\[ K\varphi = a(t_0)\varphi(t_0) + \frac{b(t_0)}{i\pi} \int_{\gamma_0} \varphi(t)\left[ \zeta(t-t_0) + \zeta(t_0-z_0) \right] dt \]

\[ + \frac{1}{i\pi} \int_{\gamma_0} \left[ K(t_0,t) - K(t_0,t_0) \right] \left[ \zeta(t-t_0) + \zeta(t_0-z_0) \right] \varphi(t) dt \]

(2)
Since \( \zeta(t) \) is uniformly convergent in any closed bounded region lying entirely in \( \mathbb{P} \),
\[
\left| \zeta(t-t_0) + \zeta(t_0 - z_0) \right| \leq M|t-t_0| + M
\]
for any \( t_0, t \in \mathcal{L}_0 \), where \( M \) is some positive finite constant. By noting that \( K(t_0,t) \in H^\alpha \) \( (0 < \alpha \leq 1) \), we obtain
\[
\left| K(t_0,t) - K(t_0,t_0) \right| \left| \zeta(t-t_0) + \zeta(t_0-z_0) \right| \leq N|t-t_0|^\alpha
\]
\( (0 < \alpha \leq 1) \), where \( N \) is some positive finite constant. Write
\[
K^0 \phi = a(t_0) \phi(t_0) + \frac{b(t_0)}{\pi i} \int_{t_0} \phi(t) \left[ \zeta(t-t_0) + \zeta(t_0-z_0) \right] \, dt,
\]

equal to \( k(t_0,t) = \frac{1}{\pi i} \int \left[ K(t_0,t) - K(t_0,t_0) \right] \zeta(t-t_0) + \zeta(t_0-z_0) \, dt \)
then (1) can be rewritten in the form
\[
\left( K^0 + k \right) \phi,
\]
where \( k \) is a Fredholm operator and \( K^0 \) is called the characteristic operator of \( K \). Now the index of \( K \) is
obtain
\[
S(t) = a(t) + b(t), \quad D(t) = a(t) - b(t)
\]
and for definiteness we assume that \( a^2(t) - b^2(t) \neq 0 \), namely we assume that \( K \) is an operator of normal type.
Now the associated operator of \( K^0 \) becomes
\[
K^{\omega} \phi = a(t_0) \phi(t_0) - \frac{1}{\pi i} \int b(t) \phi(t) \left[ \zeta(t-t_0) + \zeta(t_0-z_0) \right] \, dt, t_0 \in \mathcal{L}
\]
In addition, if we write
\[
J(t_0,t) = \int_{t_0} \left( k(t_0,t) - \frac{1}{\pi i} \left[ b(t) - b(t_0) \right] \left[ \zeta(t-t_0) + \zeta(t_0-z_0) \right] \right) \phi(t) \, dt, t_0 \in \mathcal{L}
\]
then (4) can be rewritten as
\[
K' \psi = a(t_0) \psi(t_0) - \frac{b(t_0)}{\pi i} \int_{t_0} \psi(t) \left[ \zeta(t-t_0) + \zeta(t_0-z_0) \right] \, dt + J(t_0) \psi(t), t_0 \in \mathcal{L}
\]
where
\[
\left[ b(t) - b(t_0) \right] \left[ \zeta(t-t_0) + \zeta(t_0-z_0) \right] \leq D |t-t_0|^\alpha \quad (0 \leq \alpha < 1, \ D \text{ is some finite constant}).
\]
So \( k_0 \psi \) is a Fredholm operator, and then the characteristic operator of \( K' \) operator becomes
\[
K'' \omega \psi = a(t_0) \psi(t_0) - \frac{b(t_0)}{\pi i} \int_{t_0} \psi(t) \left[ \zeta(t-t_0) + \zeta(t_0-z_0) \right] \, dt,
\]
\( X(\zeta,\sigma) = \zeta(z-\sigma) + \zeta(\sigma - z_0) \)
where the fixed nonzero point \( z_0 \) and the origin lie in \( S_0 \). It is not difficult to get the following results.

\textbf{Lemma 1} Suppose that \( f(t,\tau) \in H(\mathcal{L}_0 \times \mathcal{L}_0) \), and with the same \( \mathcal{L}_0 \) as mentioned before, then
\[
a) \int_{\mathcal{L}_0} \int_{\mathcal{L}_0} f(t,\tau) X(\tau,\tau) \, d\tau = \int_{\mathcal{L}_0} \int_{\mathcal{L}_0} f(t,\tau) X(\tau,\tau) \, d\tau,
\]
\[
\int_{\mathcal{L}_0} X(\tau,\tau) \, d\tau \int_{\mathcal{L}_0} f(t,\tau) \, d\tau = \int_{\mathcal{L}_0} \int_{\mathcal{L}_0} f(t,\tau) X(\tau,\tau) \, d\tau
\]
b) (Poincare-Bertrand formula)
\[ \int_{t_0} f(t,\tau)X(\tau,t)\,d\tau = -\pi^2 f(t_0,\tau) \]

3. Some Properties of Operator \( K \)

1) If \( \phi \in H \), then \( K\phi \in H \).

