Certain $pl(m,n)$-Kummer Matrix Function of Two Complex Variables under Differential Operator

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ABSTRACT

The main aim of this paper is to define and study of a new matrix functions, say, the $pl(m,n)$-Kummer matrix function of two complex variables. The radius of regularity, recurrence relation and several new results on this function are established when the positive integers $p$ is greater than one. Finally, we obtain a higher order partial differential equation satisfied by the $pl(m,n)$-Kummer matrix function and some special properties.

Keywords: Hypergeometric Matrix Function; $pl(m,n)$-Kummer Matrix Function; Matrix Differential Equation; Differential Operator

1. Introduction

Many Special matrix functions appear in connection with statistics [1], mathematical physics, theoretical physics, group representation theory, Lie groups theory [2] and orthogonal matrix polynomials are closely related [3-5]. The hypergeometric matrix function has been introduced as a matrix power series and an integral representation and the hypergeometric matrix differential equation in [6-9] and the explicit closed form general solution of it has been given in [10]. The author has earlier studied the Kummer’s and Horn’s $H_2$ matrix function of two complex variables under differential operators [11-13]. In [14-16], extension to the matrix function framework of the classical families of $p$-Kummer’s matrix function, $p$ and $q$-Appell matrix function and Humbert matrix function have been proposed.

Throughout this paper for a matrix $A$ in $C^{N \times N}$, its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of $A$. If $A$ is a matrix in $C^{N \times N}$, its two-norm denoted by $\|A\|_2$ is defined by [17]

$$\|A\|_2 = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|_2}$$

where for a vector $x$ in $C^N$, $\|y\|_2 = \left[(\sum |y_i|^2)^{\frac{1}{2}}\right]$ is the Euclidean norm of $y$.

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable $z$, defined in an open set $\Omega$ of the complex plane, and if $A$ and $B$ are a matrix in $C^{N \times N}$ with $\sigma(A) \subset \Omega$ and $\sigma(B) \subset \Omega$ also and if $AB = BA$, then from the properties of the matrix functional calculus [18], it follows that

$$f(A)g(B) = g(B)f(A). \quad (1.1)$$

The reciprocal gamma function denoted by

$$\Gamma^{-1}(z) = \frac{1}{\Gamma(z)}$$

is an entire function of the complex variable $z$. Then for any matrix $A$ in $C^{N \times N}$, the image of $\Gamma^{-1}(z)$ acting on $A$ denoted by $\Gamma^{-1}(A)$ is a well defined matrix. Furthermore, if $A + nI$ is invertible for every non-negative integer $n$ (1.2)

where $I$ is the identity matrix in $C^{N \times N}$, then $\Gamma(A)$ is invertible, its inverse coincides with $\Gamma^{-1}(A)$ and one gets [6]

$$\Gamma(A) = \Gamma(A + I)(A + 2I)\cdots(A + (n - 1)I) = \Gamma(A + nI)\Gamma^{-1}(A); \quad n \geq 1; \quad (A)_n = I. \quad (1.3)$$

Jódar and Cortés have proved in [6], that

$$\Gamma(A) = \lim_{n \to \infty} (n - 1)! \left[(A)_n\right]^{-1} n^A. \quad (1.1)$$

2. On $pl(m,n)$-Kummer Matrix Function

We define the $pl(m,n)$-Kummer matrix function

$^{x}P_2\left(A; B; z, w\right)$ of two complex variables in the form

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\[ r \Phi_2 (A; B; z, w) \]
\[ = \sum_{l(m,n) \leq 0} \frac{(A)_{l(m,n)} [(B)_{l(m,n)}]^{-1}}{(pl(m,n))!} z^{\mathbf{w}} w^{\mathbf{r}} \]  
(2.1)

where

\[ U_{m,n}(z, w) = \frac{(A)_{l(m,n)} [(B)_{l(m,n)}]^{-1}}{(pl(m,n))!}, \]

\[ V_{m,n} = \frac{(A)_{l(m,n)} [(B)_{l(m,n)}]^{-1}}{(pl(m,n))!}, \]

\[ l(m,n) = \frac{1}{2} (m + n + 1) (m + n) + n \]  
[19] and \( p, m \) and \( n \) are non-negative integer numbers. Notice that \( l(m,n) \) is a non-negative integer number.

