Some $L_p$ Inequalities for $B$-Operators

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ABSTRACT

Let $P(z)$ be a polynomial of degree at most $n$ having all its zeros in $\{z \mid |z| \geq 1\}$, then it was recently claimed by Shah and Liman ([1], estimates for the family of $B$-operators, Operators and Matrices, (2011), 79-87) that for every $R \geq 1$, $p \geq 1$, $\|B^p [P \circ \rho](z)\|_p \leq \frac{R^p}{n} \|\phi(\lambda_0, \lambda_1, \lambda_2)\|_p \|P(z)\|_p$, where $B$ is a $B_n$-operator with parameters $\lambda_0, \lambda_1, \lambda_2$ in the sense of Rahman [2], $\rho(z) = Rz$ and $\phi(\lambda_0, \lambda_1, \lambda_2) = \lambda_0 + \lambda_1 \frac{n^2}{2} + \lambda_2 \frac{n^3(n-1)}{8}$. Unfortunately the proof of this result is not correct. In this paper, we present certain more general sharp $L_p$-inequalities for $B_n$-operators which not only provide a correct proof of the above inequality as a special case but also extend them for $0 \leq p < 1$ as well.

Keywords: $L_p$-Inequalities; $B_n$-Operators; Polynomials

1. Introduction and Statement of Results

Let $\mathcal{P}_n$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree at most $n$. For $P \in \mathcal{P}_n$, define

\[
\|P(z)\|_0 := \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |P(e^{i\theta})| \, d\theta \right\},
\]

\[
\|P(z)\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta \right\}^{\frac{1}{p}}, \quad 0 < p < \infty
\]

\[
\|P(z)\|_\infty := \max_{|z|=1} |P(z)|,
\]

and denote for any complex function $\rho : \mathbb{C} \rightarrow \mathbb{C}$ the composite function of $P$ and $\rho$, defined by $(P \circ \rho)(z) := P(\rho(z)) \in \mathbb{C}$, as $P \circ \rho$.

A famous result known as Bernstein’s inequality (for reference, see [3, p. 531], [4, p. 508] or [5]) states that if $P \in \mathcal{P}_n$, then

\[
\|P'(z)\|_\infty \leq n \|P(z)\|_\infty, \quad (1.1)
\]

whereas concerning the maximum modulus of $P(z)$ on the circle $|z| = R > 1$, we have

\[
\|P(Rz)\|_\infty \leq R^n \|P(z)\|_\infty, \quad R \geq 1, \quad (1.2)
\]

Inequalities (1.1) and (1.2) can be obtained by letting $p \to \infty$ in the inequalities

\[
\|P'(z)\|_p \leq n \|P(z)\|_p, \quad p \geq 1 \quad (1.3)
\]

and

\[
\|P(Rz)\|_p \leq R^n \|P(z)\|_p, \quad R > 1, \quad p > 0, \quad (1.4)
\]

respectively. Inequality (1.3) was found by Zygmund [7] whereas inequality (1.4) is a simple consequence of a result of Hardy [8] (see also [9, Th. 5.5]). Since inequality (1.3) was deduced from M. Riesz’s interpolation formula [10] by means of Minkowski’s inequality, it was not clear, whether the restriction on $p$ was indeed essential. This question was open for a long time. Finally Arestov [11] proved that (1.3) remains true for $0 < p < 1$ as well.

If we restrict ourselves to the class of polynomials $P \in \mathcal{P}_n$ having no zero in $|z| < 1$, then Inequalities (1.1) and (1.2) can be respectively replaced by

\[
\|P'(z)\|_\infty \leq \frac{n^2}{2} \|P(z)\|_\infty, \quad (1.5)
\]

and

\[
\|P(Rz)\|_\infty \leq \frac{R^n+1}{2} \|P(z)\|_\infty, \quad R > 1. \quad (1.6)
\]

Inequality (1.5) was conjectured by Erdös and later verified by Lax [12], whereas Inequality (1.6) is due to
Ankey and Ravilin [13].

Both the Inequalities (1.5) and (1.6) can be obtained by letting $p \to \infty$ in the inequalities

$$\|P'(z)\|_p \leq n \frac{\|P(z)\|_p}{\|1+z\|_p}, \quad p \geq 0$$  \hspace{1cm} (1.7)

and for $R > 1, p > 0$,

$$\|P(Rz)\|_p \leq \frac{\|R^p z + 1\|_p}{\|1+z\|_p} \|P(z)\|_p.$$  \hspace{1cm} (1.8)

Inequality (1.7) is due to De-Brujin [14] for $p \geq 1$. Rahman and Schmeisser [15] extended it for $0 \leq p < 1$ whereas the Inequality (1.8) was proved by Boas and Rahman [16] for $p \geq 1$ and later it was extended for $0 \leq p < 1$ by Rahman and Schmeisser [15].

Q. I. Rahman [2] (see also Rahman and Schmeisser [4, p. 538]) introduced a class $B_n$ of operators $B$ that carries a polynomial $P \in P_n$ into

$$B[P](z) = \lambda_0 P(z) + \lambda_1 \frac{(nz_2)}{2} P'(z),$$  \hspace{1cm} (1.9)

where $\lambda_0, \lambda_1$ and $\lambda_2$ are such that all the zeros of

$$U(z) = \lambda_0 + \lambda_2 C(n,1) z + \lambda_2 C(n,2) z^2$$  \hspace{1cm} (1.10)

where $C(n,r) = \frac{n!}{r!(n-r)!}$, lie in half plane $|z| \leq z - n/2$.

As a generalization of Inequality (1.1) and (1.5), Q. I. Rahman [2, inequality 5.2 and 5.3] proved that if $P \in P_n$ and $B \in B_n$ then for $|z| \geq 1$,

$$|B[P](z)| \leq \left| \phi_n \left( \lambda_0, \lambda_1, \lambda_2 \right) \right| \|P(z)\|_p,$$  \hspace{1cm} (1.11)

and if $P \in P_n$, $P(z) \neq 0$ in $|z| < 1$, then $|z| \geq 1$,

$$|B[P](z)| \leq \frac{1}{2} \left( | \phi_n \left( \lambda_0, \lambda_1, \lambda_2 \right) | + | \lambda_0 | \right) \|P(z)\|_p,$$  \hspace{1cm} (1.12)

where

$$\phi_n \left( \lambda_0, \lambda_1, \lambda_2 \right) = \lambda_0 + \lambda_1 \frac{n^2}{2} + \lambda_2 \frac{n^3 (n-1)}{8}.$$  \hspace{1cm} (1.13)

As a corresponding generalization of Inequalities (1.2) and (1.4), Rahman and Schmeisser [4, p. 538] proved that if $P \in P_n$, then $|z| = 1$,

$$|B[P \circ \rho](z)| \leq R^p \left| \phi_n \left( \lambda_0, \lambda_1, \lambda_2 \right) \right| \|P(z)\|_p$$  \hspace{1cm} (1.14)

and if $P \in P_n$, $P(z) \neq 0$ in $|z| < 1$, then as a special case of Corollary 14.5.6 in [4, p. 539], we have

$$|B[P \circ \rho](z)| \leq \frac{1}{2} \left( R^n | \phi_n \left( \lambda_0, \lambda_1, \lambda_2 \right) | + | \lambda_0 | \right) \|P(z)\|_p,$$  \hspace{1cm} (1.15)

where $\rho(z) = Rz, R \geq 1$ and $\phi_n \left( \lambda_0, \lambda_1, \lambda_2 \right)$ is defined by (1.13).