Proof Through calculation and estimation, we have

\[
\left| \int_{t_0} K(t_1,t)\phi(t)\zeta(t)\,dt - \int_{t_0} K(t_2,t)\phi(t)\zeta(t)\,dt \right| \leq M|t_1-t_2|^\alpha + N|\zeta(t_1) - \zeta(t_2)|
\]  
(7)

for any \( t_1, t_2 \in L_0 \), where \( M \) and \( N \) are all finite constant. While for any \( t_1, t_2 \in L_0 \), we have

\[
|\zeta(t_1) - \zeta(t_2)| \leq \frac{|t_1-t_2|}{t_{1,2}} + \sum \frac{|t_1-t_2|}{(t_1-\Omega_m(t_2-\Omega_m))} + \sum \frac{|t_1-t_2|}{\Omega_m^2} \leq Q|t_1-t_2|
\]  
(8)

where \( Q \) is some finite constant. Substituting (8) into (7), we obtain

\[ \int_{t_0} K(t_0,t)\phi(t)\zeta(t_0)\,dt \in H \]  
(9)

Similarly we know that

\[ \int_{t_0} K(t_0,t)\phi(t)\zeta(t-t_0)\,dt, \quad \int_{t_0} K(t_0,t)\phi(t)\zeta(t-t_0-z_0)\,dt, \quad a(t)\phi(t_0) \in H \]

Consequently, we have \( K\phi \in H \).

2) If \( K_1, K_2 \) are singular integral operator, then \( K_1K_2 \) is also a singular integral operator. That is, if \( K_1 \phi = a_j(t_0)\phi(t_0) + \frac{1}{\pi i} \int_{t_0} \phi(t)K_j(t_0,t)\left[ \zeta(t-t_0) + \zeta(t-t_0-z_0) \right]dt, \quad j = 1, 2 \)

then

\[
K_1K_2\phi = \left[ a_1(t_0)K_2(t_0,t) + b_1(t_0)b_2(t_0) \right]\phi(t_0)
\]

\[
+ \frac{1}{\pi i} \int_{t_0} \left[ \frac{a_1(t_0)K_2(t_0,t) + a_2(t_0)K_1(t_0,t)}{a_1(t_0)K_2(t_0,t) + a_2(t_0)K_1(t_0,t)} \right] \phi(t)\left[ \zeta(t-t_0) + \zeta(t-t_0-z_0) \right]dt
\]

\[
+ \frac{1}{\pi i} \int_{t_0} \left[ \frac{a_1(t_0)K_2(t_0,t) + a_2(t_0)K_1(t_0,t)}{a_1(t_0)K_2(t_0,t) + a_2(t_0)K_1(t_0,t)} \right] \phi(t)\left[ \zeta(t-t_0) + \zeta(t-t_0-z_0) \right]dt
\]

where the sum of the former two terms in the right hand of Equation (10) are the characteristic operator, and the remainder in that is a Fredholm operator.

Proof By definition, we deduce that

\[
K_1K_2\phi = a_1(t_0)K_2(t_0,t)\phi(t_0) + \frac{1}{\pi i} \int_{t_0} a_1(t_0)K_2(t_0,t)\phi(t)\left[ \zeta(t-t_0) + \zeta(t-t_0-z_0) \right]dt
\]

\[
+ \frac{1}{\pi i} \int_{t_0} a_2(t_0)K_1(t_0,t)\phi(t)\left[ \zeta(t-t_0) + \zeta(t-t_0-z_0) \right]dt + C(t_0)
\]

where

\[
C(t_0) = \frac{1}{\pi^2} \int_{t_0} K_1(t_0,t)\left[ \zeta(t-t_0) + \zeta(t-t_0-z_0) \right]dt
\]

By virtue of Lemma 1 (b), \( C(t_0) \) can be rewritten in the form

\[ C(t_0) = b_1(t_0)b_2(t_0)\phi(t_0) + \frac{1}{\pi i} \int_{t_0} \left[ \frac{K_1(t_0,t_0)K_1(t_0,t)\left[ \zeta(t-t_0) + \zeta(t-t_0-z_0) \right]}{K_1(t_0,t_0)K_1(t_0,t) + K_1(t_0,t)} \right] \phi(t_0) dt_0 \]