For simplicity, we can write the \( r \Phi_2 (A \pm I; B; z, w) \) in the form \( r \Phi_2 (A \pm I) \), \( r \Phi_2 (A; B \pm I; z, w) \) in the form \( r \Phi_2 (B \pm I) \) and \( r \Phi_2 (A \pm I; B \pm I; z, w) \) in the form \( r \Phi_2 (A \pm I, B \pm I) \).

We begin the study of this function by calculating its radius of regularity of such function for this purpose we recall relation (1.3.10) of [19] and keeping in mind that \( 1 \leq \sigma_{m,n} \leq 2^{-^{m+n}} \). Hence

\[ \frac{1}{R} = \lim_{m \to +\infty} \sup_{n \to +\infty} \left( \frac{V_{m,n}}{\sigma_{m,n}} \right) = \lim_{m \to +\infty} \sup_{n \to +\infty} \left( \frac{(A)_{l(m,n)} [(B)_{l(m,n)}]^{-1}}{(pl(m,n))!} \sigma_{m,n} \right) \]

\[ = \lim_{m \to +\infty} \sup \left[ 2\pi \left( A + (l(m,n) - 1)I \right) e^{-\frac{(A + (l(m,n) - 1)I)}{e}} \right]^{\frac{1}{m+n}} \]

\[ \times \left[ 2\pi \left( B + (l(m,n) - 1)I \right) e^{-\frac{(B + (l(m,n) - 1)I)}{e}} \right]^{\frac{1}{m+n}} \]

\[ \times \left[ 2\pi l(m,n) \left( \frac{pl(m,n)}{e} \right)^{\sigma_{m,n}} \right]^{\frac{1}{m+n}} \]

\[ \leq \lim_{m \to +\infty} \sup \left[ e^{-\frac{(A + (l(m,n) - 1)I)}{e}} \right]^{\frac{1}{m+n}} \]

\[ \times \left[ e^{-\frac{(B + (l(m,n) - 1)I)}{e}} \right]^{\frac{1}{m+n}} \]

\[ \times \left[ \frac{(A + (l(m,n) - 1)I)}{e} \right]^{-\frac{1}{m+n}} \]

\[ \leq \lim_{m \to +\infty} \sup \left[ \frac{(A + (l(m,n) - 1)I)}{l(m,n)} \right]^{\frac{1}{m+n}} \times \left[ \frac{(B + (l(m,n) - 1)I)}{l(m,n)} \right]^{\frac{1}{m+n}} \]

\[ \times \left( e \right)^{\frac{1}{m+n}} \]

\[ \leq \lim_{m \to +\infty} \sup \left[ \frac{(A + (l(m,n) - 1)I)}{l(m,n)} \right]^{\frac{1}{m+n}} \times \left[ \frac{(B + (l(m,n) - 1)I)}{l(m,n)} \right]^{\frac{1}{m+n}} \times \left( e \right)^{\frac{1}{m+n}} = 0. \]
where

\[
\sigma_{m,n} = \begin{cases} 
\left(\frac{m+n}{m}\right)^{\frac{n}{2}} \left(\frac{m+n}{n}\right)^{\frac{m}{2}}, & m,n \neq 0; \\
1, & m,n = 0.
\end{cases}
\]

Summarizing, the following result has been established.

**Theorem 2.1.** Let \( A \) and \( B \) be matrices in \( C^{N \times N} \) such that \( B + l(m,n)I \) are invertible for all integer \( l(m,n) \geq 0 \). Then, the \( p(m,n) \)-Kummer matrix function is an entire function.

For \( p = 1 \), we have

\[
\lim_{m \to \infty} \sup_{\sigma_{m,n}} \left( \frac{W_{m,n}}{\sigma_{m,n}} \right) = 0
\]

i.e., the \( l(m,n) \)-Kummer matrix function is an entire function.