Inequality (1.15) also follows by combining the Inequalities (5.2) and (5.3) due to Rahman [2].

As an extension of Inequality (1.14) to $L_\infty$-norm, recently Shah and Liman [1, Theorem 1] proved:

**Theorem A.** If $P \in P_n$, then for every $R \geq 1$ and $p \geq 1$,

$$|B[P \circ \rho](z)| \leq R^p \left| \phi_n \left( \lambda_0, \lambda_1, \lambda_2 \right) \right| \|P(z)\|_p,$$  \hspace{1cm} (1.16)

where $B \in B_n$, $\rho(z) = Rz$ and $\phi_n \left( \lambda_0, \lambda_1, \lambda_2 \right)$ is defined by (1.13).

While seeking the analogous result of (1.15) in $L_\infty$ norm, they [1, Theorem 2] have made an incomplete attempt by claiming to have proved the following result:

**Theorem B.** If $P \in P_n$, and $P(z)$ does not vanish for $|z| \leq 1$, then for each $p \geq 1$, $R \geq 1$,

$$|B[P \circ \rho](z)| \leq R^p \left| \phi_n \left( \lambda_0, \lambda_1, \lambda_2 \right) \right| \|P(z)\|_p,$$  \hspace{1cm} (1.17)

where $B \in B_n$, $\rho(z) = Rz$ and $\phi_n \left( \lambda_0, \lambda_1, \lambda_2 \right)$ is defined by (1.13).

Further, it has been claimed in [1] to have proved the Inequality (1.17) for self-inversive polynomials as well.

Unfortunately the proof of Inequality (1.17) and other related results including the key lemma [1, Lemma 4] given by Shah and Liman is not correct. The reason being that the authors in [1] deduce:

1) line 10 from line 7 on page 84,
2) line 19 on page 85 from Lemma 3 [1] and,
3) line 16 from line 14 on page 86,
by using the argument that if $P^* (z) := z^n \overline{P(1/z)}$, then for $\rho(z) = Rz, R \geq 1$ and $|z| = 1$,

$$|B[P^* \circ \rho](z)| = |B[(P^* \circ \rho)^*](z)|,$$

which is not true, in general, for every $R \geq 1$ and $|z| = 1$. To see this, let

$$P(z) = a_n z^n + \cdots + a_1 z + a_0$$

be an arbitrary polynomial of degree $n$, then

$$P^* (z) := z^n \overline{P(1/z)} = \overline{a}_n z^n + \cdots + \overline{a}_1 z + \overline{a}_0.$$

Now with $a_0 := \lambda_0 n/2$ and $a_2 := \lambda_2 n^2/8$, we have

$$B[P^* \circ \rho](z) = \sum_{k=0}^{n} (\lambda_0 + \omega_1(n-k) + \omega_2(n-k)(n-k-1)) \overline{a}_k z^{-k} R^{n-k},$$
In this direction, we next present the following and
\[ p R z n R P z \in (1.20), \]
we get the following result which extends Theorem A to
\[ 01 \]
Inequalities (1.1), (1.2), (1.14) and (1.16) and also extend
where \( \lambda_1 \) and \( \lambda_2 \) are given

\[ \lambda_0 + \omega_1 (n-k) + \omega_2 (n-k)(n-k-1) a_k \left( \frac{z}{R} \right)^k, \]
whence
\[ \left| B \left[ P^* \circ \rho \right] \right| (z) \in R^* \left( \sum_{k=0}^n \left( \lambda_0 + \omega_1 (n-k) + \omega_2 (n-k)(n-k-1) a_k \left( \frac{z}{R} \right)^k \right) \right), \]
but
\[ \left| B \left[ \left( P^* \circ \rho \right)^* \right] \right| (z) = R^* \left( \sum_{k=0}^n \left( \lambda_0 + \omega_1 k + \omega_2 k (k-1) a_k \left( \frac{z}{R} \right)^k \right) \right), \]
so the asserted identity does not hold in general for every \( R \geq 1 \) and \( \left| z \right| = 1 \) as e.g. the immediate counterexample of \( P(z) = z^* \) demonstrates in view of \( P^* (z) = 1 \),
\[ \left| B \left[ P^* \circ \rho \right] \right| (z) = \left| \lambda_0 \right| \quad \text{and} \quad \left| B \left[ \left( P^* \circ \rho \right)^* \right] \right| (z) = \left| \lambda_0 + \lambda_1 (n^2/2) + \lambda_2 n^3 (n-1)/8 \right| \]
for \( \left| z \right| = 1 \).

Authors [1] have also claimed that Inequality (1.17) and its analogue for self-inversive polynomials are sharp has remained to be verified. In fact, this claim is also wrong.

The main aim of this paper is to establish \( L_p \)-mean extensions of the inequalities (1.14) and (1.15) for \( 0 \leq p < \infty \) and present correct proofs of the results mentioned in [1]. In this direction, we first present the following result which is a compact generalization of the Inequalities (1.1), (1.2), (1.14) and (1.16) and also extend Inequality (1.17) for \( 0 \leq p < 1 \) as well.

**Theorem 1.** If \( P \in \mathcal{P}_n \) then for \( \alpha \in \mathbb{C} \) with \( |\alpha| \leq 1 \), \( 0 \leq p < \infty \) and \( R > r \geq 1 \),
\[ \left| B \left[ P \circ \rho \right] \right| (z) - \alpha B \left[ P \circ \rho \right] (z) \leq \left( R^* - \alpha R^* \right) \left| \phi_\alpha (\lambda_0, \lambda_1, \lambda_2) \right| \left| P(z) \right|, \]
(1.18)
where \( B \in \mathcal{B}_\alpha \), \( \rho(z) = t \) and \( \phi_\alpha (\lambda_0, \lambda_1, \lambda_2) \) is given by (1.13). The result is best possible and equality holds in (1.18) for \( P(z) = z^n \).

If we choose \( \alpha = 0 \) in (1.18), we get the following result which extends Theorem A to \( 0 \leq p < 1 \).

**Corollary 1.** If \( P \in \mathcal{P}_n \) then for \( 0 \leq p < \infty \) and \( R > 1 \),
\[ \left| B \left[ P \circ \rho \right] \right| (z) \leq R^* \left| \phi_0 (\lambda_0, \lambda_1, \lambda_2) \right| \left| P(z) \right|, \]
(1.19)
where \( B \in \mathcal{B}_0 \), \( \rho(z) = Rz \) and \( \phi_0 (\lambda_0, \lambda_1, \lambda_2) \) is given by (1.13).