Consequently, (10) is established.
\[ \int_{t_0} K_1(t_0, t_t) K_2(t_t, t) \zeta(t_t - t_0) \zeta(t - t_t) dt_t = \{1\} + \{2\} + \{3\} + \{4\} \]

where
\[ \{1\} = \int_{t_0} K_1(t_0, t_1) K_2(t_1, t) \frac{1}{(t_t - t_0)(t - t_t)} dt_t , \quad \{2\} = \int_{t_0} K_1(t_0, t_1) K_2(t_1, t) \sum_{m_n} \left( \frac{1}{t_t - t_0 - \Omega_{m_n}} + \frac{1}{\Omega_{m_n}^2} \right) dt_t , \]
\[ \{3\} = \int_{t_0} K_1(t_0, t_1) K_2(t_1, t) \sum_{m_n} \left( \frac{1}{t_t - t_0 - \Omega_{m_n}} + \frac{t_t - z_{m_n}}{\Omega_{m_n}^2} \right) dt_t , \]
\[ \{4\} = \int_{t_0} K_1(t_0, t_1) K_2(t_1, t) \sum_{m_n} \left[ \frac{1}{(t_t - t_0 - \Omega_{m_n}) + \Omega_{m_n}} + \frac{(t_t - t_0)}{\Omega_{m_n}^2} \right] \sum_{m_n} \left[ \frac{1}{(t_t - t_0 - \Omega_{m_n}) + \Omega_{m_n}} + \frac{(t_t - t_0)}{\Omega_{m_n}^2} \right] dt_t . \]

By [1], we know that \{1\} is a Fredholm integral. For \{4\}, we know from
\[ K_1(t_0, t_1), K_2(t_1, t) \in H \left( L_0 \times L_0 \right) \]
that \{4\} is continuous about the variable \( t \in L_0 \), and so that \[ \int_{t_0} \phi(t) \frac{d}{dt} \] is also a Fredholm integral. By nothing that \{2\} and \{3\} have the same form, we only need to discuss either one of them. Here we consider the integral \{2\}. Write
\[ h(z) = \sum_{m_n} \left[ \frac{1}{(t_t - z - \Omega_{m_n}) + \Omega_{m_n}} + \frac{t_t - z}{\Omega_{m_n}^2} \right] \]
then \( h(t_t) \in H \left( L_0 \right) \) is analytic in \( P \) and so that \( h(t_t) \in H \left( L_0 \right) \). Consequently, we read from
\[ K_1(t_0, t_1), K_2(t_1, t) \in H \left( L_0 \times L_0 \right) \]
that \{2\} \in \( H \left( L_0 \right) \) and so that \{2\} is continuous on \( L_0 \), therefore \[ \int_{t_0} \phi(t) \frac{d}{dt} \] is also a Fredholm integral.

So far, we conclude that \( K_1, K_2 \) is a singular integral operator.

3) Let \( K_3 = K_1 K_2 \), where \( K_1 \) denotes the indices of \( K_j \) \( (j = 1, 2, 3) \), then \( K_3 = K_1 + K_2 \).

Proof From 2), we know
\[ \int_{t_0} \psi K \phi dt = \int_{t_0} \psi t \left( \phi(t) + \frac{1}{\pi i} \int_{t_0} K(t, t_0) \phi(t) \zeta(t - t_0) + (t - z_0) \right) dt_t \]
\[ = \int_{t_0} \phi(t) \psi(t) dt + \frac{1}{\pi i} \int_{t_0} \psi(t) \int_{t_0} K(t, t_0) \phi(t) \zeta(t - t) + \zeta(t - z_0) dt_t dt \]

Whereas
\[ \int_{t_0} \phi K' \psi dt = \int_{t_0} a(t) \phi(t) \psi(t) dt - \frac{1}{\pi i} \int_{t_0} \phi(t) \int_{t_0} K(t, t_0) \psi(t) \zeta(t - t_0) + \zeta(t - z_0) dt_t dt \]

Let
\[ W = \int_{t_0} \phi(t) \int_{t_0} K(t, t_0) \psi(t) \zeta(t - t_0) + \zeta(t - z_0) dt_t dt \]

then by Lemma 1(a), we have
\[ W = \int_{t_0} \int_{t_0} K(t, t_0) \phi(t) \psi(t) \zeta(t - t_0) + \zeta(t - z_0) dt_t dt_t \]
\[ = -\int_{t_0} \psi(t) \left| \int_{t_0} K(t, t_0) \phi(t) \zeta(t - t) - \zeta(t - t_0) \right| dt_t \]
Substituting (13) into (12), we see that
\[
\int_{i_0} \phi K' \psi dt = \left\{ a(t) \psi(t) \phi_i dt + \frac{1}{\pi i} \int_{i_0} \psi(t) \left[ \int_{i_0} K(t,t_i) \phi(t) \left[ \zeta(t_i - t) - \zeta(t_i - z_0) \right] dt_i \right] dt \right\} dt.
\]
(14)

Therefore, \( \int_{i_0} \psi K \phi dt = \int_{i_0} \phi K' \psi dt \) cannot be established.

REFERENCES