Some matrix recurrence relations are carried out on the \( p(m,n) \)-Kummer matrix function. In this connection the following matrix contiguous functions relations follow, directly by increasing or decreasing one in original

\[
p \Phi_2(A--;B; z, w) = \sum_{l(m,n) \geq 0} (A-I) [B + l(m,n)I]^{-1} U_{m,n}(z, w),
\]

\[
p \Phi_2(A--;B; z, w) = \sum_{l(m,n) \geq 0} (B - I) [A + l(m,n)I]^{-1} U_{m,n}(z, w).
\]

By the same way, we have

\[
p \Phi_2(A++;B; z, w) = \sum_{l(m,n) \geq 0} (B - I) [A + l(m,n)I]^{-1} U_{m,n}(z, w),
\]

\[
D = \frac{1}{2}(D_1 + D_2) = \frac{1}{2}(D^2 + D) + D_2
\]

where \( D = d_1 + d_2 \), \( d_1 = \frac{\partial}{\partial z} \) and \( d_2 = w \frac{\partial}{\partial w} \).

It is clear that

\[
D^p \Phi_2(A;B; z, w) = \left[ \frac{1}{2}(D^2 + D) + D_2 \right] \Phi_2(A;B; z, w) = \sum_{l(m,n) \geq 0} \left[ \frac{1}{2}((m+n)^2 + (m+n)) + n \right] (A)(B)_{l(m,n)}^{-1} \left( pl(m,n) \right)! z^{m} w^{n},
\]

\[
= \sum_{l(m,n) \geq 0} l(m,n) (A)(B)_{l(m,n)}^{-1} \left( pl(m,n) \right)! z^{m} w^{n}.
\]
So that

\[ D^p \Phi_2(A; B; z, w) = \sum_{l(m,n) \geq 0} \frac{l(m,n)}{(pl(m,n))!} \left( \frac{(A)_{l(m,n)}}{(B)_{l(m,n)}} \right)^{-1} \cdot z^m w^n \]  

Putting in this relation \( m - 1 \) and \( n + 1 \) instead of \( m \) and \( n \) respectively, then

\[ l(m-1, n+1) = \frac{1}{2} (m+n)(m+n+1) + n+1 = l(m, n) + 1 \]

and so that we can be written the relation \( m - \frac{1}{p} \) and

\[ n + \frac{1}{p} \]

instead of \( m \) and \( n \) yields

\[ l \left( m - \frac{1}{p}, n + \frac{1}{p} \right) = l(m, n) + \frac{1}{p} \]

and

\[ pl \left( m - \frac{1}{p}, n + \frac{1}{p} \right) = pl(m, n) + 1. \]

Therefore, the power series \( ^p \Phi_2(A; B; z, w) \), as follows

\[ \left[ D \left( D - \frac{1}{p} \right) \cdots \left( D - \frac{p-1}{p} \right) \right] ^p \Phi_2(A; B; z, w) \]

\[ = \sum_{l(m,n) \geq 0} \frac{l(m,n)}{(pl(m,n))!} \left( \frac{(A)_{l(m,n)}}{(B)_{l(m,n)}} \right)^{-1} \cdot z^m w^n \]  

i.e., the \( pl(m, n) \)-Kummer matrix function is a solution of the matrix differential equation

\[ D^p \Phi_2(A; B; z, w) \]

\[ = \frac{1}{p} \sum_{l(m,n) \geq 0} \frac{(A)_{l(m,n)}}{(B)_{l(m,n)}} \left( \frac{(A)_{l(m,n)}}{(B)_{l(m,n)}} \right)^{-1} \cdot z^m w^n \]  

\[ = \frac{1}{p} \int \frac{1}{z} (A) \left( \frac{(B)}{z} \right) \Phi_2 \left( A + \frac{1}{p}; B + \frac{1}{p}; z, w \right) \]  

(2.7)