**Remark 1.** Taking \( \lambda_0 = 0 \leq \lambda_2 \) in (1.19) and noting that in this case all the zeros of \( U(z) \) defined in (1.10) lie in \( |z| \leq |z-n/2| \), we get for \( R > 1 \) and \( 0 \leq p < \infty \),
\[ \left\| P^* (Rz) \right\|_p \leq n R^{p-1} \left\| P(z) \right\|_p, \]
which includes (1.4) as a special case. Next if we choose \( \lambda_0 = 0 = \lambda_2 \) in (1.19), we get inequality (1.4). Inequality (1.11) also follows from Theorem 1 by letting \( p \to \infty \) in (1.18).

Theorem 1 can be sharpened if we restrict ourselves to the class of polynomials \( P(z) \) which does not vanish in \( |z| < 1 \). In this direction, we next present the following interesting compact generalization of Theorem B which yields \( L_p \) mean extension of the inequality (1.12) for \( 0 \leq p < \infty \) which among other things includes a correct proof of inequality (1.17) for \( 1 \leq p < \infty \) as a special case.

**Theorem 2.** If \( P \in \mathcal{P}_n \) and \( P(z) \) does not vanish for \( |z| < 1 \) then for \( \alpha \in \mathbb{C} \) with \( |\alpha| \leq 1 \), \( 0 \leq p < \infty \) and \( R > r \geq 1 \),
\[ \left| B \left[ P \circ \rho \right] \right| (z) - \alpha B \left[ P \circ \rho \right] (z) \leq \left( R^* - \alpha R^* \right) \left| \phi_\alpha (\lambda_0, \lambda_1, \lambda_2) \right| z + (1-\alpha) \lambda_0 \left| P(z) \right|, \]
(1.20)
where \( B \in \mathcal{B}_\alpha \), \( \rho(z) = t \) and \( \phi_\alpha (\lambda_0, \lambda_1, \lambda_2) \) is defined by (1.13). The result is best possible and equality holds in (1.18) for \( P(z) = az^n + b \), \( |a| = |b| = 1 \).
We take \( \alpha = 0 \) in (1.20), we get the following result which is the generalization of Theorem B for \( p \geq 1 \) but also extends it for \( 0 \leq p < \infty \).

**Corollary 2.** If \( P \in \mathcal{P}_n \) and \( P(z) \) does not vanish for \( |z| < 1 \) then for \( 0 \leq p < \infty \) and \( R > 1 \),
\[ \left| B \left[ P \circ \rho \right] \right| (z) \leq \left( R^* \left| \phi_0 (\lambda_0, \lambda_1, \lambda_2) \right| z + \lambda_0 \right) \left| P(z) \right|, \]
(1.21)
where \( B \in \mathcal{B}_0 \), \( \rho(z) = Rz \) and \( \phi_0 (\lambda_0, \lambda_1, \lambda_2) \) is defined by (1.13).

By triangle inequality, the following result is an immediately follows from Corollary 2.

**Corollary 3.** If \( P \in \mathcal{P}_n \) and \( P(z) \) does not vanish for \( |z| < 1 \) then for \( 0 \leq p < \infty \) and \( R > 1 \),
\[ \left| B \left[ P \circ \rho \right] \right| (z) \leq \left( R \left| \phi_0 (\lambda_0, \lambda_1, \lambda_2) \right| z + \lambda_0 \right) \left| P(z) \right|, \]
(1.22)
where \( B \in \mathcal{B}_0 \), \( \rho(z) = Rz \) and \( \phi_0 (\lambda_0, \lambda_1, \lambda_2) \) is defined by (1.13).

**Remark 2.** Corollary 3 establishes a correct proof of a result due to Shah and Liman [1, Theorem 3] for \( p \geq 1 \) and also extends it for \( 0 \leq p < 1 \) as well.
Remark 3. If we choose \( \lambda_0 = 0 = \lambda_2 \) in (1.21), we get for \( R > 1 \) and \( 0 \leq p < \infty \),

\[
\|P'(Rz)\|_p \leq n R^{r-1} \|P(z)\|_p,
\]

which, in particular, yields Inequality (1.7). Next if we take \( \lambda_1 = 0 = \lambda_2 \) in (1.21), we get Inequality (1.8). Inequality (1.12) can be obtained from corollary 2 by letting \( p \to \infty \) in (1.20).

By using triangle inequality, the following result immediately follows from Theorem 2.

Corollary 4. If \( P \in \mathcal{P}_n \) and \( P(z) \) does not vanish for \( |z| < 1 \), then for \( x \in \mathbb{C} \) with \( |x| \leq 1 \) \( 0 \leq p < \infty \) and \( R > r > 1 \),

\[
\left\| B[P \circ \rho_{x}](z) - \alpha B[P \circ \rho_{x}](z) \right\|_p
\]

\[
\leq \left\| \left(R^\alpha - \alpha^\alpha \right) \phi_n(\lambda_0, \lambda_1, \lambda_2) + \left(1 - \alpha\right) \lambda_0 \right\|_p \left\| P(z) \right\|_p,
\]

(1.23)

where \( B \in \mathcal{B}_n \), \( \rho_{x}(t) = i z \) and \( \phi_n(\lambda_0, \lambda_1, \lambda_2) \) is defined by (1.13).

A polynomial \( P \in \mathcal{P}_n \) is said be self-inversive if \( P(z) = \overline{P}(z) \) where \( \overline{P} \) is the conjugate polynomial of \( P(z) \), that is, \( \overline{P}(z) := z^n \overline{P}(\overline{z}) \).

Finally in this paper, we establish the following result for self-inversive polynomials, which includes a correct proof of an another result of Shah and Liman [1, Theorem 2] as a special case.

Theorem 3. If \( P \in \mathcal{P}_n \) and \( P(z) \) is a self-inversive polynomial, then for \( \alpha \in \mathbb{C} \) with \( |\alpha| \leq 1 \) \( 0 \leq p < \infty \) and \( R > r > 1 \),

\[
\left\| B[P \circ \rho_{x}](z) - \alpha B[P \circ \rho_{x}](z) \right\|_p
\]

\[
\leq \left\| \left(R^\alpha - \alpha^\alpha \right) \phi_n(\lambda_0, \lambda_1, \lambda_2) + \left(1 - \alpha\right) \lambda_0 \right\|_p \left\| P(z) \right\|_p,
\]

(1.24)

where \( B \in \mathcal{B}_n \), \( \rho_{x}(t) = i z \) and \( \phi_n(\lambda_0, \lambda_1, \lambda_2) \) is defined by (1.13). The result is sharp and an extremal polynomial is \( P(z) = c (az^n + \overline{z}) \), \( ac \neq 0 \).

For \( \alpha = 0 \), we get the following result.

Corollary 5. If \( P \in \mathcal{P}_n \) and \( P(z) \) is a self-inversive polynomial, then for \( 0 \leq p < \infty \) and \( R > 1 \),

\[
\left\| B[P \circ \rho_{x}](z) \right\|_p
\]

\[
\leq \left\| R^n \phi_n(\lambda_0, \lambda_1, \lambda_2) z + \lambda_0 \right\|_p \left\| P(z) \right\|_p,
\]

(1.25)

where \( B \in \mathcal{B}_n \), \( \rho_{x}(z) = R z \) and \( \phi_n(\lambda_0, \lambda_1, \lambda_2) \) is defined by (1.13).

The following result is an immediate consequence of

Corollary 6. If \( P \in \mathcal{P}_n \), \( P(z) \) is a self-inversive polynomial, then for \( 0 \leq p < \infty \) and \( R > 1 \),

\[
\left\| B[P \circ \rho](z) \right\|_p
\]

\[
\leq \frac{1}{2} \left\| R^n \phi_n(\lambda_0, \lambda_1, \lambda_2) + \lambda_0 \right\|_p \left\| P(z) \right\|_p,
\]

(1.26)

where \( B \in \mathcal{B}_n \), \( \rho(z) = R z \) and \( \phi_n(\lambda_0, \lambda_1, \lambda_2) \) is given by (1.13).

Remark 4. Corollary 6 establishes a correct proof of a result due to Shah and Liman [1, Theorem 3] for \( p \geq 1 \) and also extends it for \( 0 \leq p < 1 \) as well.

Remark 5. A variety of interesting results can be easily deduced from Theorem 3 in the same way as we have deduced from Theorem 2. Here we mention a few of these. Take \( \lambda_0 = 0 = \lambda_2 \) in (1.25), we get for \( R > 1 \) and \( 0 \leq p < \infty \),

\[
\left\| P'(Rz) \right\|_p \leq R n^{r-1} \left\| P(z) \right\|_p,
\]

which, in particular, results due to Dewan and Govil [17] and A. Aziz [18] for polynomials \( P \in \mathcal{P}_n \).

Next if we choose \( \lambda_1 = 0 = \lambda_2 \) in (1.25), we get for \( R < 1 \); \( 0 \leq p < \infty \),

\[
\left\| P(Rz) \right\|_p \leq R z + \left\| P(z) \right\|_p.
\]

The above inequality is a special case of a result proved by Aziz and Rather [19].

Lastly letting \( p \to \infty \) in (1.25), it follows that if \( P(z) \), is a self-inversive polynomial then

\[
\left\| B[P \circ \rho](z) \right\|_p
\]

\[
\leq \frac{1}{2} \left\| R^n \phi_n(\lambda_0, \lambda_1, \lambda_2) + \lambda_0 \right\|_p \left\| P(z) \right\|_p,
\]

(1.27)

where \( B \in \mathcal{B}_n \), \( \rho(z) = R z \) and \( \phi_n(\lambda_0, \lambda_1, \lambda_2) \) is defined by (1.13). The result is sharp.

Inequality (1.27) is a special case of a result due to Rahman and Schmeisser [4, Cor. 14.5.6].

2. Lemma

For the proof of above theorems we need the following Lemmas:

The following lemma follows from Corollary 18.3 of [20, p. 86].

Lemma 1. If \( P \in \mathcal{P}_n \) and \( P(z) \) has all zeros in \( |z| \leq 1 \), then all the zeros of \( B[P](z) \) also lie in \( |z| \leq 1 \).

Lemma 2. If \( P \in \mathcal{P}_n \) and \( P(z) \) have all its zeros in \( |z| \leq 1 \) then for every \( R \geq 1 \), and \( |z| = 1 \),
where \( r_j \leq 1 \). Now for \( 0 \leq \theta < 2\pi \), \( R \geq r \geq 1 \), we have
\[
\left| \frac{\text{Re}^\theta - r_j \text{e}^{i\theta}}{\text{Re}^\theta - r_j \text{e}^\theta} \right| = \left( \frac{R^2 + r_j^2 - 2Rr_j \cos(\theta - \theta_j)}{r + r_j^2 - 2rr_j \cos(\theta - \theta_j)} \right)^{1/2}
\geq \left( \frac{R + r_j}{r + r_j} \right) \geq \left( \frac{R + 1}{r + 1} \right), \text{ for } j = 1, 2, \ldots, n.
\]
Hence
\[
\left| P(\text{Re}^\theta) \right| = \left( \prod_{j=1}^{n} \left| \frac{\text{Re}^\theta - r_j \text{e}^{i\theta}}{\text{Re}^\theta - r_j \text{e}^\theta} \right| \right) \geq \prod_{j=1}^{n} \left( \frac{R + 1}{r + 1} \right) = \left( \frac{R + 1}{r + 1} \right)^n,
\]
for \( 0 \leq \theta < 2\pi \). This implies for \( |z| = 1 \) and \( R \geq r \geq 1 \),
\[
\left| P(Rz) \right| \geq \left( \frac{R + 1}{r + 1} \right)^n \left| P(rz) \right|,
\]
which completes the proof of Lemma 2.

**Lemma 3.** If \( P \in \mathcal{P}_n \) and \( P(z) \) has no zero in \( |z| < 1 \), then for every \( \alpha \in \mathbb{C} \) with \( |\alpha| \leq 1 \), \( R \geq 1 \) and \( |z| = 1 \),
\[
|B[P \circ \rho_k](z) - \alpha B[P \circ \rho_k](z)| \leq B[P^* \circ \rho_k](z) - \alpha B[P^* \circ \rho_k](z),
\]
where \( P^*(z) := z^n P(1/z) \) and \( \rho_k(z) = tz \).

**Proof.** Since the polynomial \( P(z) \) has all its zeros in \( |z| \geq 1 \), therefore, for every real or complex number \( \lambda \) with \( |\lambda| > 1 \), the polynomial \( f(z) = P(z) - \lambda P^*(z) \), where \( P^*(z) := z^n P(1/z) \) has all zeros in \( |z| \leq 1 \). Applying Lemma 2 to the polynomial \( f(z) \), we obtain for every \( R > r \geq 1 \) and \( 0 \leq \theta < 2\pi \),
\[
\left| f(\text{Re}^\theta) \right| \geq \left( \frac{R + 1}{r + 1} \right)^n \left| f(\text{r} \text{e}^\theta) \right|; \quad (2.2)
\]
Since \( f(\text{Re}^\theta) \neq 0 \) for every \( R > r \geq 1 \), \( 0 \leq \theta < 2\pi \) and \( R + 1 > r + 1 \), it follows from (2.2) that
\[
\left| f(\text{Re}^\theta) \right| > \left( \frac{R + 1}{r + 1} \right)^n \left| f(\text{r} \text{e}^\theta) \right| \geq \left| f(\text{r} \text{e}^\theta) \right|,
\]
for every \( R > r \geq 1 \) and \( 0 \leq \theta < 2\pi \). This gives
\[
\left| f(rz) \right| < \left| f(Rz) \right| \text{ for } |z| = 1, \text{ and } R > r \geq 1.
\]
Using Rouche’s theorem and noting that all the zeros of \( f(Rz) \) lie in \( |z| \leq 1/R < 1 \), we conclude that the polynomial
\[
T(z) = f(Rz) - \alpha^* f(rz)
\]
has all its zeros in \( |z| < 1 \) for every real or complex \( \alpha \) with \( |\alpha| \geq 1 \) and \( R > r \geq 1 \).