In this paper, we affect by differential operator \( D \) the \( pl(m, n) \)-Kummer matrix function, successively, then we have

\[ \left[ D \left( D - \frac{1}{p} \right) \cdots \left( D - \frac{p-1}{p} \right) \right] ^p \Phi_2(A; B; z, w) \]

\[ = \sum_{l(m,n) \geq 0} \frac{l(m,n)}{(pl(m,n))!} \left( \frac{(A)_{l(m,n)}}{(B)_{l(m,n)}} \right)^{-1} \cdot z^m w^n \]  

(2.6)
i.e. the \((m, n)\)-Kummer matrix function is a solution to this matrix differential equation

\[
D \left( D - \frac{1}{p} \right) \left( D - \frac{2}{p} \right) \cdots \left( D - \frac{p-1}{p} \right) \frac{w}{z^p} \Phi_2(A; B; z, w) = 0. \tag{2.8}
\]

Then

\[
D \left( D - \frac{1}{p} \right) \left( D - \frac{2}{p} \right) \cdots \left( D - \frac{p-1}{p} \right) (DI + B - I) \frac{w}{z^p} \Phi_2(A; B; z, w)
\]

\[
= \sum_{i, a \geq 0} (B + (l(m, n) - 1)I)(A)_{l(m, n)} \left( B \right)_{l(m, n)}^{-1} \frac{z^a w^p}{(pl(m, n))!}.
\]

Therefore, the following result has been established.

**Theorem 2.2.** Let \( A \) and \( B \) be matrices in \( C^{N \times N} \). Then the \( p(m, n) \)-Kummer matrix function is solution of this matrix differential equation

\[
D \left( D - \frac{1}{p} \right) \left( D - \frac{2}{p} \right) \cdots \left( D - \frac{p-1}{p} \right) (DI + B - I) = 0.
\]

The \( \alpha(D) \) differential operator has been defined by Sayed [19] in the form

\[
\alpha(D) = 1 + \sum_{k=1}^{N} D^k; \quad D^k = DD^{k-1}.
\]

From (2.1), (2.3) and (2.5), we obtain

\[
(DI + B - I) \frac{w}{z^p} \Phi_2(A; B; z, w)
\]

\[
= \sum_{i, a \geq 0} (B + (l(m, n) - 1)I)(A)_{l(m, n)} \left( B \right)_{l(m, n)}^{-1} \frac{z^a w^p}{(pl(m, n))!}.
\]

hence

\[
D \Phi_2(A; B; z, w)
\]

\[
= (B - I) \left[ D \Phi_2(A; B; z, w) - \frac{w}{z^p} \Phi_2(A; B; z, w) \right] \tag{2.10}
\]

and

\[
D^2 \Phi_2(A; B; z, w)
\]

\[
= (B - I) \left( B - 2I \right) \frac{w}{z^p} \Phi_2(A; B; z, w)
\]

\[
= \left( B - I \right) \left( B - 2I \right) \frac{w}{z^p} \Phi_2(A; B; z, w) \tag{2.12}
\]

Thus by mathematical induction, we have the following general form

\[
\alpha(D) \Phi_2(A; B; z, w) = 1 + \sum_{k=1}^{N} D^k \Phi_2(A; B; z, w)
\]

\[
= \Phi_2(A; B; z, w) + \sum_{k=1}^{N} \left( B - jI \right) \Phi_2(A; B - jI; z, w)
\]

\[
+ \sum_{k=1}^{N} \left( B - jI \right) \Phi_2(A; B - jI; z, w) \tag{2.13}
\]

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where \( N \) is a finite positive integer.

Special cases: we can be written the matrix function \( \Phi_2(A; z, w) \) in the form
\[
\rho \Phi_2(A; z, w) = \sum_{l(n, m) \in \mathbb{N}} (A)_{l(n, m)} z^w w^l (n, m) (l(n, m))! \quad (2.14)
\]
we see that
\[
D^n \Phi_2(A; z, w) = \frac{w}{z} (D I + A)^n \Phi_2(A; z, w).
\]

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