Applying Lemma 1 to polynomial \( T(z) \) and noting that \( B \) is a linear operator, it follows that all the zeros of polynomial
\[
B[T](z) = B[f \circ \rho_k](z) - \alpha B[f \circ \rho_k](z)
\]
lie in \( |z| < 1 \) where \( \rho_k(z) = tz \). This implies
\[
B[P \circ \rho_k](z) - \alpha B[P \circ \rho_k](z)
\]
\[
\leq B[P^* \circ \rho_k](z) - \alpha B[P^* \circ \rho_k](z), \quad (2.3)
\]
for \( |z| \geq 1 \) and \( R > r \geq 1 \). If Inequality (2.3) is not true, then there exits a point \( z = z_0 \) with \( |z_0| \geq 1 \) such that
\[
B[P \circ \rho_k](z_0) - \alpha B[P \circ \rho_k](z_0)
\]
\[
\leq B[P^* \circ \rho_k](z_0) - \alpha B[P^* \circ \rho_k](z_0), \quad (2.4)
\]
But all the zeros of \( P^*(Rz) \) lie in \( |z| < 1/R < 1 \), therefore, it follows (as in case of \( f(z) \)) that all the zeros of \( P^*(Rz) - \alpha P^*(rz) \) lie in \( |z| < 1 \). Hence, by Lemma 1, we have
\[
B[P^* \circ \rho_k](z_0) - \alpha B[P^* \circ \rho_k](z_0) \neq 0.
\]
We take
\[
\lambda = \frac{B[P \circ \rho_k](z_0) - \alpha B[P \circ \rho_k](z_0)}{B[P^* \circ \rho_k](z_0) - \alpha B[P^* \circ \rho_k](z_0)},
\]
then \( \lambda \) is well defined real or complex number with \( |\lambda| > 1 \) and with this choice of \( \lambda \), we obtain
\[
B[T](z_0) = 0 \text{ where } |z_0| \geq 1. \text{ This contradicts the fact that all the zeros of } B[T](z) \text{ lie in } |z| < 1. \text{ Thus (2.3) holds true for } |\alpha| \leq 1 \text{ and } R > r \geq 1.
\]
Next we describe a result of Arestov [11].
For \( \delta = (\delta_1, \delta_2, \ldots, \delta_n) \in \mathbb{C}^{n+1} \) and
\[
P(z) = \sum_{j=1}^{n} a_j z^j \in \mathcal{P}_n \text{, we define}
\]
\[
\Lambda_{\delta}\mathcal{P}(z) = \sum_{j=0}^{n} \delta_j a_j z^j.
\]
The operator \( \Lambda_{\delta} \) is said to be admissible if it pre-
serves one of the following properties:
1) \( P(z) \) has all its zeros in \( \{ z \in \mathbb{C} : |z| \leq 1 \} \).
2) \( P(z) \) has all its zeros in \( \{ z \in \mathbb{C} : |z| \geq 1 \} \).

The result of Arestov [11] may now be stated as follows.

**Lemma 4.** [11, Theorem 4] Let \( \phi(x) = \psi(\log x) \) where \( \psi \) is a convex non-decreasing function on \( \mathbb{R} \). Then for all \( P \in \mathcal{P}_n \) and each admissible operator \( \Lambda_\delta \),

\[
\int_0^{2\pi} \phi\left( |\Lambda_\delta P(e^{i\theta})| \right) d\theta \\
\leq \int_0^{2\pi} \phi\left( C(\delta,n) |P(e^{i\theta})| \right) d\theta,
\]

where \( C(\delta,n) = \max\{ |\partial_\delta|, |\partial_n| \} \).

In particular, Lemma 4 applies with \( \phi : x \to x^p \) for every \( p \in (0, \infty) \). Therefore, we have

\[
\left( \int_0^{2\pi} \left| \Lambda_\delta P(e^{i\theta}) \right|^p d\theta \right)^{1/p} \\
\leq C(\delta,n) \left( \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right)^{1/p}.
\]  \tag{2.5}

We use (2.5) to prove the following interesting result.

**Lemma 5.** If \( P \in \mathcal{P}_n \) and \( P(z) \) does not vanish in \( |z| < 1 \), then for every \( p > 0 \), \( R > r \geq 1 \) and for \( \sigma \) real, \( 0 \leq \sigma < 2\pi \),

\[
\int_0^{2\pi} \left| B[\Lambda, P(z)] - \alpha B[P^* \circ \rho_r](z) \right|^p d\theta \\
+ \left\| B[P^* \circ \rho_r](z) - \alpha B[P^* \circ \rho_r](z) \right\|^p d\theta \\
\leq \left\| \left( R^* - \alpha \sigma \right) \phi(\lambda_0, \lambda_1, \lambda_2) e^{i\sigma} + (1 - \alpha \sigma) \lambda_0 \phi(\lambda_0, \lambda_1, \lambda_2) \right\|^p
\]

\[
\int_0^{2\pi} |P(e^{i\theta})|^p d\theta,
\]

where \( B \in \mathcal{B}_n \), \( \rho_r(z) = rz \),

\[
B[P^* \circ \rho_r](z) := \left( B[P^* \circ \rho_r](z) \right)^* \quad \text{and} \quad \phi(\lambda_0, \lambda_1, \lambda_2)
\]

is defined by (1.13).

**Proof.** Since \( P \in \mathcal{P}_n \) and \( P^*(z) = z^n P(1/z) \), by Lemma 3, we have for \( |z| \geq 1 \),

\[
B[P^* \circ \rho_r](z) - \alpha B[P^* \circ \rho_r](z) \\
\leq \left( B[P^* \circ \rho_r](z) - \alpha B[P^* \circ \rho_r](z) \right).
\]  \tag{2.7}

Also, since

\[
P^*(Rz) - \alpha P^*(rz) \\
= R^* z^n P(1/|z|^2) - \alpha R^n z^n P(1/|z|^2),
\]

and therefore,

\[
B[P^* \circ \rho_r](z) - \alpha B[P^* \circ \rho_r](z) \\
= \left( B[P^* \circ \rho_r](z) - \alpha B[P^* \circ \rho_r](z) \right)^*
\]

\[
= \left( B[P^* \circ \rho_r](z) - \alpha B[P^* \circ \rho_r](z) \right)^*.
\]  \tag{2.8}

Also, for \( |z| = 1 \),

\[
B[P^* \circ \rho_r](z) - \alpha B[P^* \circ \rho_r](z) \\
= B[P^* \circ \rho_r](z) - \alpha B[P^* \circ \rho_r](z).
\]

Using this in (2.7), we get for \( |z| = 1 \),

\[
B[P^* \circ \rho_r](z) - \alpha B[P^* \circ \rho_r](z) \\
\leq \left( B[P^* \circ \rho_r](z) - \alpha B[P^* \circ \rho_r](z) \right)^*.
\]

As in the proof of Lemma 3, the polynomial \( P^*(z) - \alpha P^*(z) \) has all its zeros in \( |z| < 1 \), and by Lemma 1, \( B[P^* \circ \rho_r](z) - \alpha B[P^* \circ \rho_r](z) \), also has all its zeros in \( |z| < 1 \), therefore,

\[
B[P^* \circ \rho_r](z) - \alpha B[P^* \circ \rho_r](z) \quad \text{has all its zeros in} \quad |z| \geq 1.
\]  \tag{2.9}
A direct application of Rouche’s theorem shows that
with \( P(z) = a_n z^n + \cdots + a_0 \),
\[
\Lambda_\sigma P(z) = \left\{ B\left[ P \circ \rho_{\pi}\right](z) - a B\left[ P \circ \rho_{\pi}\right](z) \right\} e^{\sigma z} + B\left[ P \circ \rho_{\pi}\right]^* (z) - e^{-\sigma z} B\left[ P \circ \rho_{\pi}\right]^* (z)
\]  
\[
= \left\{ \left( R^n - \alpha r^n \right)(\lambda_0 + \lambda_1 n^2 \frac{2}{2} + \lambda_2 n^4 \frac{(n-1)}{8}) e^{\sigma z} + (1-\alpha) \lambda_0 \right\} - a z^n + \cdots
\]  
\[
\left( R^n - \alpha r^n \right)(\lambda_0 + \lambda_1 n^2 \frac{2}{2} + \lambda_2 n^4 \frac{(n-1)}{8}) e^{\sigma z} + (1-\alpha) \lambda_0,
\]
has all its zeros in \(|z| \geq 1\) for every real \( \sigma \), \(0 \leq \sigma \leq 2\pi\). Therefore, \( \Lambda_\sigma \) is an admissible operator. Applying (2.5) of Lemma 4, the desired result follows immediately for each \( p > 0 \).

From Lemma 5, we deduce the following more general result.

**Lemma 6.** If \( P \in \mathcal{P}_0 \), then for every \( p > 0 \), \( R > r \geq 1 \) and \( \sigma \) real, \( 0 \leq \sigma \leq 2\pi \),
\[
\int_0^{2\pi} \left| B\left[ P \circ \rho_{\pi}\right]\left(e^{i\theta}\right) - a B\left[ P \circ \rho_{\pi}\right]\left(e^{i\theta}\right) \right| e^{\sigma z} d\theta
\]  
\[
+ \left\{ B\left[ P \circ \rho_{\pi}\right]^* (e^{i\theta}) - e^{-\sigma z} B\left[ P \circ \rho_{\pi}\right]^* (e^{i\theta}) \right\} \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^p d\theta,
\]
(2.10)
\[
\leq \left\{ \left( R^n - \alpha r^n \right) \phi(\lambda_0, \lambda_1, \lambda_2) e^{\sigma z} + (1-\alpha) \lambda_0 \right\}
\]  
\[
\cdot \int_0^{2\pi} \left| e^{i\theta}\right|^p d\theta,
\]
Proof. Let \( P \in \mathcal{P}_0 \) and let \( z_1, z_2, \ldots, z_n \) be the zeros of \( P(z) \). If \( |z_j| \geq 1 \) for all \( j = 1, 2, \ldots, n \), then the result follows by Lemma 5. Henceforth, we assume that \( P(z) \) has at least one zero in \(|z| < 1\) so that we can write
\[
P(z) = P_{k}(z) P_{l}(z),
\]
\[
a \prod_{j=1}^{k} (z - z_j) \prod_{j=k+1}^{n} (z - z_j),
\]
where the zeros \( z_1, z_2, \ldots, z_k \) of \( P_{k}(z) \) lie in \(|z| \geq 1\) and the zeros \( z_{k+1}, z_{k+2}, \ldots, z_n \) of \( P_{l}(z) \) lie in \(|z| < 1\). First we suppose that \( P_{k}(z) \) has no zero on \(|z| = 1\) so that all the zeros of \( P_{k}(z) \) lie in \(|z| > 1\). Since all the zeros of \((n-k)\) th degree polynomial \( P_{l}(z) \) lie in \(|z| < 1\), all the zeros of its conjugate polynomial
\[
P_{l}^*(z) = z^{n-k} \prod_{j=k+1}^{n} (z - z_j),
\]
\[
\left| P_{l}^*(z) \right| = \left| P_{l}(z) \right| \text{ for } |z| = 1.
\]
Now consider the polynomial
\[
f(z) = P_{l}^*(z) P_{l}(z)
\]
\[
= a \prod_{j=1}^{k} (z - z_j) \prod_{j=k+1}^{n} (1 - z_j),
\]
then all the zeros of \( f(z) \) lie in \(|z| > 1\), and for \(|z| = 1\),
\[
\left| f(z) \right| = \left| P_{l}(z) \right| \left| P_{l}^*(z) \right| = \left| P_{l}(z) \right| \left| P_{l}^*(z) \right| = \left| P(z) \right|.
\]
(2.11)

Therefore, it follows by Rouche’s Theorem that the polynomial \( g(z) = P(z) + \beta f(z) \) has all its zeros in \(|z| > 1\) for every \( \beta \), with \(|\beta| > 1\) so that all the zeros of \( T(z) = g(\tau z) \) lie in \(|z| \geq 1\) for some \( \tau > 1 \). Applying (2.9) and (2.8) to the polynomial \( T(z) \), we get for \( R > 1 \) and \(|z| < 1\),
\[
\int_0^{2\pi} \left| B\left[ T \circ \rho_{\pi}\right]\left(e^{i\theta}\right) - a B\left[ T \circ \rho_{\pi}\right]\left(e^{i\theta}\right) \right| e^{\sigma z} d\theta
\]  
\[
+ \left\{ B\left[ T \circ \rho_{\pi}\right]^* (e^{i\theta}) - e^{-\sigma z} B\left[ T \circ \rho_{\pi}\right]^* (e^{i\theta}) \right\} \int_0^{2\pi} \left| T\left(e^{i\theta}\right) \right|^p d\theta,
\]
that is,
\[
\int_0^{2\pi} \left| B\left[ T \circ \rho_{\pi}\right]\left(e^{i\theta}\right) - a B\left[ T \circ \rho_{\pi}\right]\left(e^{i\theta}\right) \right| e^{\sigma z} d\theta
\]  
\[
\leq \left\{ \left( R^n - \alpha r^n \right) \phi(\lambda_0, \lambda_1, \lambda_2) e^{\sigma z} + (1-\alpha) \lambda_0 \right\}
\]  
\[
\cdot \int_0^{2\pi} \left| e^{i\theta}\right|^p d\theta,
\]
(2.12)
where \( \rho_1(z) = iz \).

Since \( g(z) \) has all its zeros in \( |z| > 1 \), it follows that \( g'(z) \) has its zeros in \( |z| < 1 \) and hence (proceeding similarly as in proof of Lemma 3) the polynomial \( g^* \circ \rho_1(z) - g^* \circ \rho_1(z) \) also has all its zeros in \( |z| < 1 \). By Lemma 1,

\[
\frac{B[g \circ \rho_k](z) - \alpha B[g \circ \rho_k](z)}{|B[g \circ \rho_k](z) - \alpha B[g \circ \rho_k](z)|} + \frac{B[g \circ \rho_k](z) - \alpha B[g \circ \rho_k](z)}{|B[g \circ \rho_k](z) - \alpha B[g \circ \rho_k](z)|} = \alpha B[g \circ \rho_k](z)
\]

for \( |z| \leq 1 \). If Inequality (2.15) is not true, then there exists a point \( z = z_0 \) with \( |z_0| \leq 1 \) such that

\[
\frac{B[f \circ \rho_k](z_0) - \alpha B[f \circ \rho_k](z_0)}{|B[f \circ \rho_k](z_0) - \alpha B[f \circ \rho_k](z_0)|} + \frac{B[f \circ \rho_k](z_0) - \alpha B[f \circ \rho_k](z_0)}{|B[f \circ \rho_k](z_0) - \alpha B[f \circ \rho_k](z_0)|} = \alpha B[f \circ \rho_k](z_0)
\]

so that \( \beta \) is a well-defined real or complex number with \( |\beta| > 1 \) and with this choice of \( \beta \), from (2.14), we get \( L(z_0) = 0 \). This clearly is a contradiction to the fact that \( L(z) \) has all its zeros in \( |z| > 1 \). Thus (2.15) holds, which in particular gives for each \( p > 0 \) and \( \sigma \) real,

\[
\int_0^{2\pi} \left| B[f \circ \rho_k](e^{\theta}) - \alpha B[f \circ \rho_k](e^{\theta}) \right| e^{i\theta} \left| e^{i\theta} \right| d\theta \leq \int_0^{2\pi} \left| B[f \circ \rho_k](e^{\theta}) - \alpha B[f \circ \rho_k](e^{\theta}) \right| e^{i\theta} \left| e^{i\theta} \right| d\theta.
\]
Lemma 4 and (2.7) applied to \( f \), gives for each \( p > 0 \),
\[
\int_0^{2\pi} \left| B[P \circ \rho_\alpha \theta](e^{i\theta}) - \alpha B[P \circ \rho_\alpha \theta](e^{i\theta}) \right|^p d\theta + \left| B[P^* \circ \rho_\alpha \theta^*](e^{i\theta}) - \alpha B[P^* \circ \rho_\alpha \theta^*](e^{i\theta}) \right|^p d\theta \\
\leq \left( R^* - \alpha r^* \right) \phi(\lambda_0, \lambda_1, \lambda_2) e^{i\theta} + (1 - \alpha) \overline{\lambda}_0 \right] \times \int_0^{2\pi} f(e^{i\theta}) \left| f(e^{i\theta}) \right|^p d\theta.
\]
(2.16)

Now if \( \tilde{P}(z) \) has a zero on \( |z| = 1 \), then applying (2.16) to the polynomial \( \tilde{P}(z) = P_1(\mu z) P_2(z) \) where \( 0 < \mu < 1 \), we get for each \( p > 0, \ R > r \geq 1 \) and \( \sigma \) real,
\[
\int_0^{2\pi} \left| B[P \circ \rho_\alpha \theta](e^{i\theta}) - \alpha B[P \circ \rho_\alpha \theta](e^{i\theta}) \right|^p d\theta + \left| B[P^* \circ \rho_\alpha \theta^*](e^{i\theta}) - \alpha B[P^* \circ \rho_\alpha \theta^*](e^{i\theta}) \right|^p d\theta \\
\leq \left( R^* - \alpha r^* \right) \phi(\lambda_0, \lambda_1, \lambda_2) e^{i\theta} + (1 - \alpha) \overline{\lambda}_0 \right] \times \int_0^{2\pi} \tilde{P}(e^{i\theta}) \left| \tilde{P}(e^{i\theta}) \right|^p d\theta.
\]
(2.17)

Letting \( \mu \to 1 \) in (2.17) and using continuity, the desired result follows immediately and this proves Lemma 6.

**Lemma 7.** If \( P \in \mathcal{P}_r \), then for every \( p > 0, \ R > r \geq 1 \) and \( 0 \leq \sigma < 2\pi \),
\[
\int_0^{2\pi} \int_0^{2\pi} \left| B[P \circ \rho_\alpha \theta](z) - \alpha B[P \circ \rho_\alpha \theta](z) \right|^p d\theta d\sigma + \left| B[P^* \circ \rho_\alpha \theta^*](z) - \alpha B[P^* \circ \rho_\alpha \theta^*](z) \right|^p d\theta d\sigma \\
\leq \int_0^{2\pi} \int_0^{2\pi} \left( R^* - \alpha r^* \right) \phi(\lambda_0, \lambda_1, \lambda_2) e^{i\theta} + (1 - \alpha) \overline{\lambda}_0 \right] \times \int_0^{2\pi} P(e^{i\theta}) \left| P(e^{i\theta}) \right|^p d\theta.
\]
(2.18)

where \( B \in \mathcal{B} \), \( \rho_\alpha(z) = \frac{z}{\alpha} \) and \( \phi(\lambda_0, \lambda_1, \lambda_2) \) is defined by (1.13). The result is best possible and \( P(z) = b z^p \) is an extremal polynomial for any \( b \neq 0 \).

**Proof.** By Lemma 6, for each \( p > 0, \ 0 \leq \alpha < 2\pi \) and \( R > r \geq 1 \), the Inequality (2.6) holds. Since \( B[P^* \circ \rho_\alpha \theta^*](z) - \alpha B[P^* \circ \rho_\alpha \theta^*](z) \) is the conjugate polynomial of \( B[P^* \circ \rho_\alpha \theta^*](z) - \alpha B[P^* \circ \rho_\alpha \theta^*](z) \),
\[
\left| B[P^* \circ \rho_\alpha \theta^*](z) - \alpha B[P^* \circ \rho_\alpha \theta^*](z) \right| = \left| B[P^* \circ \rho_\alpha \theta^*](z) - \alpha B[P^* \circ \rho_\alpha \theta^*](z) \right|,
\]
and therefore for each \( p > 0, \ R > r \geq 1 \) and \( 0 \leq \alpha < 2\pi \), we have
\[
\int_0^{2\pi} \left| B[P \circ \rho_\alpha \theta](z) - \alpha B[P \circ \rho_\alpha \theta](z) \right|^p d\sigma + \left| B[P^* \circ \rho_\alpha \theta^*](z) - \alpha B[P^* \circ \rho_\alpha \theta^*](z) \right|^p d\sigma \\
\leq \int_0^{2\pi} \left( R^* - \alpha r^* \right) \phi(\lambda_0, \lambda_1, \lambda_2) e^{i\theta} + (1 - \alpha) \overline{\lambda}_0 \right] \times \int_0^{2\pi} P(e^{i\theta}) \left| P(e^{i\theta}) \right|^p d\sigma.
\]
3. Proof of Theorems

Proof of Theorem. By hypothesis \( P \in \mathcal{P}_n \), we can write
\[
P(z) = P_1(z)P_2(z) = a_k \prod_{j=1}^{n} (z - z_j),
\]
where the zeros \( z_1, z_2, \ldots, z_n \) of \( P_1(z) \) lie in \( |z| \leq 1 \) and the zeros \( z_1, z_2, \ldots, z_n \) of \( P_2(z) \) lie in \( |z| > 1 \). First, we suppose that all the zeros of \( P_1(z) \) lie in \( |z| < 1 \). If all the zeros of \( P_1(z) \) lie in \( |z| > 1 \), the polynomial \( P_2(z) = z^{-k}P_2(1/z) \) has all its zeros in \( |z| < 1 \) and \( |P_2(z)| = |P_2(1/z)| \) for \( |z| = 1 \). Now consider the polynomial
\[
M(z) = P(z)P_2(z) = a_k \prod_{j=1}^{n} (z - z_j) (1 - z^{-1}),
\]
then all the zeros of \( M(z) \) lie in \( |z| < 1 \), and for \( |z| = 1 \),
\[
|M(z)| = |P(z)||P_2(z)| = |P(z)||P_2(z)| = |P(z)|. \tag{3.1}
\]

Observe that \( P(z)/M(z) \to 1/\prod_{j=1}^{n} (z - z_j) \) when \( z \to \infty \), so it is regular even at \( \infty \) and thus from (3.1) and by the maximum modulus principle, it follows that
\[
|P(z)| \leq |M(z)| \text{ for } |z| \geq 1.
\]

Since \( M(z) \neq 0 \) for \( |z| \geq 1 \), a direct application of Rouche’s theorem shows that the polynomial \( H(z) = P(z) + \lambda M(z) \) has all its zeros in \( |z| < 1 \) for every \( \lambda \) with \( |\lambda| > 1 \). Applying Lemma 2 to the polynomial \( H(z) \) and noting that the zeros of \( H(Rz) \) lie in \( |z| < 1/R < 1 \), we deduce (as in Lemma 3) that for every real or complex \( \alpha \) with \( |\alpha| \leq 1 \), all the zeros of
\[
G(z) = H(Rz) - \alpha H(Rz) = P(Rz) - \alpha P(Rz) - \lambda \{M(Rz) - \alpha M(Rz)\}
\]
lie in \( |z| < 1 \). Applying Lemma 1 to \( G(z) \) and noting that \( B \) is a linear operator, it follows that all the zeros of
\[
B[G](z) = B[P \circ \rho_{\lambda}](z) - \alpha B[P \circ \rho_{\lambda}](z)
\]
lie in \( |z| < 1 \) for every \( \lambda \) with \( |\lambda| > 1 \). This implies for \( |z| > 1 \),
\[
B[P \circ \rho_{\lambda}](z) - \alpha B[P \circ \rho_{\lambda}](z) \leq B[M \circ \rho_{\lambda}](z) - \alpha B[M \circ \rho_{\lambda}](z),
\]
which, in particular, gives for each \( p > 0 \), \( R > r \geq 1 \) and \( 0 \leq \theta < 2\pi \),
\[
\int_{0}^{2\pi} \left[ B[P \circ \rho_{\lambda}](e^{i\theta}) - \alpha B[P \circ \rho_{\lambda}](e^{i\theta}) \right] d\theta \leq \int_{0}^{2\pi} \left[ B[M \circ \rho_{\lambda}](e^{i\theta}) - \alpha B[M \circ \rho_{\lambda}](e^{i\theta}) \right] d\theta. \tag{3.2}
\]

Again, (as in case of \( H(z) \)) \( M(Rz) - \alpha M(Rz) \) has all its zeros in \( |z| < 1 \), thus by Lemma 1,
\[
B[P \circ \rho_{\lambda}](z) - \alpha B[P \circ \rho_{\lambda}](z) \leq \int_{0}^{2\pi} \left[ B[M \circ \rho_{\lambda}](e^{i\theta}) - \alpha B[M \circ \rho_{\lambda}](e^{i\theta}) \right] d\theta.
\]

Combining Inequalities (3.3), (3.4) and noting that \( M(\xi) = |P(\xi)| \), we obtain for each \( p > 0 \) and \( R > 1 \),
\[
\int_{0}^{2\pi} \left[ B[P \circ \rho_{\lambda}](e^{i\theta}) - \alpha B[P \circ \rho_{\lambda}](e^{i\theta}) \right] d\theta \leq \int_{0}^{2\pi} \left[ B[M \circ \rho_{\lambda}](e^{i\theta}) - \alpha B[M \circ \rho_{\lambda}](e^{i\theta}) \right] d\theta. \tag{3.5}
\]
In case \( P_1(z) \) has a zero on \( |z| = 1 \), then Inequality (3.5) follows by continuity. This proves Theorem 1 for \( p > 0 \). To obtain this result for \( p = 0 \), simply make \( p \to 0 + \).

Proof of Theorem 2. By hypothesis \( P(z) \) does not vanish in \( |z| < 1 \), \( \rho_{\lambda}(z) = rz \) and \( R > r \geq 1 \), therefore, for \( 0 \leq \theta < 2\pi \), (2.1) holds. Also, for each \( p > 0 \) and \( \sigma \) real, (2.18) holds.

Now it can be easily verified that for every real number \( \sigma \) and \( s \geq 1 \),
\[
|s e^{i\sigma}| \geq 1 + e^{i\sigma}.
\]
This implies for each \( p > 0 \),
\[
\int_{0}^{2\pi} \left[ s e^{i\sigma}|e^{i\theta}|^p d\sigma \geq \left( e^{i\sigma}|e^{i\theta}|^p \right) d\sigma. \tag{3.6}
\]
If \( B[P \circ \rho_{\lambda}](e^{i\theta}) - \alpha B[P \circ \rho_{\lambda}](e^{i\theta}) \neq 0 \), we take
\[
s = \frac{B[P \circ \rho_{\lambda}](e^{i\theta}) - \alpha B[P \circ \rho_{\lambda}](e^{i\theta})}{B[P \circ \rho_{\lambda}](e^{i\theta}) - \alpha B[P \circ \rho_{\lambda}](e^{i\theta})},
\]
then by (2.1), \( s \geq 1 \) and we get with the help of (3.6),
\[
\int_0^{2\pi} \left[ B[P \circ \rho_R](e^{i\sigma}) - \alpha B[P \circ \rho_R](e^{i\sigma}) \right] \, d\sigma \\
\leq \int_0^{2\pi} \left\{ \left\| R \alpha P \right\| \phi(\lambda_0, \lambda_1, \lambda_2) e^{i\sigma} + (1-\alpha) \lambda_0 \right\|^p \, d\sigma
\]

From which Theorem 2 follows for \( p > 0 \). To establish this result for \( p = 0 \), we simply let \( p \to 0^+ \).

**Proof of Theorem 3.** Since \( P(z) \) is a self-inversive polynomial, then we have for some \( \nu \), with \( |\nu| = 1 \)
\( P(z) = \nu P^*(z) \) for all \( z \in \mathbb{C} \), where \( P^*(z) \) is the conjugate polynomial \( P(z) \). This gives, for \( 0 \leq \theta < 2\pi \)
\[
\left\| B[P \circ \rho_R](e^{i\sigma}) - \alpha B[P \circ \rho_R](e^{i\sigma}) \right\| = \left| B[P^* \circ \rho_R](e^{i\sigma}) - \alpha B[P^* \circ \rho_R](e^{i\sigma}) \right|. \\
\]

Using this in place of (2.1) and proceeding similarly as in the proof of Theorem 2, we get the desired result for each \( p > 0 \). The extension to \( p = 0 \) obtains by letting \( p \to 0^+ \).

**REFERENCES**


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